Decidability results concerning tree transducers I

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A tree transducer is called functional if its induced transformation is a partial mapping. We show that the functionality of tree transducers is decidable. Consequently, the equivalence problem for deterministic tree transducers is also decidable. The latter result was independently achieved by Z. ZACHAR in [12] for bottom-up tree transducers and a restricted class of top-down tree transducers. The solvability of the equivalence problem for generalized deterministic sequential machines is known from [2] and [4]. It was proved in [11] that this positive result can not be generalized for arbitrary, i.e. generalized nondeterministic; sequential machines. Therefore, the equivalence problem for nondeterministic tree transducers is undecidable.

Our result can be used to minimize deterministic tree transducers in an effective manner. However, the minimal realizations of a deterministic tree transducer are not isomorphic. We investigate conditions assuring the uniqueness (up to isomorphism) of minimal realizations in certain classes of tree transducers.

Part of the results of the present paper have been announced in [8]. The terminology is used in the sense of [5].

1. Notions and notations

By a *type* $F=\bigcup_{n<\omega} F_n$ we mean a finite type such that $F_0\neq\emptyset$. For the type F, $\nu(F)=\max\{n\,|\,F_n\neq\emptyset\}$. An *F-algebra* is a system $\mathbf{A}=(A, \{(f)_{\mathbf{A}}\,|\,f\in F\})$, or shortly, (A, F), where for every nonnegative integer n and $f\in F_n$ $(f)_{\mathbf{A}}\colon A^n\to A$ is the realization of the n-ary operational symbol f.

Let Y be an arbitrary set. Then $\mathbf{T}_{F,Y}=(T_{F,Y}, F)$ denotes the *free F-algebra generated by Y*. The elements of $T_{F,Y}$ are called *trees* and they can be obtained by induction as follows: $T_{F,Y}$ is the smallest set satisfying

(i) $F_0, Y\subseteq T_{F,Y}$,

(ii) if $n>0$, $f\in F_n$, $t_1, \dots, t_n\in T_{F,Y}$ then $f(p_1, \dots, p_n)\in T_{F,Y}$.

In particular, if $Y=X_n$, the set of the first n variables $x_1, \ldots, x_n$ for a nonnegative integer n, $T_{F,Y}$ is denoted by $T_{F,n}$ and $T_{F,0}$ is written T_F. Each *n-ary tree* $p \in T_{F,n}$ induces a mapping $(p)_{\mathbf{A}}\colon A^n \to A$ in an F-algebra $\mathbf{A}$. If $\mathbf{A}$ is the free algebra $\mathbf{T}_{F,Y}$ then $(p)_{\mathbf{A}}(t_1, \ldots, t_n) = p(t_1, \ldots, t_n)$, i.e. the tree obtained by substituting t_i for x_i $(i=1, \ldots, n)$ in p.

The *depth* (dp), *rank* (rn) and *frontier* (fr) of trees are defined as usually. For a tree $p \in T_{F,Y}$ we have

(i) $\mathrm{dp}(p) = 0, \quad \mathrm{rn}(p) = 1, \quad \mathrm{fr}(p) = p \quad \text{if} \quad p \in Y,$

(ii) $\mathrm{dp}(p) = 0, \quad \mathrm{rn}(p) = 1, \quad \mathrm{fr}(p) = \lambda \quad \text{if} \quad p \in F_0,$

(iii) $\mathrm{dp}(p) = 1 + \max\{\mathrm{dp}(p_i) \mid i = 1, \ldots, n\}, \quad \mathrm{rn}(p) = 1 + \sum_{i=1}^{n} \mathrm{rn}(p_i),$
$\mathrm{fr}(p) = \mathrm{fr}(p_1) \ldots \mathrm{fr}(p_n) \quad \text{if} \quad p = f(p_1, \ldots, p_n), \quad f \in F_n,$

$p_1, \ldots, p_n \in T_{F,Y}$ and $n>0$. Here λ denotes the empty string.

In connection with the elements of $T_{F,n}(n \geqq 0)$ we shall also use the concept of *path*. For an arbitrary $i(1 \leqq i \leqq n)$ and $p \in T_{F,n}$ $\mathrm{path}_i(p)$ is given by

(i) $\mathrm{path}_i(p) = \{\lambda\} \quad \text{if} \quad p = x_i,$

(ii) $\mathrm{path}_i(p) = \emptyset \quad \text{if} \quad p \in F_0 \cup X_n - \{x_i\},$

(iii) $\mathrm{path}_i(p) = \{jw \mid w \in \mathrm{path}_i(p_j),\ 1 \leqq j \leqq m\} \quad \text{if} \quad p = f(p_1, \ldots, p_m),$

$m>0$, $f \in F_m$, $p_1, \ldots, p_m \in T_{F,n}$. If $\mathrm{path}_i(p)$ is a singleton then it is identified with its unique element. For $w \in \mathrm{path}_i(p)$ we denote by $|w|$ the *length* of w. $\mathrm{path}(p) = \bigcup_{i=1}^{n} \mathrm{path}_i(p)$. For arbitrary two strings v and w v/w denotes the *derivative of v with respect to w*, i.e. $v/w=u$ if and only if $v=wu$.

