# Modal logics with function symbols

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We prove completeness theorems for modal logics with function symbols. These logics are generalizations of the well-known non-classical logical systems.

Our work was deeply influenced by a paper of K. SCHÜTTE [2].

## § 1. Preliminaries

We shall use the following symbols: parantheses, commas, variables, function symbols, relation symbols, logical symbols ($\sim, \wedge, \Box, \forall$). The set of terms is defined by the usual recurrence:

(i) If $x$ is a variable, then $x$ is a term.
(ii) If $f$ is an $n$-ary function symbol and $\tau_1, \ldots, \tau_n$ are terms, then $f(\tau_1, \ldots, \tau_n)$ is also a term. In the case of $n=0$ the parantheses will be omitted.

The set of atoms is defined in the standard way: if $r$ is an $n$-ary relation symbol and $\tau_1, \ldots, \tau_n$ are terms, then $r(\tau_1, \ldots, \tau_n)$ is an atom. Also, definition of the set of formulae is well-known:

(i) If $\mathscr{A}$ is an atom, then $\mathscr{A}$ is a formula.
(ii) If $\mathscr{A}, \mathscr{B}$ are formulae, then so are $(\mathscr{A} \wedge \mathscr{B})$, $\sim\mathscr{A}$ and $\Box\mathscr{A}$.
(iii) If $\mathscr{A}$ is a formula and $x$ is a variable, then $\forall x \mathscr{A}$ is a formula.

We shall use the abbreviations: $(\mathscr{A} \vee \mathscr{B})$ for $\sim(\sim\mathscr{A} \wedge \sim\mathscr{B})$; $(\mathscr{A} \to \mathscr{B})$ for $\sim(\sim\mathscr{B} \wedge \mathscr{A})$; $\Diamond\mathscr{A}$ for $\sim\Box\sim\mathscr{A}$; $\exists x\mathscr{A}$ for $\sim\forall x \sim\mathscr{A}$. Parantheses will be omitted if no confusion can occur. If $\mathscr{K}$ is a formula or term $x$ is a free variable (defined in the well-known way) and $\tau$ is a term, then $\mathscr{K}[x/\tau]$ will denote the result of substitution of $\tau$ for $x$ everywhere in $\mathscr{K}$. By a classical model $A$ we shall mean a function if it associates

(i) a non-empty set $|A|$ to 0 (zero),
(ii) a function $f_A\colon |A|^n \to |A|$ to each $n$-ary function symbol $f$,
(iii) a relation $r_A \subseteq |A|^n$ to each $n$-ary relation symbol $r$.

**Definition.** By a modal model we mean a quintuple $\langle S, N, O, R, P\rangle$ where $S$ is an arbitrary set, $N \subseteqq S$, $O \in S$, $R \subseteqq S \times S$, $P$ is a function with domain $S$ and $P(A)$ is a classical model, provided $A \in S$, furthermore $|P(A)| \subseteqq |P(B)|$ if $A, B \in S$, $ARB$.

**Definition.** A modal model is simple if for every $n$-ary function symbol $f$, there exists a function $\bar{f}$ with domain $\bigcup_{A \in S} |P(A)|$, such that $f_A$ is a restriction of $\bar{f}$ to $|P(A)|$ where $A \in S$.

**Definition.** If $|P(A)| = |P(B)|$ for every $A, B \in S$, $ARB$, then the model is called stable.

Let $\langle S, N, O, R, P\rangle$ be a modal model. By an interpretation we mean a function $k$ such that to each variable $k$ associates an element of $\bigcup_{A \in S} |P(A)|$.

Let a model $\langle S, N, O, R, P\rangle$ and an interpretation $k$ be given. By a valuation $\varkappa$ a partial function is meant with the following properties:

(i) $\varkappa(x, A) = k(x)$ if $A \in S$ and $x$ is variable such that $k(x) \in |P(A)|$.
(ii) $\varkappa(f(\tau_1, ..., \tau_n), A) = f_{P(A)}(\varkappa(\tau_1, A), ..., \varkappa(\tau_n, A))$ if $A \in S$ and $\tau_1, ..., \tau_n$ are terms such that for every variable $x$ occuring in any of them, $k(x) \in |P(A)|$.
(iii) $\varkappa(\tau, A)$ is undefined if $A \in S$ and there exists a variable $x$ in the term $\tau$ such that $k(x) \notin |P(A)|$.

Let $\mathscr{A}$ be an expression (i.e. a term or formula) and assume a model $\langle S, N, O, R, P\rangle$ is given. Let us fix $A \in S$ and an interpretation $k$. We say that $\mathscr{A} \in \mathscr{H}_k(A)$ if for every variable $x$ occuring free in $\mathscr{A}$ we have $k(x) \in |P(A)|$.

Let $A \in S$, $\mathscr{A}$ be a formula and $k$ an interpretation. We define the satisfaction relation $A \models \mathscr{A}[k]$ by the following clauses:

(i) $A \models r(\tau_1, ..., \tau_n)[k]$ if and only if $\tau_1, ..., \tau_n \in \mathscr{H}_k(A)$ and $r_{P(A)}(\varkappa(\tau_1, A), ...$ $..., \varkappa(\tau_n, A))$;
(ii) $A \models (\mathscr{A} \wedge \mathscr{B})[k]$ if and only if $A \models \mathscr{A}[k]$ and $A \models \mathscr{B}[k]$;
(iii) $A \models \sim\mathscr{A}[k]$ if and only if $\mathscr{A} \in \mathscr{H}_k(A)$ and $A \models \mathscr{A}[k]$ does not hold;
(iv) $A \models \Box\mathscr{A}[k]$ if and only if $A \in N$ and for every $B \in S$, $ARB$ implies $B \models \mathscr{A}[k]$;
(v) $A \models \forall x \mathscr{A}[k]$ if and only if for every interpretation $k'$, such that $k'(x) \in |P(A)|$ and $k'(y) = k(y)$ if $y \neq x$, we have $A \models \mathscr{A}[k']$.

We put $\mathscr{I}$ into the set of relation symbols with the following meaning:

$A \models \mathscr{I}(\tau_1, \tau_2)[k]$ if and only if $\tau_1, \tau_2 \in \mathscr{H}_k(A)$ and $\varkappa(\tau_1, A) = \varkappa(\tau_2, A)$, i.e. $\mathscr{I}$ denotes the identity.

Let $\mathscr{A}$ be a formula, $\langle S, N, O, R, P\rangle$ a modal model and $k$ an interpretation. $\mathscr{A}$ is valid in $\langle S, N, O, R, P\rangle$ under the interpretation $k$ if $O \models \mathscr{A}[k]$.

The reader can consult with [1] for notions and notations not explained here.

## § 2. Modal logics

To give a modal logic we have to give a classical formula $\mathscr{T}$ with the properties:

(i) no free variable occurs in $\mathscr{T}$,

(ii) $\mathscr{T}$ is in the classical language of the following non-logical symbols: $o$, 0-ary function symbol; $n$, unary relation symbol; $r$, binary relation symbol; $i$, binary relation symbol.

