Schützenberger's monoids*

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In [1], Schützenberger proposed the following problem. "Give an algorithm to construct inductively all finite monoids M which contain a submonoid P satisfying

$$(U_s)\; m, m' \in M \,\&\, mm', m'm \in P \Leftrightarrow m, m' \in P$$

$$(N_d)\; m \in M \Rightarrow P \cap MmM \neq \emptyset$$

and (to limit the problem to its essential) which are such that P is not a union of classes of a nontrivial congruence on M."

Definition 1. *A* (U_s, N_d)*-submonoid* of a monoid M is a submonoid P satisfying the two conditions (U_s) and (N_d). Such a submonoid is *simple* if P is not a union of classes of a nontrivial congruence on M.

Theorem 1. *Let P be a simple (U_s, N_d)-submonoid of a finite monoid M. Then P contains all invertible elements of M. If $x \in P$ and the $\mathcal{H}$-class of x is a group then P contains the entire $\mathcal{H}$-class of x. Some element of the lowest $\mathcal{D}$-class of M belongs to P. All $\mathcal{H}$-classes of the lowest $\mathcal{D}$-class D_0 of M contain only one element. And P contains the centralizer of any element of $D_0 \cap P$.*

Proof. Let P be a simple (U_s, N_d)-submonoid of M, and let D_0 be the lowest $\mathcal{D}$-class of M. Condition (N_d) is equivalent to stating that P contains some element z of D_0. Suppose x belongs to P and the $\mathcal{H}$-class of x is a group. Let e be the identity element of this $\mathcal{H}$-class and y any other element of the $\mathcal{H}$-class. Then $ex=xe=x$ implies e belongs to P. And $yy^{-1}=y^{-1}y=e$ implies y belongs to P. Therefore P contains the entire $\mathcal{H}$-class of x, and also the $\mathcal{H}$-classes of all elements of P in D_0. Also P contains the $\mathcal{H}$-class of the identity element of M. Therefore it contains all invertible elements of M.

Let α be the equivalence relation $x\mathcal{L}y$ if and only if $x=y$ or $x, y \in D_0$ and $x\mathcal{H}y$. We claim α is a congruence. Let $x, y \in D_0$ and $x\mathcal{H}y$. Let e be the idempotent of this $\mathcal{H}$-class. Let $a \in M$. Then $ax=(ae)x$ and $ay=(ae)y$. The $\mathcal{D}$-class D_0 is a finite simple semigroup, and $ae \in D_0$ and $x\mathcal{H}y$ in D_0. By the structure of

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finite simple semigroups (Suschkevitch's theorem) this implies $(ae)x\mathscr{H}(ae)y$. Likewise $xa\mathscr{H}ya$. Therefore α is a congruence. If the $\mathscr{H}$-classes of D_0 contain more than one element the congruence α is nontrivial. Since P is a union of classes of α, this would mean P is not simple. Therefore the $\mathscr{H}$-classes of P contain only a single element.

Let c belong to the centralizer of $z\in D_0\cap P$. Then $cz=zc=zzc=zcz=z(zcz)z$ since z is idempotent. But $z(zcz)z$ lies in the $\mathscr{H}$-class of z. Since this $\mathscr{H}$-class contains only one element $cz=zc=z$. Therefore $c\in P$. This proves the theorem.

NOTATION. (1) Let $|X|$ denote the cardinality of a set X.
(2) Let B_n denote the semigroup of binary relations on an n-element set.
(3) Let T_n denote the semigroup of transformations on an n-element set.

Corollary. *Let M, P be as in the preceding theorem and let $|M|>1$, and let 1 be the monoid identity. Then M cannot be abelian, contain a zero, be an inverse semigroup, B_n, T_n, or $GLS(n, F)$.*

Proof. The preceding theorem implies that D_0 must contain more than one $\mathscr{H}$-class, else D_0 would be a single zero element and $P=M$. For $|M|>1$, P would not be simple. In particular M cannot contain a zero. This rules out all the above types of semigroups except T_n.

Suppose $M=T_n$, $n>1$. Then the symmetric group belongs to P. Therefore all rank 1 transformations belong to P. This implies all transformations belong to P, by condition (U_s). Therefore for $n>1$, M is not simple.

Proposition 2. *Let P be a (U_s, N_d)-submonoid of the finite monoid M. Let α be the relation $x\alpha y$ if and only if for all $u, v\in M$ ($uxv\in P$ if and only if $uyv\in P$). Then α is a congruence on M and P is a union of classes of α. Let M_0, P_0 be the quotients of M, P by α. Then P_0 is a simple (U_s, N_d)-submonoid of M_0.*

Proof. It is immediate that α is an equivalence relation, and a computation shows that α is a congruence. Suppose $x\alpha y$ and $y\in P$. Take $u=v=1$, the identity of the monoid. Then $x\in P$. Therefore P is a union of classes of α. Let M_0, P_0 be the quotients of M, P by α. Suppose P_0 is a union of classes of some congruence β. Let M_1, P_1 be the quotients of M_0, P_0 by β. Let $h_1\colon M\to M_0$ and $h_2\colon M_0\to M_1$ be the quotient homomorphisms. Let γ be the congruence on M such that $x\gamma y$ if and only if $(x)h_1h_2=(y)h_1h_2$. If β is a nontrivial congruence, there exist x, y such that $x\gamma y$ but not $x\alpha y$. By symmetry we may assume that for some $u, v\in M$, $uxv\in P$ and $uyv\notin P$. Therefore $(uxv)h_1h_2\in P_1$ but $(uyv)h_1h_2\notin P_1$. But $(x)h_1h_2= =(y)h_1h_2$. Therefore $(uxv)h_1h_2=(uyv)h_1h_2$. This is a contradiction. This proves the proposition.

