The cardinality of closed sets in pre-complete classes in k-valued logics

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Introduction

Let $E_k = \{0, 1, \ldots k-1\}$. By a k-valued function we shall mean a function $f: E_k^n \to E_k$, and by P_k we denote the set of all those functions. If A is a subset of P_k, $[A]$ will stand for the set of all superpositions over A. (The definition of a superposition over A is the following:

1. $f \in A$ is a superposition over A.

2. If $g_0(x_1, \ldots, x_n), g_1(x_{11}, \ldots, x_{1m_1}), \ldots g_n(x_{n1}, \ldots, x_{nm_n})$ are either superpositions over A or $g_i(x_{i1}, \ldots, x_{im_i}) = x_j$ $(i=1, \ldots, n)$ then $g_0(g_1(x_{11} \ldots x_{1m_1}), \ldots, g_n(x_{n1}, \ldots x_{nm_n}))$ is a superposition over A.)

The set $A \subset P_k$ is closed if $A = [A]$. We call A complete if $[A] = P_k$. The closed set $\mathcal{M}$ is precomplete if $\mathcal{M} \subsetneqq A \subseteqq P_k$ implies $[A] = P_k$. I. Rosenberg [8] has given a complete description of the precomplete classes in P_k. In order to formulate his theorem we need some definitions. An h-ary relation R is a subset of E_k^h. If g is an n-ary k-valued function and R is an h-ary relation we say that f preserves R if $(f(x_1^1, \ldots, x_1^n), \ldots, f(x_h^1, \ldots, x_h^n)) \in R$ whenever $(x_1^1, \ldots, x_h^1) \in R, \ldots$ $\ldots, (x_1^n, \ldots, x_h^n) \in R$ an h-ary relation R is called central if it fulfils the following conditions:

1. $(a_1, \ldots, a_h) \in R$ whenever not all of $a_1, \ldots, a_h$ are distinct,

2. for each permutation π of $1, 2, \ldots, h$, $(a_1, \ldots, a_h) \in R$ if and only if $(a_{\pi(1)}, \ldots, a_{\pi(h)}) \in R$,

3. $\emptyset \neq \bigcap_{(a_1, \ldots, a_{h-1}) \in E_k^{h-1}} \{c \mid (a_1, \ldots, a_{h-1}, c) \in R\} \neq E_k$.

For $a \in E_k$ we denote by $[a]_l$ the l-th digit $(l=0, \ldots, m-1)$ in the expansion $a = \sum_{l=0}^{m-1} [a]_l \cdot h^l$ of a in the scale of h.

We may now state the theorem of Rosenberg as follows:

There are *6* types of precomplete classes in P_k and every proper closed subset of P_k is contained in at least one precomplete class.

This *6* types are the following:

1. $\mathcal{M}_\mu$ the set of all functions which preserve a partial order μ of E_k with greatest and least element.

2. S_π, the set of all functions which preserve the graph of a nonidentical permutation π where π is the product of cycles with the same prime length.

3. L_σ the set of all functions which preserve the quaternary relation

$$\sigma = \{(a_1, a_2, a_3, a_4)/a_1+a_2 = a_3+a_4\}$$

where $\langle E_k, +\rangle$ is an elementary Abelian p-group.

4. K_θ, the set of all functions which preserve the non trivial equivalence-relation θ of E_k^2.

5. C_ϱ, the set of all functions which preserve the h-ary central relation ϱ $(1 \leqq h \leqq k)$.

6. H_R, the set of all functions which preserve the relation R, where R is for some h $(3 \leqq h \leqq k)$ and for some surjection $\Phi\colon E_k \to E_{h^m}$ the h-ary relation

$$|\{[\Phi(x_1)]_l, \dots, [\Phi(x_n)]_l\}| < h \quad \text{for} \quad l = 0, \dots, m-1.$$

(Such a relation R is called h-regular.)

If A is a closed subset of P_k, $\nu(A)$ will denote the cardinality of the set of all closed sets contained in A. Let us denote by $\mathfrak{c}$ the cardinality of the continuum.

Ju. I. Yanov and A. A. Mučnik [5] have proved that $\nu(P_k)=\mathfrak{c}$ for $k>2$. The general result of E. Post [10] implies that $\nu(P_2)=\aleph_0$.

It is a natural question to determine $\nu(A)$ when A is a precomplete class. In this paper we shall prove the following three statements:

I. if $k>2$ and M is a precomplete class of type 1., 4., 5., or 6. then $\nu(M)=\mathfrak{c}$,

II. if $k>2$ then $\nu(S_\pi)\geqq\aleph_0$ for all precomplete classes of type 2.,

III. $\nu(S_\pi)=\mathfrak{c}$ if k is not prime.

The precomplete class L_σ was investigated by many authors. A. Salomaa [8] J. Demetrovics and J. Bagyinszki ([2] and [3]) proved $\nu(L_\sigma)<\aleph_0$ in the case if k is prime. J. Bagyinszki [1] and A. Szendrei [9] showed that if k is square-free then there are finitely many closed linear classes in P_k. A. Salooma [8] proved, that $\nu(L_\sigma)\geqq\aleph_0$ if k is not square-free and D. Lau [7] showed that $\nu(L_\sigma)=\aleph_0$ in this case.