This classical formula, called parameter of the logic, is meant to formalize a property of the structure $\langle S, N, O, R\rangle$ provided $\langle S, N, O, R, P\rangle$ is a model of the intended modal logic.

If we restrict ourselves to modal logic with only simple/stable models then we call them simple/stable modal logics.

Let a modal logic be given. A formula $\mathscr{A}$ is satisfiable if there exist a model $\langle S, N, O, R, P\rangle$ and a interpretation $k$ such that:

(i) $\langle S, N, O, R, P\rangle$ is simple/stable if the given logic is simple/stable;

(ii) the parameter of the logic is valid in the classical model $A$ defined by: $|A|=S$, $O_A=o$, $n_A(B)\Leftrightarrow B\in N$, if $B, C\in S$ then $r_A(B, C)\Leftrightarrow BRC$ and $i_A(B, C)\Leftrightarrow B=C$;

(iii) $O\models\mathscr{A}[k]$.

A formula $\mathscr{A}$ is a tautology if $\sim\mathscr{A}$ is not satisfiable.

In this paper we treat some special logics, the parameter of which is an arbitrary (may be empty) conjunction of the following formulae:

K1. $\forall x n(x)$

K2. $\forall x r(x, x)$

K3. $\forall x\forall y\forall z(r(x, y)\wedge r(y, z)\rightarrow r(x, z))\wedge\forall x(n(x)\rightarrow\forall y(r(x, y)\rightarrow n(y)))$.

These logics will be axiomatized with suitable subsets of the following axioms:

A1. $\mathscr{A}\rightarrow\mathscr{A}\wedge\mathscr{A}$

A2. $\mathscr{A}\wedge\mathscr{B}\rightarrow\mathscr{A}$

A3. $(\mathscr{A}\rightarrow\mathscr{B})\rightarrow(\sim(\mathscr{B}\wedge\mathscr{C})\rightarrow\sim(\mathscr{C}\wedge\mathscr{A}))$

A4. $\forall x(\mathscr{A}\rightarrow\mathscr{B})\rightarrow(\forall x\mathscr{A}\rightarrow\forall x\mathscr{B})$

A5. $\mathscr{A}\rightarrow\forall x\mathscr{A}$ where $x$ is not free in $\mathscr{A}$;

A6.a. $\forall x\mathscr{A}\rightarrow\mathscr{A}[x/y]$ where $y$ is a variable and it is free with respect to $x$ in $\mathscr{A}$;

A6.b. $\forall x\mathscr{A}\rightarrow\mathscr{A}[x/\tau]$ where $\tau$ is a term and it is free with respect to $x$ in $\mathscr{A}$ and $\mathscr{A}$ is a classical formula;

A7. $\Box(\mathscr{A}\rightarrow\mathscr{B})\rightarrow(\Box\mathscr{A}\rightarrow\Box\mathscr{B})$

A8. $\Box(\mathscr{A}\rightarrow\mathscr{A})$ if K1 appears in the parameter of the logic as a conjunct;

A9. $\Box\mathscr{A}\rightarrow\mathscr{A}$ if K2 is a conjunct in the parameter of the logic;

A10. $\Box\mathscr{A}\rightarrow\Box\Box\mathscr{A}$ if K3 occurs in the parameter of the logic;

A11. $\forall x\mathscr{I}(x, x)$

$\forall x\forall y(\mathscr{I}(x, y)\rightarrow(\mathscr{A}[x/y]\rightarrow\mathscr{A}))$

$\Box\mathscr{A}\rightarrow\forall x\forall y(\mathscr{I}(x, y)\rightarrow\Box\mathscr{I}(x, y))$

$\Box\mathscr{A}\rightarrow\forall x\forall y(\sim\mathscr{I}(x, y)\rightarrow\Box\sim\mathscr{I}(x, y))$ if $\mathscr{I}$ occurs in the logic;

A12. $\forall x\Box\mathscr{A}\rightarrow\Box\forall x\mathscr{A}$ if the logic is stable.

If the logic is simple, then axioms A6.a and A6.b are replaced by the more general axiom.

A6. $\forall x\mathscr{A}\to\mathscr{A}[x/\tau]$ where $\tau$ is a term free with respect to $x$ in $\mathscr{A}$ and $\mathscr{A}$ is arbitrary.

We fix the following rules of inference:

R1. From $\mathscr{A}$ and $\mathscr{A}\to\mathscr{B}$ we infer $\mathscr{B}$.
R2. From $\mathscr{A}$ we infer $\forall x\mathscr{A}$.
R3. From $\mathscr{A}\to\mathscr{B}$ we infer $\Box\mathscr{A}\to\Box\mathscr{B}$.

This last rule can be used in a logic in which $\forall x(r(o,x)\wedge\mathscr{T}\to\mathscr{T}[o/x])$ is a tautology, where $\mathscr{T}$ is the parameter of the logic. This holds for K1, K2, K3.

The notion of derivability is used in the usual sense (denotation: $\vdash$).

**Theorem 1.** *(Soundness.)* Let a modal logic be given. If a formula $\mathscr{A}$ is derivable in this modal logic, then it is a tautology.

*Proof.* Trivial.

## § 3. Metatheorems

The proofs of metatheorems will only be sketched.

**Assertion 1.** Every tautology of classical sentential logic is derivable.

*Proof.* A1—A3 and R1 is a complete formalization of classical sentential logic.

**Assertion 2.** $\vdash\Box(\mathscr{A}\wedge\mathscr{B})\to\Box\mathscr{A}\wedge\Box\mathscr{B}$.

*Proof.*
$\vdash\mathscr{A}\wedge\mathscr{B}\to\mathscr{A}$ (classical theorem)
$\vdash\Box(\mathscr{A}\wedge\mathscr{B})\to\Box\mathscr{A}$ (R3)
$\vdash\mathscr{A}\wedge\mathscr{B}\to\mathscr{B}$ (classical theorem)
$\vdash\Box(\mathscr{A}\wedge\mathscr{B})\to\Box\mathscr{B}$ (R3)
$\vdash(\Box(\mathscr{A}\wedge\mathscr{B})\to\Box\mathscr{A})\to((\Box(\mathscr{A}\wedge\mathscr{B})\to\Box\mathscr{B})\to(\Box(\mathscr{A}\wedge\mathscr{B})\to\Box\mathscr{A}\wedge\Box\mathscr{B}))$
(classical theorem)
$\vdash\Box(\mathscr{A}\wedge\mathscr{B})\to\Box\mathscr{A}\wedge\Box\mathscr{B}$ (R1)

**Assertion 3.** $\vdash\Box\mathscr{A}\wedge\Box\mathscr{B}\to\Box(\mathscr{A}\wedge\mathscr{B})$.

*Proof.*
$\vdash\mathscr{A}\to(\mathscr{B}\to\mathscr{A}\wedge\mathscr{B})$ (classical theorem)
$\vdash\Box\mathscr{A}\to\Box(\mathscr{B}\to\mathscr{A}\wedge\mathscr{B})$ (R3)
$\vdash\Box(\mathscr{B}\to\mathscr{A}\wedge\mathscr{B})\to(\Box\mathscr{B}\to\Box(\mathscr{A}\wedge\mathscr{B}))$ (A7)
$\vdash\Box\mathscr{A}\to(\Box\mathscr{B}\to\Box(\mathscr{A}\wedge\mathscr{B}))$ (by classical theorems)
$\vdash\Box\mathscr{A}\wedge\Box\mathscr{B}\to\Box(\mathscr{A}\wedge\mathscr{B})$ (by classical theorems)

