Completeness in non-simple and stable modal logics

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In my work [1] I have defined the syntax and semantics of modal logics. Also, inference systems and completeness theorems for simple, non-stable logics have been included. Unfortunately, the methods used there cannot apply directly to non-simple and stable logics. In this paper I give a modification of the method and prove completeness theorems for the cases not covered in [1]. In fact, this paper is a continuation to [1], all non-common notions and notations are introduced there.

§ 1. Completeness in non-simple logics

The notion of consistency is defined in [1].

Definition. The set of formulae is complete if the following conditions are satisfied:

(i) α is consistent;

(ii) If $\mathscr{A}$ contains variables only from $\pi(\alpha)$, then either $A\in\alpha$ or $\sim\mathscr{A}\in\alpha$;

(iii) Let $\mathscr{A}$ contain variables only from $\pi(\alpha)$. If $\exists x\mathscr{A}\in\alpha$, then there exists a variable $a\in\pi(\alpha)$ such that a is free for x and $\mathscr{A}[x/a]\in\alpha$;

(iv) Let f be n-argument function symbol and let $x_1, \ldots, x_n\in\pi(\alpha)$. There exists a variable $a\in\pi(\alpha)$ such that for all classical formula $\mathscr{A}$ the fact $f(x_1, \ldots, x_n)$ is free for a in $\mathscr{A}$ implies that the two assertions $\mathscr{A}\in\alpha$ and $\mathscr{A}[a/f(x_1, \ldots, x_n)]\in\alpha$ are equivalent.

Theorem 1. If α is consistent, then there exists a complete set β such that $\alpha\subseteqq\beta$.

Proof. Parallel to the proof of Theorem 5 in [1] using the following Lemma.

Lemma. Let f be an n-argument function symbol, α a consistent set and $a\notin\pi(\alpha)$. Moreover, let $\alpha'=\alpha\cup\{\mathscr{A}: \mathscr{A}$ is a classical formula, $f(x_1, \ldots, x_n)$ is free for a in $\mathscr{A}$ and $\mathscr{A}[a/f(x_1, \ldots, x_n)]\in\alpha\}$. Then α' is consistent.

Proof. In contrary, let us suppose that there exist the formulae $\mathscr{A}_1, \ldots, \mathscr{A}_k$, $\mathscr{B}_1, \ldots, \mathscr{B}_l$ such that $\mathscr{A}_1, \ldots, \mathscr{A}_k\in\alpha$, $\mathscr{B}_1[a/f(x_1, \ldots, x_n)], \ldots, \mathscr{B}_l[a/f(x_1, \ldots, x_n)]\in\alpha$

and $\vdash \sim(\mathscr{A}_1\wedge\ldots\wedge\mathscr{A}_k\wedge\mathscr{B}_1\wedge\ldots\wedge\mathscr{B}_l)$. Applying R2 and A6.b we have

$$\vdash \forall a \sim (\mathscr{A}_1\wedge\ldots\wedge\mathscr{A}_k\wedge\mathscr{B}_1\wedge\ldots\wedge\mathscr{B}_l)$$

$$\vdash \sim(\mathscr{A}_1\wedge\ldots\wedge\mathscr{A}_k\wedge\mathscr{B}_1[a/f(x_1, \ldots, x_n)]\wedge\ldots\wedge\mathscr{B}_l[a/f(x_1, \ldots, x_n)])$$

which is a contradiction.

Definition of a complete system of formula-sets is just the same as in [1]-however, item (iii) can be omitted by the remark above.

The theorem remains valid for the new concept:

Theorem 2. If α is a complete set of formulae, then there exists a complete system of sets M such that $\alpha\in M$.

The completeness result follows easily from this theorem.

Theorem 3. Let a non-simple, non-stable modal logic be given. If $\mathscr{A}$ cannot be derived in this logic, then $\sim\mathscr{A}$ is satisfiable.

Proof. By the previous theorems, there exist a complete set α and a complete system of sets M such that $\sim\mathscr{A}\in\alpha$, $\alpha\in M$. We assume, by the definition of a complete set, that is a function for which the following property holds: if $\alpha\in M$, f is n-argument function symbol, $x_1, \ldots, x_n\in\pi(\alpha)$, then $v(\alpha, f(x_1, \ldots, x_n))$ is a variable, such that for all classical formula $\mathscr{B}$, if $f(x_1, \ldots, x_n)$ is free for $v(\alpha, f(x_1, \ldots, x_n))$ in $\mathscr{B}$, then the two assertions $\mathscr{B}\in\alpha$ and $\mathscr{B}[v(\alpha, f(x_1, \ldots, x_n))/f(x_1, \ldots, x_n)]\in\alpha$ are equivalent.

