Linear parallel maps of tessellation automata

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The s-state cellular automaton, where the local map computes the modulo-s sum of all neighbour-states, has very useful algebraic properties (see [1]). The principle of this cellular automaton will be generalized in this paper, as far as it is possible, applying inhomogeneous tessellation automata (defined in [2], [6] and below), so we get the concept of *linear parallel maps.* A wide class of parallel maps is obtained in such a way, keeping the good algebraic properties of the structure in [1].

The present work discusses the fundamental characteristics of linear parallel maps, giving some examples and applications to demonstrate the theoretical results. Finally, we mention an open problem showing that many further investigations are possible in the area.

1. Definitions

In [6] a general conception is developed for practical construction and application of cellular automata. On the basis of it we have given a strong generalization for the concept of cellular automaton in [2], those definitions are repeated below.

DEFINITION. An *inhomogeneous cellular automaton* (in short CA) is a fourtuple (C, A, N, Φ), where

$C=\{c_1, \ldots, c_m\}$ is the finite *set of cells,*

$A=\{0, 1, \ldots, s-1\}$ is the finite *set of cell-states,*

$N: c_i \mapsto (c_{i_1}, \ldots, c_{i_{n_i}})$ is the *neighbourhood function,*

which assigns to each cell its neighbours. The specification of neighbours may be different cell by cell, i.e. the cellular automaton has a totally arbitrary topology.

$\Phi: c_i \mapsto f_i$ is the *function-system,* which assigns to each cell an $f_i\colon A^{n_i} \to A$ local transition function (*local map* in short). The local maps also may be different cell by cell.

Note that a nearly homogeneous topology with only a few different local maps is sufficient in practice, yet the theoretical studies need no such a restriction.

The CA works as usual: at time $t=0$ each cell has an initial state. From t to $t+1$ each cell changes its state synchromously so that the new state of a cell depends only on its neighbours, according to the local map.

DEFINITION. An *inhomogeneous tessellation automaton* (in short TA) is a triple (C, A, N), where the components correspond to the above-mentioned ones. In this case the function-system is time-varying: at time t the TA executes a function-system Φ_t.

FURTHER USUAL DEFINITIONS. A *configuration* is a possible global state of the TA, formally a mapping $\alpha: C \to A$. It will be denoted always by Greek letters, and the notation $\alpha=(a_1, \ldots, a_m)$, where $a_i=\alpha(c_i)$, will be used too. We denote by $\mathcal{A}$ the set of all configurations.

The *parallel map* induced by a given function-system is a mapping $F: \mathcal{A} \to \mathcal{A}$, where $F(\alpha)=\beta$ if for all i $f_i(a_{i_1}, \ldots, a_{i_{n_i}})=\beta(c_i)$ (the n_i-tuple $(a_{i_1}, \ldots, a_{i_{n_i}})$ denotes the neighbourhood of c_i in α). A mapping $F: \mathcal{A} \to \mathcal{A}$ will be called a parallel map, if and only if there exists a function-system inducing F.

2. Linear parallel maps

In [1] an s-state homogeneous CA was investigated, with arbitrary dimension and neighbourhood-index, where the local map of any cell computes the sum (in the sense modulo s) of all neighbour-states. The parallel map of this CA is *linear* in the following sense: for any configurations α, β: $F(\alpha+\beta)=F(\alpha)+F(\beta)$, where $\alpha+\beta$ denotes the configuration satisfying $(\alpha+\beta)(c_i)=\alpha(c_i)+\beta(c_i)$ for any i. This property is indispensable to prove the most important characteristic of this CA: it reproduces an arbitrary pattern in s^q steps if q is great enough (see [1]).

In the following we define the concept of linear parallel maps in general.

Let (C, A, N) be a TA such that the set A forms a *finite commutative ring* with identity element, whose operations are denoted by $+$ and $\cdot$. In the most simple case A is a residue-class-ring, namely $A=\{0, 1, \ldots, s-1\}$ and the two operations are the modulo-s addition and multiplication.