**Assertion 4.** $\vdash\Box\mathscr{A}\vee\Box\mathscr{B}\to\Box(\mathscr{A}\vee\mathscr{B})$.

*Proof.*
$\vdash\mathscr{A}\to\mathscr{A}\vee\mathscr{B}$ (classical theorem)
$\vdash\Box\mathscr{A}\to\Box(\mathscr{A}\vee\mathscr{B})$ (R3)
$\vdash\Box\mathscr{B}\to\Box(\mathscr{A}\vee\mathscr{B})$ (similarly)
$\vdash\Box\mathscr{A}\vee\Box\mathscr{B}\to\Box(\mathscr{A}\vee\mathscr{B})$ (by classical theorems)

**Theorem 2.** If $\vdash \mathscr{A} \to \mathscr{B}$ and $\vdash \mathscr{B} \to \mathscr{A}$ then $\mathscr{A}$ can be replaced by $\mathscr{B}$ in an arbitrary formula without loss of its derivability.

*Proof.* One can proceed by induction from the following facts:

$\vdash \mathscr{A} \to \mathscr{B}$ implies that $\vdash \sim\mathscr{B} \to \sim\mathscr{A}$

$\vdash \mathscr{A} \to \mathscr{B}$ implies that $\vdash \mathscr{A} \wedge \mathscr{C} \to \mathscr{B} \wedge \mathscr{C}$ and $\vdash \mathscr{C} \wedge \mathscr{A} \to \mathscr{C} \wedge \mathscr{B}$

$\vdash \mathscr{A} \to \mathscr{B}$ implies that $\vdash \Box\mathscr{A} \to \Box\mathscr{B}$

$\vdash \mathscr{A} \to \mathscr{B}$ implies that $\vdash \forall x\mathscr{A} \to \forall x\mathscr{B}$.

**Assertion 5.** A8 and R3 can be omitted if the following rule is added to the system: If $\vdash \mathscr{A}$ then $\vdash \Box\mathscr{A}$.

*Proof.* (a) Let $\vdash \mathscr{A}$. Since $\vdash A \to ((\mathscr{B} \to \mathscr{B}) \to \mathscr{A})$ implies $\vdash (\mathscr{B} \to \mathscr{B}) \to \mathscr{A}$, by R3, we have $\vdash \Box(\mathscr{B} \to \mathscr{B}) \to \Box\mathscr{A}$. By A8, $\vdash \Box(\mathscr{B} \to \mathscr{B})$, i.e., $\vdash \Box\mathscr{A}$.

(b) Let $\vdash \mathscr{A} \to \mathscr{B}$, then $\vdash \Box(\mathscr{A} \to \mathscr{B})$. By A7, $\vdash \Box\mathscr{A} \to \Box\mathscr{B}$. But $\vdash \mathscr{A} \to \mathscr{A}$, so $\vdash \Box(\mathscr{A} \to \mathscr{A})$ holds.

**Assertion 6.** $\vdash \Box\forall x\mathscr{A} \to \forall x\Box\mathscr{A}$.

| | |
|---|---|
| *Proof.* $\vdash \forall x\mathscr{A} \to \mathscr{A}$ | (A6.a) |
| $\vdash \Box\forall x\mathscr{A} \to \Box\mathscr{A}$ | (by R3) |
| $\vdash \forall x\Box\forall x\mathscr{A} \to \forall x\Box\mathscr{A}$ | (by R2 and A4) |
| $\vdash \Box\forall x\mathscr{A} \to \forall x\Box\forall x\mathscr{A}$ | (A5) |
| $\vdash \Box\forall x\mathscr{A} \to \forall x\Box\mathscr{A}$ | (by classical theorems) |

**Assertion 7.** $\vdash \Diamond\forall x\mathscr{A} \to \forall x\Diamond\mathscr{A}$.

| | |
|---|---|
| *Proof.* $\vdash \sim\mathscr{A} \to \exists x\sim\mathscr{A}$ | (from A6.a) |
| $\vdash \Box\sim\mathscr{A} \to \Box\exists x\sim\mathscr{A}$ | (by R3) |
| $\vdash \Diamond\forall x\mathscr{A} \to \Diamond\mathscr{A}$ | (by classical theorems) |
| $\vdash \Diamond\forall x\mathscr{A} \to \forall x\Diamond\mathscr{A}$ | (similarly). |

## § 4. Completeness theorems

A set of formulae $\alpha$ is consistent if for every $\mathscr{A}_1, \ldots, \mathscr{A}_n \in \alpha$, $\sim(\mathscr{A}_1 \wedge \ldots \wedge \mathscr{A}_n)$ is not a theorem.

We introduce the following notation: $\alpha^+ = \{\mathscr{A} : \Box\mathscr{A} \in \alpha\}$.

**Theorem 3.** Let $\alpha$ be a consistent set of formulae and assume $\alpha^+ \neq \emptyset$. If $\Diamond\mathscr{B} \in \alpha$, then $\alpha^+ \cup \{\mathscr{B}\}$ is consistent.

*Proof.* Assume the contrary, i.e. $\alpha^+ \cup \{\mathscr{B}\}$ is not consistent. Then there exist $\mathscr{A}_1, \ldots, \mathscr{A}_n \in \alpha^+$ such, that $\vdash \sim(\mathscr{A}_1 \wedge \ldots \wedge \mathscr{A}_n \wedge \mathscr{B})$. It means that $\vdash \mathscr{A}_1 \wedge \ldots \wedge \mathscr{A}_n \to \sim\mathscr{B}$ (using the hypothesis $\alpha^+ \neq \emptyset$ and that $\vdash \sim\mathscr{A}$ implies $\vdash \sim(\mathscr{A} \wedge \mathscr{C})$ for arbitrary $\mathscr{C}$). By R3, we obtain $\vdash \Box\mathscr{A}_1 \wedge \ldots \wedge \Box\mathscr{A}_n \to \Box\sim\mathscr{B}$, i.e. $\vdash \sim(\Box\mathscr{A}_1 \wedge \ldots \wedge \Box\mathscr{A}_n \wedge \Diamond\mathscr{B})$. This contradicts the assumption, that $\alpha$ is consistent.

If $\alpha$ is a set of formulae, then let us denote the set of variables occuring in $\alpha$ by $\Pi(\alpha)$.