Let us introduce the notations:

$$N = \{\beta\colon \beta\in M \text{ and } \beta^+ \neq \emptyset\};$$

$$\text{If } \beta, \gamma\in M, \quad \text{then} \quad \beta R\gamma \Leftrightarrow ((\beta^+ \subseteqq \gamma \text{ and } \beta^+ \neq \emptyset) \quad \text{or} \quad (\beta^+ = \emptyset \text{ and } \gamma=\beta));$$

$$|P(\beta)| = \pi(\beta);$$

$$f_{P(\beta)}(x_1, \ldots, x_n) = v(\beta, f(x_1, \ldots, x_n)), \quad \text{where} \quad x_1, \ldots, x_n\in\pi(\beta);$$

$$r_{P(\beta)}(x_1, \ldots, x_n) \Leftrightarrow r(x_1, \ldots, x_n)\in\beta, \quad \text{where} \quad x_1, \ldots, x_n\in\pi(\beta).$$

It is clear that $\langle M, N, \alpha, R, P\rangle$ is a model. Let us extend the domain of v as follows: let $v(\beta, x)=x$, where $x\in\pi(\beta)$; and let $v(\beta, f(\tau_1, \ldots, \tau_n))=v(\beta, f(v(\beta, \tau_1), \ldots, \ldots, v(\beta, \tau_n)))$, where $\tau_1, \ldots, \tau_n$ are terms containing variables from $\pi(\beta)$ exclusively. The following assertions can be proved by (the usual) induction:

Let k be an interpretation and $\varkappa$ the corresponding valuation.

(i) If $\tau\in\mathscr{H}_k(\beta)$, then $\varkappa(\tau, \beta)=v(\beta, \tau[x_1, \ldots, x_m/k(x_1), \ldots, k(x_m)])$, where $x_1, \ldots, x_m$ are all variables occuring in τ.

(ii) If $\mathscr{B}\in\mathscr{H}_k(\beta)$, then $\beta\models\mathscr{B}[k]\Leftrightarrow\mathscr{B}[x_1, \ldots, x_m/k(x_1), \ldots, k(x_m)]\in\beta$; where $x_1, \ldots, x_m$ are all variables occuring in $\mathscr{B}$.

In particular, it follows that $\sim\mathscr{A}$ is valid in the model $\langle M, N, \alpha, R, P\rangle$.

Properties K1—K3 can be proved just as in [1, Theorem 7].

§ 2. Completeness in stable logics

Theorem 4. If α is complete, then there exist a complete set β and a complete system of sets M such that $\alpha\subseteqq\beta$, $\beta\in M$ and for every $\gamma\in M$, $\pi(\beta)=\pi(\gamma)$.

Before proving this theorem we give the completeness result for stable logics.

Theorem 5. Let a stable modal logic be given. If the formula $\mathscr{A}$ cannot be derived in this logic, then $\sim\mathscr{A}$ is satisfiable.

Proof. Very similar to the proof of Theorem 3 above or Theorem 7 of [1] provided complete system of sets M, given by Theorem 4, is used in the construction.

§ 3. Proof of Theorem 4

We introduce the following notations: let α be a set of formulae. By $\psi(\alpha)$ we shall mean the set of all formulae which contain variables only from $\pi(\alpha)$.

Let R be a two-argument relation. We define the relation R^n, n finite, by the following recurrence: R^0 is the identity relation and let R^{n+1} be defined by $AR^{n+1}B$ if there exists C such that AR^nC and CRB.

Then, $\bar{R}=\bigcup_{n=0}^{\infty}R^n$, where $\bar{R}$ is the reflexive, transitive closure of R.

In the following we shall deal with certain ordered triplets $\langle\alpha, M, R\rangle$. Without further mentioning we always suppose that the following conditions hold for $\langle\alpha, M, R\rangle$:

(i) M is a set of complete sets, $\alpha\in M$, R is a binary relation on M.

(ii) For every $\beta\in M, \alpha\bar{R}\beta$ and if $S\subseteqq R$ and for all $\beta\in M$ $\alpha\bar{S}\beta$, then $R=S$.