Further on we shall often use *vector notations* to simplify the treatment. Vectors, except possibly the one dimensional ones, are always denoted by boldfaced letters. For each k dimensional vector $\mathbf{a} \in A^k$ $(k \geqq 0)$ and i $(1 \leqq i \leqq k)$ a_i denotes the ith component of $\mathbf{a}$. Conversely, if $a \in A$ then $\mathbf{a}^k \in A^k$ is the k dimensional vector whose each component is equal to a. The product $\mathbf{ab}$ of the k dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by $\mathbf{ab} = (a_1b_1, \ldots, a_kb_k)$ where a_ib_i are short notations for (a_i, b_i) $(i=1, \ldots, k)$. For the vectors of trees $\mathbf{p} \in T_{F,n}^k$ and $\mathbf{q} \in T_{F,m}^n$ we denote by $\mathbf{p}(\mathbf{q})$ the vector $(p_1(\mathbf{q}), \ldots, p_k(\mathbf{q}))$.

According to the function fr one can distinguish the subset $\hat{T}_{F,n}$ of $T_{F,n}$. This consists of those elements of $T_{F,n}$ whose frontier is a *permutation* of the variables in X_n. We may extend this definition to vectors as follows: $\hat{T}_{F,n}^k = \{\mathbf{p} \in T_{F,n}^k \mid \mathrm{fr}(p_1) \ldots \mathrm{fr}(p_k) \text{ is a permutation of } X_n\}$. Observe that $\hat{T}_{F,n}^k$ is not the kth power of $\hat{T}_{F,n}$.

We now turn to the definition of tree transducers. Following [5] a *top-down tree transducer* is a system $\mathbf{A} = (F, A, G, A_0, \Sigma)$, where F and G are types, A is a finite, nonvoid set, the set of states, $A_0 \subseteq A$ is the set of initial states, finally, Σ is a finite set of top-down rewriting rules. A *top-down rule* has the form $af \to p$ — or equivalently $af(x_1, \ldots, x_n) \to p$, where $n \geqq 0, a \in A, f \in F_n, p \in T_{G, A \times X_n}$. A *bottom-up tree transducer* $\mathbf{A} = (F, A, G, A_0, \Sigma)$ has a similar structure except A_0 is called the set of final states and Σ contains bottom-up rewriting rules. A typical *bottom-up rewriting rule* is of form $f(a_1x_1, \ldots, a_nx_n) \to ap$ where $n \geqq 0$, $f \in F_n$, $p \in T_{G,n}$, $a, a_1, \ldots, a_n \in A$. By a tree transducer we mean a top-down or bottom-up transducer.

Take an arbitrary tree transducer $\mathbf{A} = (F, A, G, A_0, \Sigma)$ and let Y be an arbitrary set. Σ can be used to define a binary relation $\overset{*}{\Rightarrow}_{\mathbf{A},Y}$ on $T_{G, A \times T_{F,Y}}$ in the top-down case and on the set $T_{F, A \times T_{G,Y}}$ in the bottom-up case. It is called *derivation* and its exact definition can be found in [5]. If there is no danger of confusion $\mathbf{A}$ is omitted in $\overset{*}{\Rightarrow}_{\mathbf{A},Y}$. It can be seen that if $Y_1 \subseteq Y_2$ and $p, q \in T_{G, A \times T_{F,Y_1}}$ then $p \overset{*}{\Rightarrow}_{Y_1} q$ if and only if $p \overset{*}{\Rightarrow}_{Y_2} q$. Similar equivalence is valid in the bottom-up case. Thus we may omit Y in $\overset{*}{\Rightarrow}_Y$.

Again take the tree transducer $\mathbf{A}$. This induces a *transformation* $\tau_{\mathbf{A}} \subseteq T_F \times T_G$:

$$\tau_{\mathbf{A}} = \{(p, q) \mid \exists a_0 \in A_0 \;\; a_0 p \overset{*}{\Rightarrow} q\}$$

in the top-down case, and

$$\tau_{\mathbf{A}} = \{(p, q) \mid \exists a_0 \in A_0 \;\; p \overset{*}{\Rightarrow} a_0 q\}$$

for bottom-up $\mathbf{A}$. If $\tau_{\mathbf{A}}$ is a (partial) function $\mathbf{A}$ is called *functional*. This is always the case if $\mathbf{A}$ is *deterministic,* i.e. different rules have different left sides, moreover, A_0 is a singleton in the top-down case. Two tree transducers are called *equivalent* if their induced transformations coinside. For a tree transducer $\mathbf{A} = (F, A, G, A_0, \Sigma)$ and a state $a \in A$ we denote by $\mathbf{A}(a)$ the transducer $\mathbf{A}(a) = (F, A, G, \{a\}, \Sigma)$.