**Definition.** Let $\alpha$ be a set of formulae. $\alpha$ is complete if the following conditions are satisfied:

(i) $\alpha$ is consistent;
(ii) If $\mathscr{A}$ contains variables from $\Pi(\alpha)$ only, then either $\mathscr{A}\in\alpha$ or $\sim\mathscr{A}\in\alpha$;
(iii) Let $\mathscr{A}$ contain variables from $\Pi(\alpha)$ only and let $x$ be the only variable occuring free in $\mathscr{A}$. If $\exists x\mathscr{A}\in\alpha$ then there exists $a$ such that $a\in\Pi(\alpha)$ and $a$ is free for $x$, moreover $\mathscr{A}[x/a]\in\alpha$.

**Theorem 4.** Let $\alpha$ be a complete set of formulae. Then

(i) $\mathscr{A}\in\alpha$ and $\mathscr{A}\to\mathscr{B}\in\alpha$ imply $\mathscr{B}\in\alpha$;
(ii) $\mathscr{A}\wedge\mathscr{B}\in\alpha$ if and only if $\mathscr{A}\in\alpha$ and $\mathscr{B}\in\alpha$;
(iii) $\forall x\mathscr{A}\in\alpha$ if and only if for every $a\in\Pi(\alpha)$ free for $x$ we have $\mathscr{A}[x/a]\in\alpha$, where $x$ is the only variable occuring free in $\mathscr{A}$.
(iv) If $\alpha^{+}\cup\{\mathscr{A}\}$ is consistent and $\mathscr{A}$ contains variables from $\Pi(\alpha)$ only, then $\lozenge\mathscr{A}\in\alpha$.

*Proof.* (i) If $\mathscr{B}\notin\alpha$ then $\sim\mathscr{B}\in\alpha$ by completeness. But it means $\alpha$ is not consistent since $\vdash\sim(\mathscr{A}\wedge(\mathscr{A}\to\mathscr{B})\wedge\sim\mathscr{B})$.

(ii) Since $\vdash\sim((\mathscr{A}\wedge\mathscr{B})\wedge\sim\mathscr{A})$, $\vdash\sim((\mathscr{A}\wedge\mathscr{B})\wedge\sim\mathscr{B})$ and $\vdash\sim(\mathscr{A}\wedge\mathscr{B}\wedge\sim\sim(\mathscr{A}\wedge\mathscr{B}))$ hold, it is trivial.
(iii) $\vdash\sim(\forall x\mathscr{A}\wedge\sim\mathscr{A}[x/a])$, so if $\forall x\mathscr{A}\in\alpha$, then $\mathscr{A}[x/a]\in\alpha$. Conversely, if $\forall x\mathscr{A}\notin\alpha$, then $\sim\forall x\mathscr{A}\in\alpha$ by completeness, i.e. $\exists x\sim\mathscr{A}\in\alpha$. Thus there exists $a\in\Pi(\alpha)$ such that $\sim\mathscr{A}[x/a]\in\alpha$, i.e., $\mathscr{A}[x/a]\notin\alpha$.
(iv) If $\lozenge\mathscr{A}\notin\alpha$, then $\Box\sim\mathscr{A}\in\alpha$, by completeness, and $\sim\mathscr{A}\in\alpha^{+}$. But it means $\alpha^{+}\cup\{\mathscr{A}\}$ is not consistent since $\vdash\sim(\mathscr{A}\wedge\sim\mathscr{A})$.

**Theorem 5.** If $\alpha$ is consistent, then there exists a complete $\beta$ such that $\alpha\subseteq\beta$.

*Proof.* It is easy by using the following three lemmata since we can assume that the set of variables has enough elements.

**Lemma A.** Let $\alpha$ be consistent. Then at least one of $\alpha\cup\{\mathscr{A}\}$ and $\alpha\cup\{\sim\mathscr{A}\}$ will also be consistent.

*Proof.* Suppose both $\alpha\cup\{\mathscr{A}\}$ and $\alpha\cup\{\sim\mathscr{A}\}$ are inconsistent; that means there exist $\mathscr{B}_1,\dots,\mathscr{B}_n\in\alpha$ for which $\vdash\sim(\mathscr{B}_1\wedge\dots\wedge\mathscr{B}_n\wedge\mathscr{A})$ and $\vdash\sim(\mathscr{B}_1\wedge\dots\wedge\mathscr{B}_n\wedge\sim\mathscr{A})$. Then $\vdash\sim(\mathscr{C}\wedge\mathscr{A})\to(\sim(\mathscr{C}\wedge\sim\mathscr{A})\to\sim\mathscr{C})$ entails $\vdash\sim(\mathscr{B}_1\wedge\dots\wedge\mathscr{B}_n)$ so $\alpha$ is inconsistent. This completes the proof of Lemma A.

**Lemma B.** If $a\notin\Pi(\alpha\cup\{\mathscr{A}\})$ and $\alpha$ is consistent, then $\alpha\cup\{\exists x\mathscr{A}\to\mathscr{A}[x/a]\}$ is also consistent.

*Proof.* Suppose the contrary. Then there exist $\mathscr{A}_1,\dots,\mathscr{A}_n\in\alpha$ such that $\vdash\sim(\mathscr{A}_1\wedge\dots\wedge\mathscr{A}_n\wedge(\exists x\mathscr{A}\to\mathscr{A}[x/a]))$. By applying R2 we arrive at $\vdash\sim(\mathscr{A}_1\wedge\dots\wedge\mathscr{A}_n\wedge\sim(\exists x\mathscr{A}\wedge\forall a\sim\mathscr{A}[x/a]))$. Since $\vdash\sim(\exists x\mathscr{A}\wedge\forall a\sim\mathscr{A}[x/a])$, we have $\vdash\sim(\mathscr{A}_1\wedge\dots\wedge\mathscr{A}_n)$ which contradicts the assumptions. This completes the proof of Lemma B.

**Lemma C.** If $\alpha_n$ is consistent and $\alpha_n \subseteqq \alpha_{n+1}$ for every $n$, then $\bigcup_{n=1}^{\infty} \alpha_n$ is also consistent.

*Proof.* Trivial.

**Definition.** The system of sets of formulae $M$ is said to be complete if the following conditions are satisfied:

(i) Each $\alpha \in M$ is complete.
(ii) If $\alpha \in M$, $\alpha^+ \neq \emptyset$ and $\Diamond\, \mathscr{A} \in \alpha$, then there exists $\beta \in M$ such that $\alpha^+ \cup \cup \{\mathscr{A}\} \subseteqq \beta$.
(iii) If the logic has equality symbol, then

(a) If $\alpha, \beta \in M$, $a \in \Pi(\alpha) \cap \Pi(\beta)$, then there exist natural numbers $n, m \geqq 0$ and sets of formulae $\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_m \in M$ such that $\alpha_0 = \beta_0$, $\alpha_n = \alpha$, $\beta_m = \beta$, $a \in \Pi(\alpha_0)$, $\alpha_i^+ \neq \emptyset$ and $\alpha_i^+ \subseteqq \alpha_{i+1}$ $(i = 0, \ldots, n-1)$, $\beta_i^+ \neq \emptyset$ and $\beta_i^+ \subseteqq \beta_{i+1}$ $(i = 0, \ldots, m-1)$.
(b) If $\alpha_i^+ \neq \emptyset$, $\alpha_i^+ \subseteqq \alpha_{i+1}$ $(i = 0, \ldots, n-1)$, $\beta_i^+ \neq \emptyset$, $\beta_i^+ \subseteqq \beta_{i+1}$ $(i = 0, \ldots, m-1)$, $\alpha_n = \beta_m$, then there exist $\gamma_0, \ldots, \gamma_k \in M$ such that $\gamma_i^+ \neq \emptyset$ and $\gamma_i^+ \subseteqq \gamma_{i+1}$ $(i = 0, \ldots, k-1)$ and either $\gamma_0 = \alpha_0$, $\gamma_k = \beta_0$ or $\gamma_0 = \beta_0$, $\gamma_k = \alpha_0$ are true.