(iii) If $a\in\pi(\beta)$, then there exists γ such that $a\in\pi(\delta)$ if and only if $\gamma\bar{R}\delta$.

(iv) a) If $\beta R\gamma$ then $\beta^+\cap\psi(\gamma)\subseteqq\gamma$ and $\beta^+\neq\emptyset$.

b) Let $\beta\in M$, $\langle\rangle\mathscr{A}\in\beta$. If there is a $\gamma\in M$, such that $\beta R\gamma$ and $\mathscr{A}\in\psi(\gamma)$, then there also is a $\gamma\in M$ with $\beta R\gamma$ and $\mathscr{A}\in\gamma$.

Assertion 1. For arbitrary triplet $\langle\alpha, M, R\rangle$ there is no $\beta\in M$ such that $\beta R\alpha$. If $\beta R\delta$ and $\gamma R\delta$ then $\beta=\gamma$.

Proof. (a) Consider the triplet $\langle\alpha, M, S\rangle$, where S is defined by $\beta S\gamma$ if and only if $\beta R\gamma$ and $\gamma\neq\alpha$. By the second clause (ii) above, $R=S$.

(b) Let us suppose that $\beta R\delta$, $\gamma R\delta$ and $\beta\neq\gamma$. Let S be defined as $\beta_1S\beta_2$ if and only if $\beta_1R\beta_2$ and $(\beta_1,\beta_2)\neq(\beta,\delta)$. Then conditions above will hold for $\langle\alpha, M, S\rangle$, but $S\subseteqq R$ and $S\neq R$ which contradicts the second condition (ii).

Definition. $\langle\alpha, M, R\rangle$ is called n-th order triplet if for every $\beta\in M$ there is a k $(0\leqq k\leqq n)$ such that $\alpha R^k\beta$. $\langle\alpha, M, R\rangle$ is totally n-th order triplet if it is an n-th order triplet and if $0\leqq k<n$, $\alpha R^k\beta$, $\langle\rangle\mathscr{A}\in\beta$, $\beta^+\neq\emptyset$ then there exists $\gamma\in M$ for which $\beta R\gamma$ and $\mathscr{A}\in\gamma$.

It is clear, that for every m $(m\leqq n)$ the fact $\langle\alpha, M, R\rangle$ is an n-th order triplet implies that $\langle\alpha, M, R\rangle$ is an m-th order triplet too. Similarly, if $\langle\alpha, M, R\rangle$ is a zero order triplet, then $M=\{\alpha\}$, $R=\emptyset$, thus $\langle\alpha, M, R\rangle$ is totally zero order.

Let $\langle\alpha, M, R\rangle$ be arbitrary, $\beta\in M$. Let us set

$$M/\beta=\{\gamma\colon\ \beta\bar{R}\gamma\},\quad R/\beta=R\cap(M/\beta\times M/\beta).$$

Assertion 2. If $\langle\alpha, M, R\rangle$ is an n-th order triplet and $\alpha R\beta$ then $\langle\beta, M/\beta, R/\beta\rangle$ is an $(n-1)$-th order triplet.

DEFINITION. Let us define the operation L by the following items: if $\langle\alpha, M, R\rangle$ is a 0-order triplet, then $L\langle\alpha, M, R\rangle=\alpha$, if $n>0$ and $\langle\alpha, M, R\rangle$ is an n-th order triplet, then let $L\langle\alpha, M, R\rangle\doteq\alpha\cup\bigcup_{\alpha R\beta}\{\Diamond\mathscr{A}: \mathscr{A}$ is a conjunction of formulae from the set $L\langle\beta, M/\beta, R/\beta\rangle\}$.

Theorem 6. Let $\langle\alpha, M, R\rangle$ be an n-th order triplet. Then $L\langle\alpha, M, R\rangle$ is consistent.