The domain of the transformation $\tau_{\mathbf{A}}$ is denoted by dom $\tau_{\mathbf{A}}$. It is a *regular subset* of T_F, i.e. a regular forest. Regular forests are exactly the forests recognized by tree automata. A *tree automaton* is a system $\mathbf{B} = (F, B, B_0)$ with (B, F) a finite F-algebra which is denoted by $\mathbf{B}$ too, $B_0 \subseteq B$ is the set of final states. The *forest recognized by* $\mathbf{B}$ is determined by $T(\mathbf{B}) = \{p \in T_F \mid (p)_{\mathbf{B}} \in B_0\}$.

Sometimes we need to restrict a top-down tree transducer to a regular forest. If $\mathbf{A} = (F, A, G, A_0, \Sigma)$ is a top-down tree transducer and $T \subseteq T_F$ is a regular forest then the system $\mathbf{B} = (F, T, A, G, A_0, \Sigma)$ is called a *regularly restricted top-down tree transducer*. Its induced transformation is $\tau_{\mathbf{B}} = \{(p, q) \in \tau_{\mathbf{A}} \mid p \in T\}$. A similar but more general concept is the concept of *top-down tree transducer with regular look-ahead* introduced in [6]. A top-down tree transducer with regular look-ahead is a system $\mathbf{A} = (F, A, G, A_0, \Sigma)$ where F, A, G, A_0 are the same as for top-down tree transducers and Σ is a finite set of rules

$$(af(x_1, \ldots, x_n) \to p;\; R_1, \ldots, R_n)$$

where $af(x_1, \ldots, x_n) \to p$ is a top-down rewriting rule, i.e. $a \in A$, $f \in F_n$ $(n \geqq 0)$, $p \in T_{G, A \times X_n}$, and $R_i \subseteq T_F$ $(1 \leqq i \leqq n)$ are regular forests. The regular forests R_i are used to restrict the applicability of the coressponding top-down rule $af(x_1, \ldots, x_n) \to p$. The rule $(af(x_1, \ldots, x_n) \to p;\; R_1, \ldots, R_n)$ can be applied for a subtree of a tree in $T_{G, A \times T_{F,Y}}$ if and only if it is of form $af(p_1, \ldots, p_n)$ with $p_i \in R_i$ for each i $(1 \leqq i \leqq n)$. Apart from this derivation is defined as for top-down transducers. The *induced transformation* is the relation $\tau_{\mathbf{A}} = \{(p, q) \mid a_0 p \overset{*}{\Rightarrow} q$ for some $a_0 \in A_0\}$. Again, if it is a function $\mathbf{A}$ is called *functional*. It is known that every functional bottom-up or top-down tree transducer is equivalent to some deterministic top-down transducer with regular look-ahead (cf. [7]).

2. The decidability of functionality of tree transducers

First we show that the decision of functionality of bottom-up transducers is reducible to the decision of functionality of regularly restricted top-down ones.

Let $\mathbf{A}=(F, A, G, A_0, \Sigma)$ be an arbitrary bottom-up transducer. Define the top-down transducer with regular look-ahead $\mathbf{A}'$ as follows: $\mathbf{A}'=(F, A, G, A_0, \Sigma')$ where

$$\Sigma' = \{(af \to p(a_1x_1, \ldots, a_nx_n);\ R_1, \ldots, R_n) \mid f(a_1x_1, \ldots, a_nx_n) \to ap \in \Sigma,$$
$$R_i = \operatorname{dom} \tau_{\mathbf{A}(a_i)}\ (i = 1, \ldots, n)\}.$$

Lemma 1. $\mathbf{A}$ is functional if and only if $\mathbf{A}'$ is functional.

Proof. It is obvious that $\tau_{\mathbf{A}} \subseteq \tau_{\mathbf{A}'}$. Therefore if $\mathbf{A}'$ is functional then $\mathbf{A}$ is functional, too. To prove the converse first we show that if $ap \overset{*}{\Rightarrow}_{\mathbf{A}'} q$ and $a'p \overset{*}{\Rightarrow}_{\mathbf{A}'} q'$ where $a, a' \in A$, $p \in T_F$, $q, q' \in T_G$ and $q \neq q'$ then there exist different trees $r, r' \in T_G$ such that $p \overset{*}{\Rightarrow}_{\mathbf{A}} br$ and $p \overset{*}{\Rightarrow}_{\mathbf{A}} b'r'$ are also satisfied for certain choise of states b, b' with $\{b, b'\} \subseteq \{a, a'\}$. We shall prove this by induction on p. The basis, $p \in F_0$, is immediate. Suppose now that $p = f(p_1, \ldots, p_n)$ where $n > 0$, $f \in F_n$, $p_1, \ldots, p_n \in T_F$. Since $ap \overset{*}{\Rightarrow} q$ and $a'p \overset{*}{\Rightarrow} q'$ there exist rules $f(a_1x_1, \ldots, a_nx_n) \to aq_0$, $f(a_1'x_1, \ldots, a_n'x_n) \to a'q_0' \in \Sigma$ with $p_i \in \operatorname{dom} \tau_{\mathbf{A}(a_i)} \cap \operatorname{dom} \tau_{\mathbf{A}(a_i')}$ and satisfying $q_0(a_1p_1, \ldots, a_np_n) \overset{*}{\Rightarrow} q$ and $q_0'(a_1'p_1, \ldots, a_n'p_n) \overset{*}{\Rightarrow} q'$, respectively. We distinguish two cases.