**Theorem 6.** If $\alpha$ is complete, then there exists a complete system of sets of formulae $M$ such that $\alpha \in M$.

*Proof.* Let $M_0 = \{\alpha\}$. Assume that $M_n$ is a set of complete sets of formulae. Let $\beta \in M_n$, $\beta^+ \neq \emptyset$, $\Diamond \mathscr{A} \in \beta$. Then, to $(\beta, \mathscr{A})$ we associate a set of variables. This set is disjoint from $\bigcup_{\gamma \in M_n} \Pi(\gamma)$ and different pairs have disjoint associated sets of variables. There exists a complete set $\gamma$ such that $\gamma \in M_{n+1}$, $\beta^+ \cup \{\mathscr{A}\} \subseteqq \gamma$ and $\Pi(\gamma) \setminus \Pi(\beta)$ is associated to the pair $(\beta, \mathscr{A})$. It is trivial, that $\bigcup_{n=0}^{\infty} M_n$ is a complete system of sets of formulae.

**Theorem 7.** *(Completeness Theorem.)* Let us suppose that a simple non-stable and equality free modal logic is given. If a formula $\mathscr{A}$ cannot be derived then $\sim\mathscr{A}$ is satisfiable.

*Proof.* We can assume without loss of generality, that no free variable occurs in $\mathscr{A}$. Since not $\vdash \mathscr{A}$, we have not $\vdash \mathscr{A}$ i.e., $\{\sim\mathscr{A}\}$ is consistent. There exists a complete set of formulae $\alpha$ and a complete system of sets of formulae $M$ such that $\sim\mathscr{A} \in \alpha$ and $\alpha \in M$.

Let us define the following notions: $N = \{\beta : \beta \in M$ and $\beta^+ \neq \emptyset\}$; if $\beta, \gamma \in M$, then $\beta R \gamma \Leftrightarrow ((\beta^+ \subseteqq \gamma$ and $\beta^+ \neq \emptyset)$ or $(\beta^+ = \emptyset$ and $\gamma = \beta))$; $|P(\beta)| = \{\tau:$ all variables occuring in $\tau$ are from $\Pi(\beta)\}$; $f_{P(\beta)}(\tau_1, \ldots, \tau_n) = f(\tau_1, \ldots, \tau_n)$ where $\tau_1, \ldots, \tau_n \in |P(\beta)|$; $r_{P(\beta)}(\tau_1, \ldots, \tau_n) \Leftrightarrow r(\tau_1, \ldots, \tau_n) \in \beta$ if $\tau_1, \ldots, \tau_n \in |P(\beta)|$. It is easy to see that $\langle M, N, \alpha, R, P \rangle$ is a simple model.

Let $k$ be an interpretation and $\varkappa$ the corresponding valuation. The following two assertions can easily be proved by a simple induction.

If $\tau \in \mathscr{H}_k(\beta)$, $\beta \in M$, then $\varkappa(\tau, \beta) = \tau[x_1, \ldots, x_n / k(x_1), \ldots, k(x_n)]$ where $x_1, \ldots, x_n$ are all the variables occuring in $\tau$, and $\tau[x_1, \ldots, x_n / \tau_1, \ldots, \tau_n]$ is the result of the substitutions $[x_1/\tau_1], \ldots, [x_n/\tau_n]$ executed simultaneously.

If $\mathscr{B}\in\mathscr{H}_k(\beta)$, $\beta\in M$ and $x_1, ..., x_n$ are all the variables occuring in $\mathscr{B}$, then $\beta\models\mathscr{B}[k]\Leftrightarrow\mathscr{B}[x_1, ..., x_n/k(x_1), ..., k(x_n)]\in\beta$. Hence, if for every $a, k(a)=a$, then $\alpha\models\sim\mathscr{A}[k]$. Let us suppose that $\mathscr{B}$ contains variables only from $\Pi(\beta)$, where $\beta$ is a complete set.

If $\vdash\mathscr{B}$, then $\mathscr{B}\in\beta$, since in the opposite case we have $\sim\mathscr{B}\in\beta$, i.e. $\beta$ is inconsistent.

K1. If $\Box(\mathscr{B}\to\mathscr{B})$ is an axiom, then for every $\beta\in M$, $\Box(\mathscr{B}\to\mathscr{B})\in\beta$ provided no variable occurs in $\mathscr{B}$. Thus, $\beta^+\neq\emptyset$ and $N=M$.

K2. Let $\beta$ be an arbitrary formula for which $\mathscr{B}\in\beta^+$. From $\Box\mathscr{B}\in\beta$ and $\Box\mathscr{B}\to\mathscr{B}\in\beta$ we infer $\mathscr{B}\in\beta$, i.e. $\beta^+\subseteq\beta, \beta R\beta$.

K3. Let $\beta R\gamma$ and $\gamma R\delta$, moreover $\mathscr{B}\in\beta^+$. Then $\Box\mathscr{B}\in\beta, \Box\mathscr{B}\to\Box\Box\mathscr{B}\in\beta$, so $\Box\Box\mathscr{B}\in\beta$. By definition of $R$, $\Box\mathscr{B}\in\gamma$ and $\mathscr{B}\in\delta$ follow. We obtain $\beta R\delta$. Let $\beta\in N$, then for some $\mathscr{B}$, $\Box\mathscr{B}\in\beta$. If $\beta R\gamma$ then $\Box\mathscr{B}\in\gamma$, so $\gamma\in N$.

This completes the proof of Theorem 7.

In what follows we assume that a non-stable modal logic with equality is given. Let $M$ be a complete system of sets of formulae, let $N$ and $R$ be defined analogouosly to the ones in the proof of the previous theorem. We denote the reflexive and transitive closure of $R$ by $\bar{R}$.

By these notations we redefine the third clause of the last definition in the following simple way:

(iii)′ If the logic has equality symbol, then

(a) If $\alpha, \beta\in M$, $a\in\Pi(\alpha)\cap\Pi(\beta)$ then there exists $\gamma\in M$ such that $\gamma\bar{R}\alpha, \gamma\bar{R}\beta$ and $a\in\Pi(\gamma)$.

(b) If $\alpha, \beta, \gamma\in M$, $\alpha\bar{R}\gamma$ and $\beta\bar{R}\gamma$, then either $\alpha\bar{R}\beta$ or $\beta\bar{R}\alpha$ is true; in other words, $\bar{R}$ is trichotom on the set $\{\alpha: \alpha\bar{R}\gamma\}$.