Further a configuration α will be considered as a vector $(a_1, \ldots, a_m)$ over the ring A. If $\alpha=(a_1, \ldots, a_m)$ and $\beta=(b_1, \ldots, b_m)$ then the configurations $\alpha+\beta$ and $k\cdot\alpha$ are defined in the usual way: $\alpha+\beta:=(a_1+b_1, \ldots, a_m+b_m)$ and $k\cdot\alpha:=(ka_1, \ldots \ldots, ka_m)$.

It is clear that the following holds:

Theorem 1. The set $\mathcal{A}$ of all configurations forms an m-dimensional vector-space over the ring A.

DEFINITION. A parallel map F is called a *linear parallel map,* if it is a linear transformation of the vector-space $\mathcal{A}$ onto itself: $F(\alpha+\beta)=F(\alpha)+F(\beta)$ and $F(k\cdot\alpha)=k\cdot F(\alpha)$ holds for any $k\in A$ and $\alpha, \beta\in\mathcal{A}$.

It is well-known from algebra (see [3]), that a map $F: \mathcal{A} \to \mathcal{A}$ is linear iff there exists a matrix K of type $m\cdot m$ over A such that for any $\alpha(\in\mathcal{A})$, $F(\alpha)=K\cdot\alpha$ holds (here α is considered a matrix of type $m\cdot 1$). Using this fact the following result can be proved.

Theorem 2. A parallel map F is linear iff each local map has the form $f_i(a_{i_1}, \ldots, a_{i_{n_i}})=k_{ii_1}a_{i_1}+\ldots+k_{ii_{n_i}}a_{i_{n_i}}$, where $k_{ii_1}, \ldots, k_{ii_{n_i}}\in A$.

Proof. If F is a linear parallel map then $\exists K: \forall\alpha: F(\alpha)=K\cdot\alpha$. So we get: $f_{i_1}(a_{i_1}, \ldots, a_{i_{n_i}})=k_{i1}a_1+\ldots+k_{im}a_m$ for any i. However if c_j is not a neighbour of c_i, then k_{ij} must be equal to 0.

Conversely, if each f_i forms a linear combination, then the matrix K is constructable in the following way: if c_j is a neighbour of c_i then let k_{ij} be the coefficient corresponding to c_j in the formula of f_i. If c_j is not a neighbour of c_i then $k_{ij}:=0$. □

3. One-to-one parallel maps

From [3] we recall the following result: A matrix K over the ring A has an inverse K^{-1} iff the determinant of K is not a zero-divisor. (An element $a(\in A)$ is called zero-divisor if there exists a nonzero element $b(\in A)$ such that $a\cdot b=0$.)

Applying this result to parallel maps we get

Theorem 3. A linear parallel map F is one-to-one iff the determinant of its matrix is not a zero-divisor.

Considering a two-state CA we have a more concrete result

Theorem 4. A linear parallel map F, induced by a two-state CA, is one-to-one iff the number of such distinct permutations $p: C\to C$, where for any $i, p(c_i)$ is a "real" neighbour of c_i, is odd. (c_j is called a "real" neighbour of c_i, if the local map f_i depends on the state of c_j.)

Proof. The ring $A=\{0, 1\}$ is zero-divisor-free (it is a field), therefore we must prove: the determinant D of the matrix of F has the value 1 iff the right-hand condition above holds. By the definition, $D=\sum_p(-1)^I\cdot k_{1p_1}\ldots k_{mp_m}$ where p is a permutation $C\to C$ and p_i denotes $p(c_i)$. The factor $(-1)^I$ may be eliminated because in the field $\{0, 1\}$ the elements -1 and 1 are equal. Further we have: $D=1$ iff the number of permutations satisfying $k_{1p_1}\cdot\ldots\cdot k_{mp_m}=1$ is odd, where $k_{ij}=1$ means that c_j is a real neighbour of c_i. □

Example. Let (C, A, N, Φ) be a one-dimensional two-state CA interconnected into a circle (i.e., c_1 and c_m are neighbours), where $N(c_i)=(c_i, c_{i+1})$ and $f_i(a_i, a_{i+1})=a_i+a_{i+1}$ for any i. The parallel map of this CA is *not one-to-one*, because there are two permutations satisfying the condition in Theorem 4: $p_1(c_i)=c_i$ and $p_2(c_i)=c_{i+1}$ for any i. However this parallel map can be made *one-to-one* modifying only one among the local maps on $f_i(a_i, a_{i+1})=a_i$, since in this case we have only the permutation p_1.