Proof. We proceed by induction on n. If $n=0$ then the assertion clearly holds. Let $n>0$, and assume the contrary, i.e. there exist $\mathscr{A}_1, \mathscr{A}_2, \ldots$ and a conjunction $\mathscr{B}_1$ of formulae from $L\langle\beta_1, M/\beta_1, R/\beta_1\rangle$, a conjunction $\mathscr{B}_2$ of formulae from $L\langle\beta_2, M/\beta_2, R/\beta_2\rangle$ etc., such that

$$\vdash\sim(\mathscr{A}_1\wedge\mathscr{A}_2\wedge\ldots\wedge\Diamond\mathscr{B}_1\wedge\Diamond\mathscr{B}_2\wedge\ldots)$$

that is

$$\vdash\sim\mathscr{A}_1\vee\sim\mathscr{A}_2\vee\ldots\vee\Box\sim\mathscr{B}_1\vee\Box\sim\mathscr{B}_2\vee\ldots.$$

We can assume that all β_i, β_j are distinct, for if not, then can apply

$$\vdash\Box\sim\mathscr{B}_i\vee\Box\sim\mathscr{B}_j\rightarrow\Box\sim(\mathscr{B}_i\wedge\mathscr{B}_j).$$

Hence we obtain a form in which all sets β_i, β_j are distinct. Let x be a variable of $\mathscr{B}_1$ such that $x\notin\pi(\alpha)$. It follows from Assertion 1 and condition (iii) that x does not occur in the formulae $\mathscr{A}_1, \mathscr{A}_2, \ldots$ or $\mathscr{B}_2, \ldots$.

Apply rule R2 for all variables not occuring in $\pi(\alpha)$:

$$\vdash\sim\mathscr{A}_1\vee\sim\mathscr{A}_2\vee\ldots\vee\forall x_{11}\,\forall x_{12}\ldots\Box\sim\mathscr{B}_1\vee\forall x_{21}\,\forall x_{22}\ldots\Box\sim\mathscr{B}_2\vee\ldots.$$

Since the fixed logic is stable we can repeatedly apply the axiom $\forall x\Box\mathscr{A}\rightarrow\Box\forall x\mathscr{A}$ and obtain

$$\vdash\sim\mathscr{A}_1\vee\sim\mathscr{A}_2\vee\ldots\vee\Box\forall x_{11}\,\forall x_{12}\ldots\sim\mathscr{B}_1\vee\Box\forall x_{21}\,\forall x_{22}\ldots\sim\mathscr{B}_2\vee\ldots$$

where all free variables are from $\pi(\alpha)$. Since bound variables can be substitued by suitable ones from $\pi(\alpha)$ we have

$$\vdash\sim\mathscr{A}_1\vee\sim\mathscr{A}_2\vee\ldots\vee\Box\forall x'_{11}\,\forall x'_{12}\ldots\sim\mathscr{B}'_1\vee\Box\forall x'_{21}\,\forall x'_{22}\ldots\sim\mathscr{B}'_2\vee\ldots$$

α complete, so this possible only when some disjunctive terms, e.g. $\Box\forall x'_{11}\,\forall x'_{12}\ldots\ldots\sim\mathscr{B}'_1\in\alpha$. (For if $\sim\mathscr{A}_1\in\alpha$, then $\mathscr{A}_1\in\alpha$ which contradicts the completeness of α.) So

$$\forall x'_{11}\,\forall x'_{12}\ldots\sim\mathscr{B}'_1\in\beta_1\subseteqq L\langle\beta_1, M/\beta_1, R/\beta_1\rangle$$

and

$$\vdash\sim(\forall x'_{11}\,\forall x'_{12}\ldots\sim\mathscr{B}'_1\wedge\mathscr{B}_1).$$

We concluded that $L\langle\beta_1, M/\beta_1, R/\beta_1\rangle$ is consistent, wich proves the theorem.

Theorem 7. If $\langle\alpha, M, R\rangle$ is an n-th order triplet, $\alpha R\gamma$ and β is a complete set such that $L\langle\alpha, M, R\rangle\subseteqq\beta$ then $\beta^+\cup L\langle\gamma, M/\gamma, R/\gamma\rangle$ is consistent.

Proof. In contrary, let us assume that there are formulae $\mathscr{B}\in\beta^+$ and $\mathscr{C}$, a conjunction of elements from $L\langle\gamma, M/\gamma, R/\gamma\rangle$ such that $\vdash\sim(\mathscr{B}\wedge\mathscr{C})$ i.e. $\vdash\mathscr{B}\to\sim\mathscr{C}$. By R3, we obtain $\vdash\Box\mathscr{B}\to\Box\sim\mathscr{C}$ i.e. $\vdash\sim(\Box\mathscr{B}\wedge\Diamond\mathscr{C})$. In accordance with our conditions, $\Box\mathscr{B}\in\beta$ and so $\Diamond\mathscr{C}\in\beta$ which contradicts the completeness of β.