We prove some simple assertions:

**Assertion 8.** If $\beta\bar{R}\gamma$, then $\Pi(\beta)\subseteq\Pi(\gamma)$.

*Proof.* Trivial.

**Assertion 9.** If $\beta\bar{R}\gamma$ and $a, b\in\Pi(\beta)$, then $\mathscr{I}(a, b)\in\beta\Leftrightarrow\mathscr{I}(a, b)\in\gamma$.

*Proof.* If $\beta\bar{R}\gamma$, then $\Box\mathscr{I}(a, b)\vee\Diamond\mathscr{A}\in\beta$. If $\beta\bar{R}\gamma$ and $\gamma\neq\beta$, then $\beta^+\neq\emptyset$, e.g. $\Box\sim\mathscr{A}_1\in\beta$. If $\mathscr{A}$ is replaced by $\mathscr{A}_1$, then $\Box\sim\mathscr{A}_1\to\Box\mathscr{I}(a, b)\in\beta$ and so $\Box\mathscr{I}(a, b)\in\beta$. That means $\mathscr{I}(a, b)\in\beta^+$ and $\mathscr{I}(a, b)\in\gamma$ by induction.

If $\sim\mathscr{I}(a, b)\in\beta$, then by an analogouos argument we can obtain the other direction. This completes the proof the Assertion 9.

Let $a, b$ be two variables. $a\equiv b$ if and only if there exist $\alpha, \beta, \gamma\in M$ and $c\in\Pi(\gamma)$ such that $\gamma\bar{R}\alpha, \gamma\bar{R}\beta, \mathscr{I}(a, c)\in\alpha$ and $\mathscr{I}(b, c)\in\beta$. Obviously, $\equiv$ is a reflexive and symmetric relation. We shall prove that it is transitive, as well.

**Assertion 10.** If $\gamma\bar{R}\alpha, \gamma\bar{R}\beta, c\in\Pi(\gamma), \mathscr{I}(c, a)\in\alpha$ and $\mathscr{I}(a, b)\in\beta$, then $\mathscr{I}(c, b)\in\beta$.

*Proof.* It is clear, that $a\in\Pi(\alpha)\cap\Pi(\beta)$. By (iii)′ there exists $\delta\in M$ such that $a\in\Pi(\delta)$ and $\delta\bar{R}\alpha, \delta\bar{R}\beta$. Also by this definition we have either $\gamma\bar{R}\delta$ or $\delta\bar{R}\gamma$. By Assertion 8 either $a, c\in\Pi(\delta)$ or $a, c\in\Pi(\gamma)$ and so either $\mathscr{I}(c, a)\in\delta$ or $\mathscr{I}(c, a)\in\gamma$. In both cases $\mathscr{I}(c, a)\in\beta$ by Assertion 9. Then $\mathscr{I}(c, b)\in\beta$ by transitivity of equality.

**Assertion 11.** Let $\alpha_i \bar{R} \beta_i$ $(i=1, \ldots, n)$ and $\alpha_{i+1} \bar{R} \beta_i$ $(i=1, \ldots, n)$. There exists a $k$ such that $1 \leqq k \leqq n+1$ and $\alpha_k \bar{R} \alpha_i$ for every $i$ $(1 \leqq i \leqq n+1)$; furthermore, there exists an $l$ such that $1 \leqq l \leqq n+1$ and $\alpha_i \bar{R} \alpha_l$ for every $i$ $(1 \leqq i \leqq n+1)$.

*Proof.* Readily follows by definitions.

**Assertion 12.** The relation $\equiv$ is transitive.

*Proof.* Assume $a \equiv b$ and $b \equiv c$. Then, there exist $d, e$ and $\alpha_1, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4$ such that $\alpha_1 \bar{R} \beta_1$, $\alpha_1 \bar{R} \beta_2$, $\mathscr{I}(d, a) \in \beta_1$, $\mathscr{I}(d, b) \in \beta_2$, $d \in \Pi(\alpha_1)$, $\alpha_3 \bar{R} \beta_3$, $\alpha_3 \bar{R} \beta_4$, $\mathscr{I}(e, b) \in \beta_3$, $\mathscr{I}(e, c) \in \beta_4$, $e \in \Pi(\alpha_3)$. By (iii)′ (a), there exists an $\alpha_2$ such that $\alpha_2 \bar{R} \beta_2$, $\alpha_2 \bar{R} \beta_3$, $b \in \Pi(\alpha_2)$. By the previous assertion, for some $i$, $\alpha_i \bar{R} \alpha_j$ $(j=1, 2, 3)$ and $\alpha_i \bar{R} \beta_j$ $(j=1, 2, 3, 4)$. It is known, that $d \in \Pi(\alpha_1)$, $b \in \Pi(\alpha_2)$, $e \in \Pi(\alpha_3)$. Let $f$ be that variable among (of $d, b, e$), which is in $\Pi(\alpha_i)$. We have $\mathscr{I}(a, d) \in \beta_1$, $\mathscr{I}(d, b) \in \beta_2$, $\mathscr{I}(b, e) \in \beta_3$, $\mathscr{I}(e, c) \in \beta_4$. By Assertion 10, we obtain $\mathscr{I}(a, f) \in \beta_1$, $\mathscr{I}(f, c) \in \beta_4$, that means $a \equiv c$.

**Assertion 13.** If $a, b \in \Pi(\beta)$ and $a \equiv b$, then $\mathscr{I}(a, b) \in \beta$.

*Proof.* Let $\beta_1 = \beta_4 = \beta$. Since $a \equiv b$ there exist $\beta_2, \beta_3, \alpha_2 \in M$ and $c \in \Pi(\alpha_2)$ such that $\mathscr{I}(a, c) \in \beta_2$, $\mathscr{I}(b, c) \in \beta_3$, $\alpha_2 \bar{R} \beta_2$, $\alpha_2 \bar{R} \beta_3$. By (iii)′ (a) there exist $\alpha_1$ and $\alpha_3$ for which $\alpha_1 \bar{R} \beta_1$, $\alpha_1 \bar{R} \beta_2$, $d \in \Pi(\alpha_1)$, $\alpha_3 \bar{R} \beta_3$, $\alpha_3 \bar{R} \beta_4$, $\beta \in \Pi(\alpha_3)$. By Assertion 11, there exists an $i$ such that $\alpha_i \bar{R} \beta_j$ $(j=1, 2, 3, 4)$. Obviously $\mathscr{I}(a, a) \in \beta_1$, $\mathscr{I}(a, c) \in \beta_2$, $\mathscr{I}(c, b) \in \beta_3$, $\mathscr{I}(b, b) \in \beta_4$. Let $d$ be that variable among $a, b, c$ which is in $\Pi(\alpha_i)$. Applying Assertion 10, we have $\mathscr{I}(a, d) \in \beta_1$ and $\mathscr{I}(d, b) \in \beta_4$, i.e. $\mathscr{I}(a, b) \in \beta$.