Firstly assume that for each $i \in \{1, \ldots, n\}$ if x_i appears in $\operatorname{fr}(q_0)$ then there exists exactly one tree $q_i \in T_G$ with $a_ip_i \overset{*}{\Rightarrow} q_i$. Then also $p_i \overset{*}{\Rightarrow} a_iq_i$. This and $p_i \in \operatorname{dom} \tau_{\mathbf{A}(a_i)}$ $(i=1, \ldots, n)$ yield $p \overset{*}{\Rightarrow} aq$. Similarly, we get $p \overset{*}{\Rightarrow} a'q'$ if, for each x_i occuring in $\operatorname{fr}(q_0')$, there is only one tree in T_G which can be derived from $a_i'p_i$. This proves our assertion in the first case.

Secondly assume that there is an integer $i \in \{1, \ldots, n\}$ such that x_i appears in $fr(q_0)$ and there are different trees $q_i, q_i' \in T_G$ with $a_ip_i \overset{*}{\Rightarrow} q_i$ and $a_ip_i \overset{*}{\Rightarrow} q_i'$, respectively. Then, by the induction hypothesis, there exist trees $r_i \neq r_i' \in T_G$ satisfying both $p_i \overset{*}{\Rightarrow} a_ir_i$ and $p_i \overset{*}{\Rightarrow} a_ir_i'$. For each index $j (j \neq i)$ choose $r_j \in T_G$ in such a way that we have $p_j \overset{*}{\Rightarrow} a_jr_j$. This can be done by $p_j \in \operatorname{dom} \tau_{\mathbf{A}(a_j)}$. Now let $r = q_0(r_1, \ldots, r_n)$, $r' = q_0(r_1, \ldots, r_{i-1}, r_i', r_{i+1}, \ldots, r_n)$. $r \neq r'$ because $r_i \neq r_i'$. On the other hand $p \overset{*}{\Rightarrow} ar$ and $p \overset{*}{\Rightarrow} ar'$.

Now assume that $\mathbf{A}'$ is not functional. Then there exist trees $p \in T_F$, $q \neq q' \in T_G$ and initial states $a_0, a_0' \in A_0$ such that both $a_0p \overset{*}{\Rightarrow}_{\mathbf{A}'} q$ and $a_0'p \overset{*}{\Rightarrow}_{\mathbf{A}'} q'$ are satisfied. By the previous considerations it follows that there are different trees $r, r' \in T_G$ with $p \overset{*}{\Rightarrow}_{\mathbf{A}} b_0r$ and $p \overset{*}{\Rightarrow}_{\mathbf{A}} b_0'r'$ where each of the states b_0 and b_0' denotes either a_0 or a_0'. This means that both (p, r) and (p, r') are in $\tau_{\mathbf{A}}$, i.e. $\mathbf{A}$ is not functional.

Lemma 2. The decision of functionality of bottom-up tree transducers is reducible to the decision of functionality of regularly restricted top-down ones.

Proof. Let $\mathbf{A}$ be an arbitrary bottom-up transducer and $\mathbf{A}'$ the top-down transducer with regular look-ahead constructed in the previous lemma. We know that $\mathbf{A}$ is functional if and only if $\mathbf{A}'$ is functional. By Theorem 2.6 in [6] we have

$\tau_{A'}=\tau\circ\tau_B$ where τ is a deterministic bottom-up relabeling, i.e. a transformation induced by a special deterministic bottom-up transducer, and **B** is a top-down transducer. Since τ is a function **A'** is functional if and only if **B** restricted to the regular forest $\tau(\operatorname{dom}\tau_{A'})$ is functional. Note that $\operatorname{dom}\tau_A=\operatorname{dom}\tau_{A'}$. As one can construct the transducers **A'** and **B** in an effective manner this proves Lemma 2.