DEFINITION. We say that $\langle\alpha, M, R\rangle$ is a continuation of $\langle\beta, N, S\rangle$ if there exists a function $f: N\hookrightarrow M$, such that $f(\beta)=\alpha$, if $\gamma\in N$ then $\gamma\subseteq f(\gamma)$, and if $\gamma S\delta$ then $f(\gamma)Rf(\delta)$.

Theorem 8. If $\langle\alpha, M, R\rangle$ is an n-th order triplet and β is a complete set for which $L\langle\alpha, M, R\rangle\subseteq\beta$, then there exists a totally n-th order triplet $\langle\beta, N, S\rangle$ which is a continuation of $\langle\alpha, M, R\rangle$.

Proof. We proceed by induction on n. If $n=0$, the assertion follows. Let $n>0$. If $\alpha^+=\emptyset$, then for all $\mathscr{A}\in\psi(\alpha)$, $\Diamond\mathscr{A}\in\alpha$, so $\Diamond\mathscr{A}\in\beta$. In particular if $\mathscr{A}$ is a negation of a tautology, then $\vdash\Diamond\mathscr{A}\to\Diamond\mathscr{B}$, thus for every $\mathscr{B}\in\psi(\beta)$, $\Diamond\mathscr{B}\in\beta$, i.e. $\beta^+=\emptyset$. It is impossible that $\beta S\gamma$, by definition, hence $\langle\beta, \{\beta\}, \emptyset\rangle$ is a totally n-th order triplet and this is a continuation of $\langle\alpha, M, R\rangle=\langle\alpha, \{\alpha\}, \emptyset\rangle$.

Let $\alpha^+\neq\emptyset$, and so $\beta^+\neq\emptyset$. As we see $\Diamond\mathscr{A}\in\beta$ implies the consistency of $\beta^+\cup\{\mathscr{A}\}$, hence we can assume that $\beta^+\cup\{\mathscr{A}\}\subseteq\delta_{\mathscr{A}}$, $\delta_{\mathscr{A}}$ is complete. By the previous theorem, $\beta^+\cup L\langle\gamma, M/\gamma, R/\gamma\rangle$ is consistent, too, provided $\alpha R\gamma$, thus there is a complete set δ_γ such that $\beta^+\cup L\langle\gamma, M/\gamma, R/\gamma\rangle\subseteq\delta_\gamma$.

It is clear, that the new variables, introduced in these steps, may be chosen so that the sets $\pi(\delta_{\mathscr{A}})\setminus\pi(\beta), \dots, \pi(\delta_\gamma)\setminus\pi(\beta), \dots$ are pairwise disjoint. As $\langle\delta_{\mathscr{A}}, \{\delta_{\mathscr{A}}\}, \emptyset\rangle$ is an $(n-1)$-th order triplet, and since $L\langle\delta_{\mathscr{A}}, \{\delta_{\mathscr{A}}\}, \emptyset\rangle\subseteq\delta_{\mathscr{A}}$ it follows that a totally $(n-1)$-th order continuation $\langle\delta_{\mathscr{A}}, M_{\mathscr{A}}, R_{\mathscr{A}}\rangle$ of $\langle\delta_{\mathscr{A}}, \{\delta_{\mathscr{A}}\}, \emptyset\rangle$ exists. Since $L\langle\gamma, M/\gamma, R/\gamma\rangle\subseteq\delta_\gamma$ by the induction hypothesis, it follows that there exist M_γ, R_γ such that $\langle\delta_\gamma, M_\gamma, R_\gamma\rangle$ is a totally $(n-1)$-th order triplet and it is a continuation of $\langle\gamma, M/\gamma, R/\gamma\rangle$.

We may assume that the common variables of any two sets $\pi(\cup M_{\mathscr{A}}), \dots, \dots, \pi(\cup M_\gamma), \dots$ are contanied in $\pi(\beta)$.

Let $\delta\in N$, provided $\delta=\beta$, or
if there is an $\mathscr{A}$, such that $\Diamond\mathscr{A}\in\beta$ and $\delta\in M_{\mathscr{A}}$, or
if there is a γ, such that $\alpha R\gamma$ and $\delta\in M_\gamma$.