**Theorem 8.** *(Completeness Theorem.)* Let a non-stable modal logic with equality be given. If $\mathscr{A}$ is not derivable, then $\sim\mathscr{A}$ is satisfiable.

*Proof.* Let $M$ be a complete system of sets of formulae, $N, R$ as defined in the proof of Theorem 7. Let us define $P$ by the following causes: for $\beta \in M$ $|P(\beta)| = \{\bar{a} : a \in \Pi(\beta)\}$, where $\bar{a} = \{b : a \equiv b\}$; if $a_1, \ldots, a_n, a \in \Pi(\beta)$, then

$$f_{P(\beta)}(\bar{a}_1, \ldots, \bar{a}_n) = \bar{a} \Leftrightarrow \mathscr{I}(f(a_1, \ldots, a_n), a) \in \beta;$$

(By definition of completeness, this function is defined and it is unique by last assertion.) $r_{P(\beta)}(\bar{a}_1, \ldots, \bar{a}_n) \Leftrightarrow r(a_1, \ldots, a_n) \in \beta$. For an arbitrary $\beta \in M$, $\langle M, N, \beta, R, P \rangle$ is a model.

If A 6 is an axiom of the given logic, then this model is simple. We have to prove that

if $a_1, \ldots, a_n, a \in \Pi(\beta)$, $b_1, \ldots, b_n, b \in \Pi(\gamma)$, $a_1 \equiv b_1, \ldots, a_n \equiv b_n$, $\mathscr{I}(f(a_1, \ldots, a_n), a) \in \beta$ and $\mathscr{I}(f(b_1, \ldots, b_n), b) \in \gamma$, then $a \equiv b$. Let $1 \leqq i \leqq n$ be given. By definition of $\equiv$ and clause (iii)′ (a) we can assume that $\beta_1 = \beta$, $\beta_n = \gamma$, $\alpha_1 \bar{R} \beta_1$, $\alpha_1 \bar{R} \beta_2$, $\alpha_2 \bar{R} \beta_2$, $\alpha_2 \bar{R} \beta_3$, $\alpha_3 \bar{R} \beta_3$, $\alpha_3 \bar{R} \beta_4$, $a_i \in \Pi(\alpha_1)$, $\mathscr{I}(a_i, c) \in \beta_2$, $c \in \Pi(\alpha_2)$, $\mathscr{I}(c, b_i) \in \beta_3$, $b_i \in \Pi(\alpha_3)$. Let $\gamma_i$ denote the first element among $\alpha_1, \alpha_2, \alpha_3$ under $\bar{R}$. Using methods from proofs of Assertion 9—13, we get $c_i \in \Pi(\gamma_i)$, $\mathscr{I}(a_i, c_i) \in \beta$, $\mathscr{I}(c_i, b_i) \in \gamma$, $\gamma_i \bar{R} \beta$, $\gamma_i \bar{R} \gamma$. Since for every $i$, $\gamma_i \bar{R} \beta$, applying (iii)′ (b) we obtain that there exists an $i$ such that $\gamma_j \bar{R} \gamma_i$ for every $j$. By Assertion 8, for this $i$ we have $c_1, \ldots, c_n \in \Pi(\gamma_i)$. So there exists a $c$ for which $\mathscr{I}(f(c_1, \ldots, c_n), c) \in \gamma_i$. Generalizing the method used in proof of Assertion 9, we arrive to $\mathscr{I}(f(c_1, \ldots, c_n), c) \in \beta$ and $\mathscr{I}(f(c_1, \ldots, c_n), c) \in \gamma$. From $\mathscr{I}(f(a_1, \ldots, a_n), a) \in \beta$ and $\mathscr{I}(f(b_1, \ldots, b_n), b) \in \gamma$, it follows that $\mathscr{I}(a, c) \in \beta$, $\mathscr{I}(b, c) \in \gamma$ and so $a \equiv b$.

Let $k$ be an interpretation and $\varkappa$ the corresponding valuation. If for a variable $x$, $k(x)\in|P(\beta)|$, then $k(x)\cap\Pi(\beta)\neq\emptyset$. Let $x^*\in k(x)\cap\Pi(\beta)$. We extend the operation $*$ for arbitrary expressions: $\mathscr{K}^*=\mathscr{K}[x_1,\ldots,x_n/x_1^*,\ldots,x_n^*]$, where $x_1,\ldots,x_n$ are all the variables occuring in $\mathscr{K}$. By a simple induction, the following statements are easy to prove:

(i) $\varkappa(\tau,\beta)=\bar{a}$ and $a\in\Pi(\beta)\Leftrightarrow\mathscr{I}(\tau^*,a)\in\beta$;
(ii) If $\mathscr{A}$ contains variables from $\Pi(\beta)$ only then $\beta\models\mathscr{A}[k]\leftrightarrow\mathscr{A}^*\in\beta$.

From (ii) the theorem follows.

## § 5. Connections with classical logics

Let us suppose that a modal logic is given; i.e., the sets of relation symbols, function symbols and set of variables are fixed. We also suppose that the following symbols do not occur in these sets: $o, s, n, r, p, i, z, z'$. Furthermore the parameter $\mathscr{T}$ of this logic is fixed. Also we know if this logic is simple, stable or so.

Now we define a classical theory. The language of this theory contains all the relation symbols and function symbols of the modal language but if a symbol has arity $m$ in the modal language we use it with arity $m+1$ in the classical one. Also we shall use the following symbols: $o$: 0-ary function symbol, $s$ and $n$ both unary relation symbols, $r, p, i$ all of them are binary relation symbols, and two new variables: $z$ and $z'$.

We define a mapping [ ] from the set of modal expressions into the set of classical ones:

(i) if $x$ is a variable, then $[x]=x$;
(ii) $[f(\tau_1,\ldots,\tau_m)]=f([\tau_1],\ldots,[\tau_m],z)$ if $f$ is an $m$-ary function symbol in the modal language, $\tau_1,\ldots,\tau_m$ are terms;
(iii) $[r(\tau_1,\ldots,\tau_m)]=r([\tau_1],\ldots,[\tau_m],z)$ if $r$ is an $m$-ary relation symbol in the modal language, $\tau_1,\ldots,\tau_m$ are terms; in particular $[\mathscr{I}(\tau_1,\tau_2)]=i([\tau_1],[\tau_2])$;
(iv) $[\sim\mathscr{A}]=\sim[\mathscr{A}]$; $[\mathscr{A}\wedge\mathscr{B}]=[\mathscr{A}]\wedge[\mathscr{B}]$;
(v) $[\forall x\mathscr{A}]=\forall x(p(x,z)\rightarrow[\mathscr{A}])$;
(vi) $[\Box\mathscr{A}]=\forall z'(r(z,z')\rightarrow[\mathscr{A}][z/z'])\wedge n(z)$.

Let $\mathscr{A}^*=p(x_1,z)\wedge\ldots\wedge p(x_m,z)\wedge[\mathscr{A}]$, where $x_1,\ldots,x_m$ are all the free variables of $\mathscr{A}$.