In these two examples the bijectivity of F was independent of m. Note that in other cases it depends strongly on the size of the given CA (see [4]).

4. Decision procedure for bijectivity and reversibility

Definition. A parallel map F is called *(locally) reversible*, if it is one-to-one and F^{-1} is also a parallel map. The function-system, which generates F^{-1}, is called the *reverse function-system.*

In this paragraph a simple procedure is presented (realizable in practice by the simulation of the CA), which decides that whether or not a linear parallel map F induced by a CA (C, A, N, Φ) is one-to-one (i.e. bijectiv). If it is, then we can *decide its reversibility and construct the reverse function-system too.*

Let the CA start from a configuration $\varepsilon_i=(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$ until it reaches a cycle (i.e. until a time t_2, for which $\exists t_1(<t_2)$: $F^{t_1}(\varepsilon_i)=F^{t_2}(\varepsilon_i)$). If this procedure was executed for any ε_i $(i=1, \ldots, m)$, then we have one of the following two cases:

(i) There exists an ε_i such that in its cycle (from $F^{t_1}(\varepsilon_i)$ to $F^{t_2}(\varepsilon_i)$) the configuration ε_i is not occuring. This implies that there exist α, β such that $\alpha\neq\beta$ and $F(\alpha)=F(\beta)$, i.e., F is *not one-to-one.*

(ii) Each ε_i generates a cycle returning into itself. Consequently each ε_i has a predecessor $F^{-1}(\varepsilon_i)$. Since $\varepsilon_1, \ldots, \varepsilon_m$ is a basis of the vector-space $\mathcal{A}$, therefore F is one-to-one, F^{-1} is linear too and its matrix consists of the columns $F^{-1}(\varepsilon_1), \ldots$ $\ldots, F^{-1}(\varepsilon_m)$. Moreover, F^{-1} is a *parallel map* iff its matrix has zero elements for each (i, j) where c_j is not a neighbour of c_i.

Summarizing our-results, we get

Theorem 5. A linear parallel map F is *one-to-one* iff for any i the configuration $\varepsilon_i=(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$ has a predecessor $F^{-1}(\varepsilon_i)$. Furthermore F is *reversible* iff every $F^{-1}(\varepsilon_i)$ has only the neighbouring cells of c_i in nonzero state. □

Note that a *homogeneous CA* shows the same behaviour for any ε_i, consequently it is sufficient to investigate it starting from ε_1. Our procedure is powerful practically only in this case.

Example. A homogeneous CA (C, A, N, Φ) is considered (a so-called Lindenmayer-CA, investigated in [4]) where

$C=\{c_{1,1}, \ldots, c_{8,8}\}$ is a two dimensional matrix,

$A=\{0, 1\}$,

N: $c_{ij}\mapsto(c_{ij}, c_{i-1,j}, c_{i+1,j}, c_{i,j-1}, c_{i,j+1})$

for any i, j. The indexes are interpreted cyclically (for example $N(c_{1,1})=$ $=(c_{1,1}, c_{8,1}, c_{2,1}, c_{1,8}, c_{1,2})$), hereby the edges are interconnected into a torus (homogeneous topology).

Φ: $c_{ij}\mapsto f$ for any i, j, where f computes the modulo-2 sum of all neighbour-states. So Φ induces a linear parallel map.