Now let us fix an arbitrary regularly restricted top-down tree transducer $\mathbf{A}=(F, T, A, G, A_0, \Sigma)$ and a tree automaton $\mathbf{B}=(F, B, B_0)$ recognizing T. Set

$$P = \{p\in T \mid \exists q \neq q' \in T_G\ (p, q), (p, q')\in\tau_A\}.$$

In the next five lemmas we shall present five reduction rules. Each reduction rule produces a smaller tree $p'\in P$ for a tree $p\in T$ if it can be applied for p.

Lemma 3. Let $p_1, p_2\in \hat{T}_{F,1}, p_3\in T_F$, $n_1, n_1', n_2, n_2'\geqq 0$, $q_1\in\hat{T}_{G,n_1}$, $q_1'\in\hat{T}_{G,n_1'}$, $\mathbf{q}_2\in\hat{T}^{n_1}_{G,n_2}$, $\mathbf{q}_2'\in\hat{T}^{n_1'}_{G,n_2'}$, $\mathbf{q}_3\in T^{n_2}_G$, $\mathbf{q}_3'\in T^{n_2'}_G$, $a_0, a_0'\in A_0$, $\mathbf{a}_i\in A^{n_i}$, $\mathbf{a}_i'\in A^{n_i'}$ $(i=1, 2)$. Let us denote by A_i and A_i' the sets $A_i=\{a_{i,j}\mid 1\leqq j\leqq n_i\}$ and $A_i'=\{a'_{i,j}\mid 1\leqq j\leqq n_i'\}$ $(i=1, 2)$ respectively. Assume that each of the following conditions is satisfied:

(i) $p_1(p_2(p_3))\in T$,

(ii) $a_0p_1\overset{*}{\Rightarrow} q_1(\mathbf{a}_1\mathbf{x}_1^{n_1})$, $a_0'p_1\overset{*}{\Rightarrow} q_1'(\mathbf{a}_1'\mathbf{x}_1^{n_1'})$,

(iii) $\mathbf{a}_1\mathbf{p}_2^{n_1}\overset{*}{\Rightarrow}\mathbf{q}_2(\mathbf{a}_2\mathbf{x}_1^{n_2})$, $\mathbf{a}_1'\mathbf{p}_2^{n_1'}\overset{*}{\Rightarrow}\mathbf{q}_2'(\mathbf{a}_2'\mathbf{x}_1^{n_2'})$,

(iv) $\mathbf{a}_2\mathbf{p}_3^{n_2}\overset{*}{\Rightarrow}\mathbf{q}_3$, $\mathbf{a}_2'\mathbf{p}_3^{n_2'}\overset{*}{\Rightarrow}\mathbf{q}_3'$,

(v) $(p_3)_{\mathbf{B}}=(p_2(p_3))_{\mathbf{B}}$, $A_1\subseteqq A_2$, $A_1'\subseteqq A_2'$,

(vi) $q_1(\mathbf{r})\neq q_1'(\mathbf{r}')$ holds for any $\mathbf{r}\in T_G^{n_1}$ and $\mathbf{r}'\in T_G^{n_1'}$.

Then $p_1(p_3)\in P$.

Proof. First note that our assumptions imply the condition $p_1(p_2(p_3))\in P$. From now on let $[n]$ denote the set of the first n positive integers for every $n\geqq 0$. Thus $[0]$ is the empty set. Let $\varphi\colon [n_1]\to[n_2]$ and $\varphi'\colon [n_1']\to[n_2']$ be mappings with $a_{1,i}=a_{2,\varphi(i)}$ $(i\in[n_1])$ and $a'_{1,i}=a'_{2,\varphi'(i)}$ $(i\in[n_2])$, respectively. Obviously we have $\mathbf{a}_1\mathbf{p}_3^{n_1}\overset{*}{\Rightarrow}\mathbf{r}$ and $\mathbf{a}_1'\mathbf{p}_3^{n_1'}\overset{*}{\Rightarrow}\mathbf{r}'$ where $\mathbf{r}=(q_{3,\varphi(1)}, \dots, q_{3,\varphi(n_1)})$, $\mathbf{r}'=(q'_{3,\varphi'(1)}, \dots, q'_{3,\varphi'(n_1)})$. By (ii) this implies that $a_0p_1(p_3)\overset{*}{\Rightarrow}q_1(\mathbf{r})$ and $a_0'p_1(p_3)\overset{*}{\Rightarrow}q_1'(\mathbf{r}')$. On the other hand $q_1(\mathbf{r})\neq q_1'(\mathbf{r}')$ by our assumption (vi). Furthermore, $p_1(p_3)\in T$ holds by (v). Hence $p_1(p_3)\in P$.

