On the Garden-of-Eden problem for one-dimensional cellular automata

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Definitions. 1. A *cellular automaton* (further on shortly CA) is a structure

$$U = (A, Z^d, x, f)$$

where

A — finite, nonempty set. It represents the set of states that can be taken by any cell.

Z^d — set of d-tuples of integers.

x — $(\xi_1, \ldots, \xi_n)$, neighborhood function $\xi_i \in Z^d$, $(i=1, \ldots, n)$.
The neighborhood of a cell is the n-tuple

$$N(s) = s+\xi_1, s+\xi_2, \ldots, s+\xi_n$$

where $s+\xi_i$, $(1 \leqq i \leqq n)$ is the componentwise sum of the d-tuples.
We say, that the neighborhood size of the CA is n.

f — local transformation function, $f: A^n \to A$. As it is a discrete function, it can be defined by a table. This table will be called the defining table of the local transformation function.

2. The local transformation function (its defining table) will be called balanced, if each $a \in A$ occurs as a value of the function just as many times.

3. A configuration is a mapping $c: Z^d \to A$. The set of all configurations will be denoted by C $(C = (c|c: Z^d \to A))$.

4. A mapping $F: C \to C$ defined from a local transformation f as follows will be called global transformation function

$$F(c) = c^1 \quad \text{if} \quad c^1(s) = f[c(N(s))] \quad s \in Z^d$$

where $c(N(s))$ is the restriction of mapping c to $N(s)$.

5. $a_0 \in A$ is a quiescent state if $f(a_0, \ldots, a_0) = a_0$. A quiescent state is always supposed to exist.

Remark. Instead of any neighborhood an arbitrary d-dimensional array can be considered that contains the given neighborhood. The transition depends on the joined cells trivially, i.e., it is independent of them.

Further we assume such a neighborhood.

6. A configuration $c \in C$ is of finite support if only a finite number of values of $c: Z^d \to A$ differs from a_0. This finite set of cells in c will be called the support of the configuration. Denote by C_F the set of all configurations of finite support.

7. A configuration $c \in C$ will be called Garden-of-Eden or briefly Eden if there is no $c_1 \in C$ such that $F(c_1) = c$. This concept can be defined considering C_F instead of C as well.

It would be important to know whether a given CA contains Eden-configuration or not, for it is closely related to function F. More exactly, if there exists an Eden-configuration in a CA, the mapping F is not surjective.

Amoroso and Patt created an algorithm [1] by means of which it can be decided for an arbitrary one-dimensional CA whether it contains Eden or not.

Algorithm of Amoroso and Patt:

Construct a finite graph (a tree) each node of which will be a subset of all n-tuples in A^n with equal images of f. Select an element $b \in A$, let the (unique) node N at level 0 be the set of all n-tuples $(a_1, \ldots, a_n)$ such that $f(a_1, \ldots, a_n) = b$. For each node N at level i (>0) construct for each $a \in A$ a node N_a at level $i+1$ as it follows. If $(a_1, \ldots, a_n)$ is an element in the set corresponding to N, the set corresponding to N_a will consist of exactly those n-tuples $(a_2, \ldots, a_n, d)$, $(d \in A)$ for which $f(a_2, \ldots, a_n, d) = a$. A directed arc labeled by a is then drawn from node N to node N_a.

If for each $(a_1, \ldots, a_n)$ at N there are no such elements d, then this node N_a is not included in the graph and we say that N is terminal node in the graph.

If during the construction process a number of nodes appear at the same or different levels, all nodes associated with the same subset of A^n then each one will be a distinct node in the graph, but only one, arbitrarily chosen, will be extended. This process must terminate since there is a bound on the number of possible subsets of A^n.

Theorem 1. In a CA (A, Z, x, f) if there is no Eden of finite support then there is no terminal node in the graph constructed above.

Proof. If we get the terminal node across the arcs $bb_1 \ldots b_k$, it is easy to show that the configuration having this support has no preimage under F.