Starting this CA from the configuration $\varepsilon_{4,4}$ a four-step cycle is obtained (Figure 1). Since the CA is homogeneous, so each ε_{ij} shows an analogous behaviour.* This implies that *our CA generates a one-to-one parallel map F.* But, at the same time, in the configuration $F^{-1}(\varepsilon_{4,4})$ (see step 3) there are 16 cells, not neighbouring with $c_{4,4}$, in nonzero state. Consequently *the CA is not locally reversible.*

* If the torus-connection is rejected and dummy cells are used on the edges of CA (inhomogeneous topology!), then we need a test for the configurations $\varepsilon_{1,1}$, $\varepsilon_{2,1}$, $\varepsilon_{2,2}$, $\varepsilon_{3,1}$, $\varepsilon_{3,2}$, $\varepsilon_{3,3}$, $\varepsilon_{4,1}$, $\varepsilon_{4,2}$, $\varepsilon_{4,3}$, too.

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0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0    0 0 0 1 0 0 0 0
0 0 0 0 0 0 0 0    0 0 0 1 0 0 0 0    0 0 0 0 0 0 0 0
0 0 0 1 0 0 0 0    0 0 1 1 1 0 0 0    0 1 0 1 0 1 0 0
0 0 0 0 0 0 0 0    0 0 0 1 0 0 0 0    0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0    0 0 0 1 0 0 0 0
0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0

    initial             step 1             step 2

       0 0 0 1 0 0 0 0    0 0 0 0 0 0 0 0
       0 0 1 1 1 0 0 0    0 0 0 0 0 0 0 0
       0 1 0 0 0 1 0 0    0 0 0 0 0 0 0 0
       1 1 0 1 0 1 1 0    0 0 0 1 0 0 0 0
       0 1 0 0 0 1 0 0    0 0 0 0 0 0 0 0
       0 0 1 1 1 0 0 0    0 0 0 0 0 0 0 0
       0 0 0 1 0 0 0 0    0 0 0 0 0 0 0 0
       0 0 0 0 0 0 0 0    0 0 0 0 0 0 0 0

           step 3             step 4
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Figure 1

Note that we can construct a *reverse* CA to the above-mentioned one, as follows: let N be a homogeneous torus-topology on the set $C=\{c_{1,1}, \ldots, c_{8,8}\}$ such that e.g. $N(c_{4,4})$ contains the cells having the state 1 in $F^{-1}(\varepsilon_{4,4})$ (see step 3 in Figure 1), and let Φ be such that each local map computes the modulo-2 sum of all neighbour-states. This CA generates the parallel map F^{-1}.

5. The problem of the synthesis

In this point we mention an important open problem in the theory of TA (it was investigated in [5], in case of one-dimension) which, if only linear parallel maps are considered, simplifies into a matrix-theoretical problem.

It is obvious that for some linear parallel maps F_1 and F_2 with matrixes K_1 and K_2, respectively, the mapping $F_2 \circ F_1$ is linear too, and has the matrix $K_2 \cdot K_1$. We have the following problem:

If (C, A, N) is a TA and $F: \mathscr{A} \to \mathscr{A}$ is an arbitrary linear mapping, then whether or not there exist linear parallel maps $F_1, \ldots, F_n$ such that $F_n \circ \ldots \circ F_1 = F$. If we have no such a row of parallel maps, then what extension of the neighbourhood function N is needed to that?

Because in the matrix of any parallel map each element $k_{ij}=0$, for which c_j is not a neighbour of c_i, so the problem described above alters into an algebraic question: which index-sets $I=\{(i_1, j_1), \ldots, (i_r, j_r)\}$ do satisfy, that an arbitrary matrix M over A is decomposable into a product $M_1 \cdot \ldots \cdot M_n$, where each component M_i contains nonzero elements only with indexes in I?

Note that in [5] it was proved that for a usual (i.e. infinite, homogeneous) one-dimensional TA such finite index set (i.e. a finite neighbourhood) does not exist.

6. Concluding remarks

In this work a special class of cellular automata was studied having linear parallel maps and — consequently — linear algebraic properties. In Theorem 2 were characterized the local maps inducing linear parallel maps; further it was shown that many interesting problems in the theory of cellular automata (e.g. the Garden-of-Eden problem, reversibility, synthesis-problem) can be investigated easily in this linear case, applying the results of linear algebra.

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