Let $M$ be a classical model in which the following formulae are valid: $s(o)$; $s(z)\rightarrow\exists x p(x,z)$; $p(x,z)\wedge r(z,z')\rightarrow p(x,z')$; $s(z)\wedge r(z,z')\rightarrow s(z')$; $s(z)\rightarrow p(f(x_1,\ldots$ $\ldots,x_m,z),z)$ for every function symbol.

Let $0=o_M$, $S=\{a\colon a\in|M|$ and $s_M(a)\}$, $N=\{a\colon a\in S$ and $n_M(a)\}$, $aRb\Leftrightarrow a,b\in S$ and $r_M(a,b)$, $|P(a)|=\{b\colon p_M(b,a)\}$, if $a\in S$, for $a_1,\ldots,a_m\in|P(a)|$, $f_{P(a)}(a_1,\ldots,a_m)=$ $=f_M(a_1,\ldots,a_m,a)$ and $q_{P(a)}(a_1,\ldots,a_m)\Leftrightarrow q_M(a_1,\ldots,a_m,a)$.

It is obvious, that by these definitions $\langle S,N,O,R,P\rangle$ is a modal model.

Let $k$ be an interpretation for $M$ such that $k$ associates an element of $S$ to $z$ and $z'$, and $k$ associates an element of $\bigcup_{a\in S}|P(a)|$ to every variable other then $z$ or $z'$. It is clear, that $k$ is also an interpretation for $\langle S,N,O,R,P\rangle$. Let the corresponding valuations be $K$ in $M$ and $\varkappa$ in $\langle S,N,O,R,P\rangle$.

**Theorem 9.** Let $\tau$ be a term, $\mathscr{A}$ a formula and suppose $z, z'$ do not occur in them. Then

(i) $\varkappa(\tau, k(z))=K([\tau])$, provided $\varkappa(\tau, k(z))$ is defined;
(ii) $k(z)\models\mathscr{A}[k]\Leftrightarrow M\models\mathscr{A}^*[k]$.

*Proof.* The easy induction is left to the reader.

Now we give the inverse of the mapping

$$M\to\langle S, N, O, R, P\rangle.$$

Let $\langle S, N, O, R, P\rangle$ be an arbitrary modal model. We define

$$|M| = S\cup(\bigcup_{a\in S}|P(a)|);\quad o_M = 0;\quad s_M(a)\Leftrightarrow a\in S;$$

$$n_M(a)\Leftrightarrow a\in N;\quad p_M(a, b)\Leftrightarrow b\in S \text{ and } a\in|P(b)|;$$

$$r_M(a, b)\Leftrightarrow a, b\in S \text{ and } aRb,\ i_M(a, b)\Leftrightarrow a = b;$$

$$f_M(a_1, \dots, a_m, a) = \begin{cases} f_{P(a)}(a_1, \dots, a_m), & \text{if } a\in S \text{ and } a_1, \dots, a_m\in|P(a)| \\ \text{arbitrary element of } |P(a)| \text{ otherwise;} \end{cases}$$

$$q_M(a_1, \dots, a_m, a)\Leftrightarrow q_{P(a)}(a_1, \dots, a_m).$$

**Theorem 10.** Let $A, B$ classical models and $|A|\subseteqq|B|$. There are the same symbols in the languages of $A$ and $B$ the only exeption is $s$, which is used only in the language of $B$ as a unary relation symbol. Let

$$f_A(a_1, \dots, a_m) = f_B(a_1, \dots, a_m),\quad \text{if } a_1, \dots, a_m\in|A|;$$

$$q_A(a_1, \dots, a_m)\Leftrightarrow q(a_1, \dots, a_m),\quad \text{if } a_1, \dots, a_m\in|A|;$$

$$s_B(b)\Leftrightarrow b\in|A|.$$

We define the mapping $\mathscr{H}$ on the set of formulae not containing the symbol $s$:
$\mathscr{H}(\mathscr{A})=\mathscr{A}$, if $\mathscr{A}$ is an atom;
$\mathscr{H}(\mathscr{A}\wedge\mathscr{B})=\mathscr{H}(\mathscr{A})\wedge\mathscr{H}(\mathscr{B})$
$\mathscr{H}(\sim\mathscr{A})=\sim\mathscr{H}(\mathscr{A})$
$\mathscr{H}(\forall x\mathscr{A})=\forall x(s(x)\to\mathscr{H}(\mathscr{A}))$.
Let $k$ be an interpretation the range of which is in $|A|$. Then

$$A\models\mathscr{A}[k]\Leftrightarrow B\models\mathscr{H}(\mathscr{A})[k].$$

*Proof.* Trivial.

If $M\to\langle S, N, O, R, P\rangle$ is the mapping defined above, $\mathscr{T}$ is the parameter of the logic, then we have:
(i) the modal model has property $\mathscr{T}$ if and only if $M\models\mathscr{H}(\mathscr{T})$;
(ii) the modal model is simple if and only if for every function symbol $f$

$$M\models\forall x_1\dots\forall x_m\forall z\forall z'(p(x_1, z)\wedge p(x_1, z')\wedge\dots\wedge p(x_m, z)\wedge p(x_m, z')\to$$
$$\to i(f(x_1, \dots, x_m, z), f(x_1, \dots, x_m, z')));$$

(iii) the modal model is stable if and only if

$$M \models \forall z \forall z'(s(z) \wedge s(z') \to \forall x(p(x,z) \to p(x,z'))).$$

Let $\mathscr{A}$ be a modal formula and assume a modal logic is given. The formula $\mathscr{A}$ is satisfiable (in modal sense) if and only if the following formula is classically satisfiable: $s(o) \wedge (s(z) \to \exists x p(x,z)) \wedge \forall x \forall z \forall z' \; (p(x,z) \wedge r(z,z') \to p(x,z')) \wedge \wedge \forall z \forall z'(s(z) \wedge r(z,z') \to s(z')) \wedge \forall z \forall x_1, \dots, \forall x_{m_1}(s(z) \to p(f_1(x_1, \dots, x_{m_1}, z), z)) \wedge \dots \dots \wedge \forall z \forall x_1 \dots \forall x_{m_k}(s(z) \to p(f_k(x_1, \dots, x_{m_k}, z), z)) \wedge \mathscr{P} \wedge \mathscr{S} \wedge \mathscr{H}(\mathscr{T}) \wedge \mathscr{A}^*[z/o]$, where $f_1, \dots, f_k$ are all the function symbols occuring in $\mathscr{A}$; $\mathscr{P}$ is true if the logic is not simple, otherwise it is the following:

$$\bigwedge_{j=1}^{k} (\forall x_1, \dots \forall x_{m_j} \forall z \forall z'(p(x_1,z) \wedge p(x_1,z') \wedge \dots \wedge p(x_{m_j},z) \wedge p(x_{m_j},z) \to i(f(x_1, \dots \dots, x_{m_j}, z), f(x_1, \dots, x_{m_j}, z')));$$

$\mathscr{S}$ is true if the logic is not stable, otherwise it is the formula

$$\forall z \forall z'(s(z) \wedge s(z') \to \forall x(p(x,z) \to p(x,z'))).$$

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