Let $\delta_1, \delta_2\in N$ and $\delta_1 S\delta_2$ provided
if $\delta_1=\beta$ and there is an $\mathscr{A}$ such that $\Diamond\mathscr{A}\in\beta$ and $\delta_2=\delta_{\mathscr{A}}$, or
if $\delta_1=\beta$ and there is an γ for which $\alpha R\gamma$ and $\delta_2=\delta_\gamma$, or
if there is an $\mathscr{A}$, such that $\Diamond\mathscr{A}\in\beta$ and $\delta_1 R_{\mathscr{A}}\delta_2$, or
if there is an γ, such that $\alpha R\gamma$ and $\delta_1 R_\gamma\delta_2$.

It is obvious, that the conditions (i)—(iv) hold for $\langle\beta, N, S\rangle$. Also, it is a totally n-th order triplet and is a continuation of $\langle\alpha, M, R\rangle$.

Now we can return to the proof of Theorem 4: Let $\langle\alpha_0, M_0, R_0\rangle=\langle\alpha, \{\alpha\}, \emptyset\rangle$ i.e. a totally 0-order triplet. Let us suppose, that for some n, a totally n-th order triplet $\langle\alpha_n, M_n, R_n\rangle$ is defined. By Theorem 6, $L\langle\alpha_n, M_n, R_n\rangle$ is consistent, and hence there is a complete set α_{n+1}, such that $L\langle\alpha_n, M_n, R_n\rangle\subseteq\alpha_{n+1}$. By Theorem 8, there exist M_{n+1}, R_{n+1} such that $\langle\alpha_{n+1}, M_{n+1}, R_{n+1}\rangle$ is a totally $(n+1)$-th order triplet and it is a continuation of $\langle\alpha_n, M_n, R_n\rangle$. Thus, there exists a function $f_n: M_n\to M_{n+1}$ such that $f_n(\alpha_n)=\alpha_{n+1}$ and $\beta\in M_n$ implies $\beta\subseteq f_n(\beta)$ and if $\beta R_n\gamma$, then $f_n(\beta)R_{n+1}f_n(\gamma)$.

Let $\beta=\bigcup_{n=0}^{\infty}\alpha_n$ and $M=\left\{\bigcup_{n=k}^{\infty}\gamma_n\colon \gamma_k\in M_k \text{ and for all } i\geqq k,\ \gamma_{i+1}=f_i(\gamma_i)\right\}$. Since union of increasing complete sets is also complete we have that every element of M is complete.

Let $\gamma\in M$, $\gamma^+\neq\emptyset$, and $\langle\rangle\mathscr{A}\in\gamma$. For $\gamma=\bigcup_{n=k}^{\infty}\gamma_n$, there exists an l, such that $\langle\rangle\mathscr{A}\in\gamma_l$ and hence there also exists δ_l for which $\gamma_l R_l \delta_l$ and $\mathscr{A}\in\delta_l$. Let $\delta=\bigcup_{n=l}^{\infty}\delta_n$. $\gamma^+=\bigcup_{n=l}^{\infty}\gamma_n^+\subseteqq\bigcup_{n=l}^{\infty}\delta_n=\delta$ and $\mathscr{A}\in\delta$, thus M is a complete system of sets.

Let $a\in\pi(\beta)$. For some k, $a\in\pi(\alpha_k)$, and if $l>k$, then $a\in\pi(\alpha_k)$, too. If $\delta\in M$, then for some i, $\delta=\bigcup_{n=i}^{\infty}\delta_n$. We may assume that $k<i$ and so $a\in\pi(\alpha_i)$. Since $\langle\alpha_i, M_i, R_i\rangle$ is a totally i-th order triplet, we have $\pi(\alpha_i)\subseteqq\pi(\delta_i)$, and thus $a\in\pi(\delta)$.

Let $a\in\pi(\delta)$ for some $\delta\in M$. We may assume that $\delta=\bigcup_{n=k}^{\infty}\delta_k$ and $a\in\pi(\delta_k)$. For $L\langle\alpha_k, M_k, R_k\rangle\subseteqq\alpha_{k+1}$, $a\in\pi(\alpha_{k+1})$ and hence $a\in\pi(\beta)$.

We gained, that for every $\delta\in M$, $\pi(\beta)=\pi(\delta)$ which completes the proof of Theorem 4 and also the completeness theorem.

References

[1] Tóth, K., Modal logics with function symbols, *Acta Cybernet.*, v. 4, 1979, pp. 291—302.

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