1. §.

The proof of the first statement is based on the construction of Ju. I. Janov and A. A. Mučnik [5]. They have proved, that the set of functions $\{g_i\}$ defined by

$$g_i(x_1, \dots, x_i) = \begin{cases} b & \text{if} \quad |\{j | x_j = c\}| = i \quad \text{or} \\ & \qquad |\{j | x_j = b\}| = 1 \quad \text{and} \\ & \qquad |\{j | x_j = c\}| = i-1 \\ a & \text{in all other cases} \end{cases}$$

has the property

$$g_i \notin [\bigcup_{j \neq i} g_j]$$

$(a, b$ and c are pairwise distinct fixed elements of $E_k, k>2)$.

Let μ be a fixed partial order of E_k, let a be its least element, c its greatest one and $a<b<c$ such that $\{x|b<x<c\}=\emptyset$. In this case every g_i preserves μ, that is $v(\mathcal{M}_\mu)=\mathfrak{c}$. If θ is a non-trivial equivalence, then we can choose $a\neq b$ such that $a\equiv b(\theta)$. Let c be an arbitrary element of E_k $(c\neq a, c\neq b)$. Since $g_i(x_1, \dots, x_n)\in\{a, b\}$ all g_i preserve θ and $v(K_\theta)=\mathfrak{c}$. If ϱ is a central relation of E_k then g_i preserves ϱ whenever a is an element of the centre of ϱ. Hence $v(C_\varrho)=\mathfrak{c}$.

If R is an h-regular relation, then we can chose arbitrary distinct elements a, b, c. Every g_i preserves every h-regular relation of E_k.

Thus we have proved

Theorem 1. If $k>2$ then

$$v(M_\mu)=\mathfrak{c}$$
$$v(K_\theta)=\mathfrak{c}$$
$$v(C_\varrho)=\mathfrak{c}$$
$$v(H_R)=\mathfrak{c}$$

for all μ, θ, ϱ, R defined in I. ROSENBERG's theorem.

A permutation of E_k, π can be written as a product of disjoint cycles. Such a cycle will be denoted by c_i. If

$$\pi = c_1 \dots c_n \quad \text{and} \quad c_i = (a_{i1}, \dots, a_{im_i})$$

then $\{c_i\}$ will denote the set $\{a_{i1}, \dots, a_{im_i}\}$.

Lemma 1. Let $k\geqq 3$, π be a permutation in the form $\pi=c_1 \dots c_m$. If $m>1$ and there are $i, j\leqq m$ such that $i\neq j$,

$$|\{c_i\}| = k_1, \quad |\{c_j\}| = k_2 \quad \text{and} \quad k_1|k_2 \quad (k_1 \text{ devides } k_2)$$

then a set of closed classes of cardinality $\mathfrak{c}$ preserving π can be constructed.

Proof. We can assume, that

$$c_1 = (0, \dots, a_n), \quad c_2 = (1, 2, \dots, a_m) \quad \text{and} \quad |\{c_1\}|\,|\,|\{c_2\}|.$$

May be that $\{c_1\}=\{0\}$ or $\{c_2\}=\{1, 2\}$.

Let $m\geqq 3$ and

$$g_m(a_1, \dots, a_m) = \begin{cases} b\in c_2, & \text{if } \{a_1, \dots, a_m\}\subset\{c_2\} \text{ and } \big|\{j|a_j=b\}\big|=1 \text{ and} \\ & \text{all } a_j\neq b \text{ is equal to } \pi^{-1}(b), \\ d\in c_1, & \text{if } \{a_1, \dots, a_m\}\subset\{c_1\}\cup\{c_2\} \text{ and the previous} \\ & \text{condition does not hold,} \\ a_1 & \text{in all other cases.} \end{cases}$$

One can easily see, that since $|\{c_1\}|\,|\,|\{c_2\}|$, $g_m(x_1, \dots, x_m)$ preserves π.