Lemma 4. Let $p_1\in\hat{T}_{F,1}, p_2\in T_F$, $n, n'>0$, $q_1\in\hat{T}_{G,n}$, $q_1'\in\hat{T}_{G,n'}$, $\mathbf{q}_2\in T_G^n$, $\mathbf{q}_2'\in T_G^{n'}$, $a_0, a_0'\in A_0$, $\mathbf{a}\in A^n$, $\mathbf{a}'\in A^{n'}$. Let $|A|$ and $|B|$ denote the cardinality of A and B, respectively and let $\|A\|=2^{|A|}$, $K=\max\{\operatorname{dp}(q)\mid\exists a\in A, p\in T_{F,X}\ ap\to q\in\Sigma\}$. Assume that the following conditions are valid:

(i) $p_1(p_2)\in T$,

(ii) $a_0p_1\overset{*}{\Rightarrow} q_1(\mathbf{a}\mathbf{x}_1^n)$, $a_0'p_1\overset{*}{\Rightarrow} q_1'(\mathbf{a}'\mathbf{x}_1^{n'})$,

(iii) $\mathbf{a}\mathbf{p}_2^n \overset{*}{\Rightarrow} \mathbf{q}_2, \quad \mathbf{a}'\mathbf{p}_2^{n'} \overset{*}{\Rightarrow} \mathbf{q}_2'$,

(iv) $\text{path}_1(q_1)$ is a prefix of $\text{path}_1(q_1')$,

$$|\text{path}_1(q_1')| - |\text{path}_1(q_1)| > \|A\|^2 |B| K, \quad \text{dp}(p_2) \geqq \|A\|^2 |B|.$$

Then there is a tree $r \in T_F$ such that $p_1(r) \in P$ and $\text{rn}(r) < \text{rn}(p_2)$.

Proof. Let R be the forest defined by

$$R = \{r \in T_F | p_1(r) \in T, \quad \text{rn}(r) \leqq \text{rn}(p_2), \quad \exists \mathbf{s} \in T_G^n, \mathbf{s}' \in T_G^{n'} \quad \mathbf{a}\mathbf{r}^n \overset{*}{\Rightarrow} \mathbf{s}, \quad \mathbf{a}'\mathbf{r}^{n'} \overset{*}{\Rightarrow} \mathbf{s}'\}.$$

Since $p_2 \in R$ R is nonvoid. Let r be an element of R with minimal rank. We shall show that $p_1(r) \in P$ and $\text{dp}(r) < \|A\|^2 B$.

Assume that the condition $\text{dp}(r) < \|A\|^2 B$ does not hold. In this case there exist

$$r_1, r_2 \in \hat{T}_{F,1}, \ r_3 \in T_F, \ m_1, m_1', m_2, m_2' \geqq 0, \ \mathbf{s}_1 \in \hat{T}^n_{G,m_1}, \ \mathbf{s}_1' \in \hat{T}^{n'}_{G,m_1'},$$

$$\mathbf{s}_2 \in \hat{T}^{m_1}_{G,m_2}, \ \mathbf{s}_2' \in \hat{T}^{m_1'}_{G,m_2'}, \ \mathbf{s}_3 \in \hat{T}^{m_2}_G, \ \mathbf{s}_3' \in \hat{T}^{m_2'}_G, \ \mathbf{b}_i \in A^{m_i}, \ \mathbf{b}_i' \in A^{m_i'} \quad (i = 1, 2)$$

such that each of the following five conditions is satisfied:

(1) $r = r_1(r_2(r_3)), \quad r_2 \neq x_1$,

(2) $\mathbf{a}\mathbf{r}_1^n \overset{*}{\Rightarrow} \mathbf{s}_1(\mathbf{b}_1 \mathbf{x}_1^{m_1}), \quad \mathbf{a}'\mathbf{r}_1^{n'} \overset{*}{\Rightarrow} \mathbf{s}_1'(\mathbf{b}_1' \mathbf{x}_1^{m_1'})$,

(3) $\mathbf{b}_1 \mathbf{r}_2^{m_1} \overset{*}{\Rightarrow} \mathbf{s}_2(\mathbf{b}_2 \mathbf{x}_1^{m_2}), \quad \mathbf{b}_1' \mathbf{r}_2^{m_1'} \overset{*}{\Rightarrow} \mathbf{s}_2'(\mathbf{b}_2' \mathbf{x}_1^{m_2'})$,

(4) $\mathbf{b}_2 \mathbf{r}_3^{m_2} \overset{*}{\Rightarrow} \mathbf{s}_3, \quad \mathbf{b}_2' \mathbf{r}_3^{m_2'} \overset{*}{\Rightarrow} \mathbf{s}_3'$,

(5) $(r_3)_\mathbf{B} = (r_2(r_3))_\mathbf{B}, \quad B_1 \subseteq B_2, \quad B_1' \subseteq B_2'$, where

$$B_i = \{b_{i,j} | 1 \leqq j \leqq m_i\}, \ B_i' = \{b_{i,j}' | 1 \leqq j \leqq m_i'\} \quad (i = 1, 2).$$

