Recognition of monotone functions

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Let $n, k, k_1, k_2, \ldots, k_n$ be integers with $n \geqq 1$, $k \geqq 1$ and $1 \leqq k_1 \leqq k_2 \leqq \ldots \leqq k_n$. Moreover, let $E = \{0, 1, \ldots, k\}$ and $E_i = \{0, 1, \ldots, k_i\}$ for $i = 1, 2, \ldots, n$. We consider functions

$$f(\underset{\sim}{x}) = f(x_1, x_2, \ldots, x_n) \colon N = E_1 \times E_2 \times \ldots \times E_n \to E.$$

We always may assume that f takes each value of E. If $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ and $\underset{\sim}{y} = (y_1, y_2, \ldots, y_n)$ are vectors from N, let $\underset{\sim}{x} \leqq \underset{\sim}{y}$ if and only if $x_i \leqq y_i$ for $i = 1, 2, \ldots, n$. f is said to be monotonically increasing if $\underset{\sim}{x} \leqq \underset{\sim}{y}$ implies $f(\underset{\sim}{x}) \leqq f(\underset{\sim}{y})$. Let $M(k_1, k_2, \ldots, k_n, k)$ denote the set of all such monotone functions. $M(1, 1, \ldots, 1, 1)$ is the set of monotone Boolean functions.

Let $P(f)$ be a minimal set of vectors $\underset{\sim}{x}$ on which f has to be known for knowing the function completely. Let

$$\chi(k_1, k_2, \ldots, k_n, k) = \max_{f \in M(k_1, \ldots, k_n, k)} |P(f)|.$$

Furthermore, let $\varphi(k_1, k_2, \ldots, k_n, k)$ denote the minimal number of operations of the best algorithm for the recognition of an arbitrary function f of $M(k_1, k_2, \ldots, k_n, k)$. Clearly,

$$\varphi(k_1, k_2, \ldots, k_n, k) \geqq \chi(k_1, k_2, \ldots, k_n, k).$$

G. HANSEL [1] proved in case $k_n = k = 1$ that

$$\varphi(1, 1, \ldots, 1, 1) = \chi(1, 1, \ldots, 1, 1) = \binom{n}{\left[\frac{n}{2}\right]} + \binom{n}{\left[\frac{n}{2}\right]+1}.$$

It is conjectured that $\varphi = \chi$ is also true in the general case. Therefore, it is important to known χ exactly, not only a lower estimation. The aim of this note is to determine the exact value of χ. Let $m = \sum_{i=1}^{n} k_i$, $m(\underset{\sim}{x}) = \sum_{i=1}^{n} x_i$ and $S_m^l(N) = |\{\underset{\sim}{x} \colon \underset{\sim}{x} \in N,$ $m(\underset{\sim}{x}) = l\}|$. We have

Theorem 1.

$$\chi(k_1, k_2, \ldots, k_n, k) = \textit{sum of the } 2k \textit{ largest values } S_m^l(N).$$

Proof. A chain $(\underset{\sim}{x}^1, \underset{\sim}{x}^2, \ldots, \underset{\sim}{x}^m)$ of length m is a sequence of m different vectors from N satisfying $\underset{\sim}{x}^1 \leqq \underset{\sim}{x}^2 \leqq \ldots \leqq \underset{\sim}{x}^m$. $P(f)$, where f is an arbitrary function belonging to M, contains no chain of length $2k+1$. Assume the contrary. Then there are 3 consecutive members x', x'', x''' of the chain satisfying $f(x')=f(x'')==f(x''')=i$, where $i \in \{1, 2, \ldots, k-1\}$, or we have $f(x^2)=0$ or $f(x^{m-1})=k$. Since $i=f(x') \leqq f(x'') \leqq f(x''')=i$, $f(x^1) \leqq f(x^2)=0$ or $f(x^m) \geqq f(x^{m-1}) \geqq k$, $f(x'')$, $f(x^1)$ or $f(x^m)$, respectively, would follow from the others immediately, i.e. x'', x^1 or x^m could be omitted in $P(f)$, in contradiction to our supposition that P is minimal. By J. SCHÖNHEIM's result ([2], Theorem 2) we obtain for each f:

$$|P(f)| \leqq \textit{sum of the } 2k \textit{ largest values } S_m^l(N).$$

Now we consider the function

$$f(\underset{\sim}{x}) = \begin{cases} k & \text{if } \left[\frac{m}{2}\right]+k \leqq m(\underset{\sim}{x}), \\ i & \text{if } \left[\frac{m}{2}\right]+2i-k \leqq m(\underset{\sim}{x}) \leqq \left[\frac{m}{2}\right]+2i-k+1 \quad (i=1, \ldots, k-1), \\ 0 & \text{if } m(\underset{\sim}{x}) \leqq \left[\frac{m}{2}\right]-k+1. \end{cases}$$

$f(\underset{\sim}{x})$, where $\left[\frac{m}{2}\right]-k+1 \leqq m(\underset{\sim}{x}) \leqq \left[\frac{m}{2}\right]+k$, cannot be infered by f of the other vectors.

J. SCHÖNHEIM's remarks ([2], Remarks 4 and 5) complets the proof. □

In case $k_n=1$ we obtain

Corollary 1.

$$\chi(1, 1, \ldots, 1, k) = \sum_{i=\left[\frac{n}{2}\right]-k+1}^{\left[\frac{n}{2}\right]+k} \binom{n}{i}.$$

In case $k_n=k=1$ we obtain partly G. HANSEL's result.

Corollary 2.

$$\chi(1, 1, \ldots, 1, 1) = \binom{n}{\left[\frac{n}{2}\right]} + \binom{n}{\left[\frac{n}{2}\right]+1}.$$

Theorem 2.

$$\varphi(1, 1, \ldots, 1, k) = \sum_{i=\left[\frac{n}{2}\right]-k+1}^{\left[\frac{n}{2}\right]+k} \binom{n}{i}.$$

Proof. We use $\varphi \geqq \chi$ and Corollary 1 on one side and the special symmetrical chain method by G. HANSEL on the other side. Let f be known on all chains having a length $\leqq a$. Furthermore, let c be an arbitrary chain of length $a+2$. Then f is known on many of the members of c immediately. More precisely, at most on 2 vectors of c we do not know if f takes the value 0 or a value of $\{1, \ldots, k\}$. Then at most on 2 vectors of c we do not know if f takes the value 1 or a value of $\{2, \ldots, k\}$; and so on. Finally, f is unknown at most on $2k$ members of c. By HANSEL's argument the theorem follows immediately. □

Finally, we want to mention that HANSEL's special symmetrical chain method cannot be generalized to the general case $k_n \geqq 2$. N is then partitionable too, but not in HANSEL's special symmetrical chains. This can be verified easily in the case $n=2$, $k_1=1$ and $k_2=2$.

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References

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[2] SCHÖNHEIM, J., A generalization of results of P. Erdős, G. Katona, and D. J. Kleitman concerning Sperner's theorem, *J. Combinatorial Theory,* v. 11, 1971, pp. 111—117.

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