We shall prove that $g_m\notin\big[\bigcup_{m\neq j} g_j\big]=G_m$ for all $m\geqq 3$. Let us suppose that $g_k\in G_k$ i.e.

$$g_k(x_1, \dots, x_k) = \mathfrak{A}(x_1, \dots, x_k)$$

where $\mathfrak{A}$ is a superposition over G_k. Let $g_s(x_{i_1}, \ldots, x_{i_s})$ be a function in $\mathfrak{A}$. If $s<k$ then we can find an x_l such that $x_l \notin \{x_{i_1}, \ldots, x_{i_s}\}$. If we choose $x_l=1$ and $x_1=x_2, \ldots$ $\ldots, x_{l-1}=x_{l+1}, \ldots, x_k=2$ then, by the definition, $g_k(x_1, \ldots, x_k)=1$, and $g_s(x_{i_1}, \ldots, x_{i_s})\in$ $\in c_1$ that is $\mathfrak{A}\neq 1$ holds. (All g_m preserve the set $\{c_1\}\cup\{c_2\}$ and $\{a_1, \ldots, a_m\}\cap c_1\neq\emptyset$, $\{a_1, \ldots, a_m\}\subset\{c_1\}\cup\{c_2\}$ imply $g_m(a_1, \ldots, a_m)\in c_1$. If $s>k$ then we have at least one pair

$$x_{i_k}, x_{i_l} \quad \text{such that} \quad i_k = i_l.$$

Let $x_{i_k}=x_{i_l}=1$ and all $x_j=2$ with $j\neq i_k$. In this case we have also $g_s(x_{i_1}, \ldots, x_{i_s})\in c_1$ and $g_k(x_1, \ldots, x_k)=1$, which is a contradiction. Thus Lemma 1 is proved.

As a corollary of Lemma 1 we obtain

Theorem 2. If $k>2$ and k is not prime then $\nu(S_\pi)=\mathfrak{c}$ for all precomplete classes S_π.

Lemma 2. Let $k>2$, and π be a permutation which contains at least one cycle of length $q\geqq 3$. Then a set of closed classes of cardinality $\mathfrak{c}$ preserving π can be constructed.

Proof. We will give a set of functions $\{t_i\}$ such that $t_k\notin[\bigcup_{i>k} t_i]=T_k$ and t_i preserves π.

Let

$$t_m(a_1, \ldots, a_m) = \begin{cases} b & \text{if } (a_1, \ldots, a_m) = (b, b, \ldots, b) \text{ or} \\ & (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_m) = (b, b, \ldots, b) \\ & \text{and } a_j = \pi^{-1}(b) \\ \pi^{-1}(b) & \text{if } \{a_1, \ldots, a_m\}\in\{\pi^{-1}(b), b\}^m \\ & \text{and } |\{j | a_j = b\}| \subset m-1 \\ a_1 & \text{in all other cases.} \end{cases}$$

(b is an element of a cycle which has the length $q\geqq 3$).

The definition implies that t_m preserves π, and $t_m(\{\pi^{-1}(b), b\}^m)\in\{\pi^{-1}(b), b\}$. A vector $a=(a_1, \ldots, a_m)$ is called characteristic if

$$|\{j | a_j = b\}| = m-1$$

and

$$|\{j | a_j = \pi^{-1}(b)\}| = 1.$$

Let us suppose that $t_m(x_1, \ldots, x_m)=\mathfrak{A}$ where $\mathfrak{A}$ is a superposition over T_m. In this case we can choose a formula $\mathfrak{A}^*$ such, that $\mathfrak{A}^*=t_s(\mathfrak{B}_1, \ldots, \mathfrak{B}_s)$, $\mathfrak{A}^*$ equals b on all characteristic vectors and for every $\mathfrak{B}_i$ there is a characteristic vector a^i such that $\mathfrak{B}_i(a^i)\neq b$. (I.e. $\mathfrak{A}^*$ is "minimal".)

By the assumption we have $s>m$. Let v^k denote the characteristic vector with $x_k=\pi^{-1}(b)$. Consider the matrix

$$\begin{vmatrix} \mathfrak{B}_1(v^1) \ldots \mathfrak{B}_s(v^1) \\ \mathfrak{B}_1(v^2) \ldots \mathfrak{B}_s(v^2) \\ \vdots \\ \mathfrak{B}_1(v^k) \ldots \mathfrak{B}_s(v^k) \end{vmatrix}$$

By the "minimality" of $\mathfrak{A}^*$ every column of the matrix contains at least one occurrence of $\pi^{-1}(b)$. $s>m$ implies that at least one row of the matrix contains two or more occurrence of $\pi^{-1}(b)$. If the l-th row contains at least twice $\pi^{-1}(b)$ then $\mathfrak{A}^*(v^l)=\pi^{-1}(b)$ which is a contradiction as $t_m(v^j)=b$ for all $j\in\{1, 2, ..., m\}$. Thus Lemma 2 is proved.

As an immediate consequence of Lemmas 2 and 1 we have

Theorem 3. If $k\geqq 3$ then for all precomplete classes S_π, $\nu(S_\pi)\geqq\aleph_0$ holds. If k is not prime, then $\nu(S_\pi)=\mathfrak{c}$.

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(Received April 3, 1979)