Now let $\varphi\colon [m_1] \to [m_2]$, $\varphi'\colon [m_1'] \to [m_2']$ be mappings satisfying the equalities $b_{1,i} = b_{2,\varphi(i)}$ $(i \in [m_1])$, $b_{1,i}' = b_{2,\varphi'(i)}'$ $(i \in [m_1'])$. It is immediate that $\mathbf{a}\mathbf{r}_1(\mathbf{r}_3)^n \overset{*}{\Rightarrow} \mathbf{s}_1(s_{3,\varphi(1)}, \ldots, s_{3,\varphi(m_1)})$ and $\mathbf{a}'\mathbf{r}_1(\mathbf{r}_3)^{n'} \overset{*}{\Rightarrow} \mathbf{s}_1'(s_{3,\varphi'(1)}', \ldots, s_{3,\varphi'(m_1')}')$. This, together with $(r_1(r_3))_\mathbf{B} = (r)_\mathbf{B}$ yields that $r_1(r_3) \in R$, which is a contradiction because $\text{rn}(r_1(r_3)) < \text{rn}(r)$.

Therefore, $\text{dp}(r) < \|A\|^2 |B|$. This implies that for every $\mathbf{s} \in T_G^n$ and $\mathbf{s}' \in T_G^{n'}$ if the derivations $\mathbf{a}\mathbf{r}^n \overset{*}{\Rightarrow} \mathbf{s}$ and $\mathbf{a}'\mathbf{r}^{n'} \overset{*}{\Rightarrow} \mathbf{s}'$ exist then $\text{dp}(s_1), \text{dp}(s_1') \leqq \|A\| \, |B| K$, thus, by (iv), $p_1(r) \in P$. Since r was of minimal rank this ends the proof of Lemma 4.

Lemma 5. Let $p_1, p_2, p_3 \in \hat{T}_{F,1}, p_4 \in T_F, n_i, n_i', m_i \geqq 0$ $(i = 1, 2, 3)$, $q_1 \in \hat{T}_{G,n_1+1}$, $q_1' \in \hat{T}_{G,n_1'+1}$, $r_1 \in \hat{T}_{G,m_1}$, $\mathbf{q}_2 \in \hat{T}^{n_1}_{G,n_2}$, $\mathbf{q}_2' \in \hat{T}^{n_1'}_{G,n_2'}$, $\mathbf{r}_2 \in \hat{T}^{m_1}_{G,m_2}$, $\mathbf{q}_3 \in \hat{T}^{n_2}_{G,n_3}$, $\mathbf{q}_3' \in \hat{T}^{n_2'}_{G,n_3'}$, $\mathbf{r}_3 \in \hat{T}^{m_2}_{G,m_3}$, $\mathbf{q}_4 \in T_G^{n_3}$, $\mathbf{q}_4' \in T_G^{n_3'}$, $\mathbf{r}_4 \in T_G^{m_3}$, $a_0, a_0' \in A_0$, $a \in A$, $\mathbf{a}_i \in A^{n_i}$, $\mathbf{a}_i' \in A^{n_i'}$, $\mathbf{b}_i \in A^{m_i}$ $(i = 1, 2, 3)$. Finally, let $v \in T_G$ and $v' = r_1(\mathbf{r}_2(\mathbf{r}_3(\mathbf{r}_4)))$. Denote by A_i, A_i' and B_i $(i = 1, 2, 3)$ the sets of components of $\mathbf{a}_i$, $\mathbf{a}_i'$ and $\mathbf{b}_i$, respectively. Assume that the following conditions are satisfied:

(i) $p_1(p_2(p_3(p_4)))\in T$,

(ii) $a_0 p_1 \overset{*}{\Rightarrow} q_1(ax_1, \mathbf{a}_1\mathbf{x}_1^{n_1})$, $a_0' p_1 \overset{*}{\Rightarrow} q_1'(r_1(\mathbf{b}_1\mathbf{x}_1^{m_1}), \mathbf{a}_1'\mathbf{x}_1^{n_1'})$,

(iii) $\mathbf{a}_1\mathbf{p}_2^{n_1} \overset{*}{\Rightarrow} \mathbf{q}_2(\mathbf{a}_2\mathbf{x}_1^{n_2})$, $\mathbf{a}_1'\mathbf{p}_2^{n_1'} \overset{*}{\Rightarrow} \mathbf{q}_2'(\mathbf{a}_2'\mathbf{x}_1^{n_2'})$, $ap_2 \overset{*}{\Rightarrow} ax_1$, $\mathbf{b}_1\mathbf{p}_2^{m_1} \overset{*}{\Rightarrow} \mathbf{r}_2(\mathbf{b}_2\mathbf{x}_1^{m_2})$,

(iv) $\mathbf{a}_2\mathbf{p}_3^{n_2} \overset{*}{\Rightarrow} \mathbf{q}_3(\mathbf{a}_3\mathbf{x}_1^{n_3})$, $\mathbf{a}_2'\mathbf{p}_3^{n_2'} \overset{*}{\Rightarrow} \mathbf{q}_3'(\mathbf{a}_3'\mathbf{x}_1^{n_3'})$, $ap_3 \overset{*}{\Rightarrow} ax_1$, $\mathbf{b}_2\mathbf{p}_3^{m_2} \overset{*}{\Rightarrow} \mathbf{r}_3(\mathbf{b}_3\mathbf{x}_1^{m_3})$,