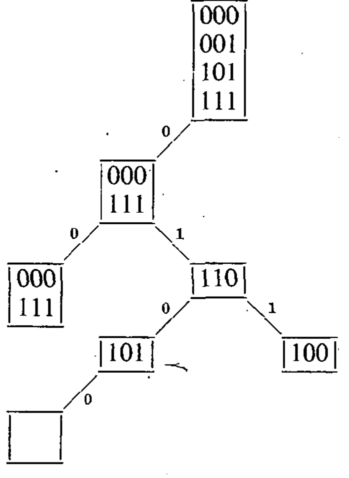
Theorem 2. In a CA (A, Z, x, f) if the defining table of the local transformation function is not balanced then there exists an Eden of finite support in the CA.

Proof. Will be given by means of the algorithm of Amoroso and Patt. Let n be the neighborhood size of the CA. Denote by $|A|$ the number of elements in the set A. We shall see if there is a node in the graph which contains k n-tuples, then continuing the process we get a node containing not more than $k-1$ n-tuples.

Suppose the contrary. Construct n levels of the graph, in this case the equal nodes too. On level n there are $|A|^n$ nodes. The number of n-tuples in each node is k because of our assumption. It should be clear from the construction process that for each $a_i \in A$ there are $|A|^{n-1}$ nodes on level n, which contains n-tuples having image a_i under f. (If it were not so, we should have a terminal node.) We show that all of such n-tuples are contained exactly k-times on level n. On the first level there are k n-tuples containing a_i as a last element, (for each $a_i \in A$) because we join all $a_i \in A$ to each n-tuple of level 0. These elements in the tailes of the n-tuples are shifting forward on each level, and to each tail every $a_i \in A$ joins. So we get all of the n-tuples of A k-times on level n. For each $a_i \neq a_j$ if the sets of n-tuples are disjoint and the multiplicity is always k then for each $a_i \in A$ there are exactly $|A|^{n-1}$ n-tuples having image a_i. So the table of the local transformation function is balanced which is contradiction.

So the statement is valid for each k, and for $k=1$ we shall get to a terminal node. Then from Theorem 1 it follows the existence of an Eden configuration.

REMARK. The converse of the theorem above is not true, there exists a transformation function which has balanced defining table, but the CA contains Eden configuration. For example: $A=2$, $n=3$, denote the two states of the cells 0 and 1. Let us construct the graph beginning with the n-tuples having image 0. (This defines the function f as well.)



We have got to a terminal node, so the CA has Eden configuration.

We have the following sufficient condition:

Theorem 3. If the defining table of the local transformation function is balanced and it holds for each $a_i \in A$ that taking away the first elements from the n-tuples having image a_i under f, there are no equal ones among the left $n-1$-tuples: or this is valid for the last elements, then the CA has no Eden configuration.

Proof. Construct the graph beginning with the n-tuples having image an arbitrary $a_i \in A$.

1. If the last $n-1$-tuples are different, on the following level all of the n-tuples appear. This is valid for all nodes so further we shall have the same nodes.

2. If the first $n-1$-tuples are different then from each n-tuple we get $|A|$ n-tuples which are in different nodes. Therefore, we shall never get to a terminal node.

Abstract

This article gives a necessary and a sufficient condition for the existence of an Eden configuration in one-dimensional cellular automata.

References

[1] Amoroso, S. and J. N. Patt, Decision procedure for surjectivity and injectivity of parallel maps for tessellation structures, *J. Comput. System Sci.*, v. 6, 1972, pp. 448—464.
[2] Amoroso, S. and G. Cooper, Tessellation structures for reproduction of arbitrary patterns, *J. Comput. System Sci.*, v. 5, 1971, pp. 455—464.
[3] Amoroso, S. and G. Cooper, The Garden-of-Eden theorem for finite configurations, *Proc. Amer. Math. Soc.*, v. 44, 1970, pp. 189—197.

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