Definition 2. Let G be a free monoid on generators $x_1, x_2, \dots, x_k$. If W is a word of G a *segment* of G is a word formed by the i-th through j-th letters of W in order, for some $i<j$. If $i=1$, the segment is called *initial.* If $j=n$ the segment is called *terminal.* Let G_n be the homomorphic image of G in which $W_1=W_2$ if and only if W_1 and W_2 have the same length n initial segment or $W_1=W_2$.

Theorem 3. *Let W_0 be a word of length n in G such that no initial segment of W_0 equals a terminal segment of W_0, other than the segment W_0 itself. Let*

$P=\{1, W_0\}$. Then P is a (U_s, N_d)-submonoid of G_n. Let α be the relation on G_n such that $x\alpha y$ if and only if for all $u, v\in G_n$: $uxv\in P$ if and only if $uyv\in P$. Then P/α is a simple (U_s, N_d)-submonoid of G_n/α. Suppose the last letter of W_0 is not x_1. Let S be the set $\{1, x_1^n$, all segments of W_0, Wx_1^{n-r} such that W is a terminal segment of length r of W_0 which also equals a nonterminal segment of $W_0\}$. Then S contains exactly one element from each class of α. Products in G_n/α can be described as follows. Take the product Y in G_n and reduce as follows. If $Y=$some element of S, the product is Y. Suppose Y does not equal an element of S. Suppose an initial segment t of Y equals a terminal segment of W_0 of length r, where r is a maximum. Then if t equals a nonterminal segment of W_0 the product in G_n/α is tx_1^{n-r}. If t does not equal a nonterminal segment of W_0 the product is t. If no initial segment of Y equals a terminal segment of W_0, then the product is x_1^n.

Proof. The set $P=\{1, W_0\}$ is a submonoid of G_n since $W_0^2=W_0$. Suppose for some $W_1, W_2\in G_n$, $W_1W_2=W_2W_1=1$. Then $W_1=W_2=1$. Suppose $W_1W_2=W_2W_1=W_0$. Then if $W_1, W_2\notin P$, some initial segment of W_0 equals a final segment. This is contrary to assumption. Therefore P satisfies condition (U_s). The lowest $\mathcal{D}$-class of G_n consists of all length n words. Therefore W_0 belongs to this lowest $\mathcal{D}$-class. Therefore P satisfies condition (N_d). It follows from Proposition 2 that G_n/α is simple. It remains to describe the relation α. Suppose x has the property that x is not a segment of W_0 and x is not 1 and no initial segment of x equals a final segment of W_0. It follows that uxv equals W_0 if and only if u equals W_0. Since x_1^n also has this property, $x\alpha x_1^n$. Suppose $x\notin S$ and an initial segment t of x equals a terminal segment of W_0 of length r, where r is maximal. Suppose t equals a nonterminal segment of W_0. If $uxv=W_0$ then $ux=W_0$ since x is not a segment of W_0. Therefore $ut=W_0$. Therefore $utx_1^{n-r}=W_0$. Suppose $utx_1^{n-r}v=W_0$. Then $utx_1^{n-r}=W_0$ since the length of tx_1^{n-r} is n. Therefore $ut=W_0$ since the last letter of W_0 is not x_1. Therefore $uxv=W_0$. This proves $x\alpha tx_1^{n-r}$. Suppose t does not equal a nonterminal segment of W_0. Then we have $x\alpha t$ by a similar argument. This proves that S contains at least one element from every class of α. Suppose $y\alpha z$ where $z=1$. Then $y\in P$. Therefore $y=1$ or W. But $zx_1\notin P$ implies $yx_1\notin P$ which implies $y\neq W$. So $y=1$. Suppose $y\alpha z$ where z is a nonterminal segment of W_0. Let $W_0=z_1zz_2$ in G. Then $z_1yz_2=W_0$. Suppose y had length greater than z. Then $z_1yz_3=W_0$ in G_n where z_3 is obtained from z_2 by omitting the last letter of z_2. But $z_1zz_3\neq W_0$. This contradicts $y\alpha z$. Therefore the length of y is not more than the length of z. So $z_1yz_2=z_1zz_2$ in G. So $y=z$. If z is a terminal segment of W_0 which does not equal a nonterminal segment of W_0, we have shown above that $z\alpha zx_1^{n-r}$ where r is the length of z. Suppose $y\alpha zx_1^{n-r}$ where z is any terminal segment of W of length r. Then y does not equal a nonterminal segment of W_0. Let $W_0=z_1z$ in G. Then $z_1y=W_0$ in G_n. Therefore $y=zz_2$ for some z_2. Suppose an initial segment of y of length greater than r equals a terminal segment of W_0. Then $y\alpha z$ will be false. This proves no two elements of S belong to the same class of α. Moreover it completely describes the relation α. The description of multiplication in G_n/α follows. This proves the theorem.

CONCLUDING REMARK. This construction can be generalized in a number of ways. For certain words W_0, P will have more than two elements. More than one word W_0 of length n can be chosen. A similar construction can be made where G_n is replaced by the free monoid band on $x_1, x_2, \ldots, x_k$ and W_0 is replaced by the word $x_1 x_2 \ldots x_k$. This will give (U_s, N_d)-simple submonoids of semigroups which are bands.

Abstract

We study pairs, P, M, $P \subset M$ of monoids such that P contains an element of the lowest $\mathcal{D}$-class of M and mm', $m'm \in P$ if and only if $m, m' \in P$ for all $m, m' \in M$. Such pairs are called simple if P is not a union of classes of a nontrivial congruence on M. We show that simple finite pairs P, M have certain characteristics which rule out most familiar semigroups. However we do construct an infinite family of simple, finite P, M pairs.

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Reference

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