(v) $\mathbf{a}_3\mathbf{p}_4^{n_3} \overset{*}{\Rightarrow} \mathbf{q}_4$, $\mathbf{a}_3'\mathbf{p}_4^{n_3'} \overset{*}{\Rightarrow} \mathbf{q}_4'$, $ap_4 \overset{*}{\Rightarrow} v$, $\mathbf{b}_3\mathbf{p}_4^{m_3} \overset{*}{\Rightarrow} \mathbf{r}_4$,

(vi) $(p_4)_{\mathbf{B}} = (p_3(p_4))_{\mathbf{B}} = (p_2(p_3(p_4)))_{\mathbf{B}}$,
$A_1 \subseteq A_2 \subseteq A_3$, $A_1' \subseteq A_2' \subseteq A_3'$, $B_1 = B_2 \subseteq B_3$,

(vii) $v \neq v'$, $\text{path}_1(q_1) = \text{path}_1(q_1')$.

Then at least one of the trees $p_1(p_2(p_4))$, $p_1(p_3(p_4))$ and $p_1(p_4)$ is in P.

Proof. First observe that by the assumptions of the lemma it follows that $p_1(p_2(p_3(p_4)))\in P$.

Let φ_i: $[n_i]\to[n_{i+1}]$, φ_i': $[n_i']\to[n_{i+1}']$ and ψ_i: $[m_i]\to[m_{i+1}]$ $(i=1, 2)$ be mappings such that we have $a_{i,j}=a_{i+1,\varphi_i(j)}$ $(i=1, 2, j\in[n_i])$, $a_{i,j}'=a_{i+1,\varphi_i'(j)}'$ $(i=1, 2, j\in[n_i'])$, $b_{i,j}=b_{i+1,\psi_i(j)}$ $(i=1, 2, j\in[m_i])$. Furthermore, let $\varphi_3=\varphi_1\circ\varphi_2$, $\varphi_3'=\varphi_1'\circ\varphi_2'$, $\psi_3=\psi_1\circ\psi_2$.

Let us introduce the following notations:

$$\mathbf{s}_1 = (q_{3,\varphi_1(1)}, \ldots, q_{3,\varphi_1(n_1)})(\mathbf{q}_4),$$
$$\mathbf{s}_1' = (q_{3,\varphi_1'(1)}', \ldots, q_{3,\varphi_1'(n_1')}')(\mathbf{q}_4'),$$
$$\mathbf{t}_1 = (r_{3,\psi_1(1)}, \ldots, r_{3,\psi_1(m_1)})(\mathbf{r}_4),$$
$$\mathbf{s}_2 = \mathbf{q}_2(q_{4,\varphi_2(1)}, \ldots, q_{4,\varphi_2(n_2)}),$$
$$\mathbf{s}_2' = \mathbf{q}_2'(q_{4,\varphi_2'(1)}', \ldots, q_{4,\varphi_2'(n_2')}'),$$
$$\mathbf{t}_2 = \mathbf{r}_2(r_{4,\psi_2(1)}, \ldots, r_{4,\psi_2(m_2)}),$$
$$\mathbf{s}_3 = (q_{4,\varphi_3(1)}, \ldots, q_{4,\varphi_3(n_1)}),$$
$$\mathbf{s}_3' = (q_{4,\varphi_3'(1)}', \ldots, q_{4,\varphi_3'(n_1')}'),$$
$$\mathbf{t}_3 = (r_{4,\psi_3(1)}, \ldots, r_{4,\psi_3(m_1)}).$$

It is easy to check that each of the following derivations is valid: $a_0p_1(p_3(p_4))\overset{*}{\Rightarrow} q_1(v, \mathbf{s}_1)$, $a_0'p_1(p_3(p_4))\overset{*}{\Rightarrow} q_1'(r_1(\mathbf{t}_1), \mathbf{s}_1')$, $a_0p_1(p_2(p_4))\overset{*}{\Rightarrow} q_1(v, \mathbf{s}_2)$, $a_0'p_1(p_2(p_4))\overset{*}{\Rightarrow} q_1'(r_1(\mathbf{t}_2), \mathbf{s}_2')$, $a_0p_1(p_4)\overset{*}{\Rightarrow} q_1(v, \mathbf{s}_3)$, $a_0'p_1(p_4)\overset{*}{\Rightarrow} q_1'(r_1(\mathbf{t}_3), \mathbf{s}_3')$. On the other hand $p_1(p_3(p_4)), p_1(p_2(p_4)), p_1(p_4)\in T$.

Assume that $p_1(p_2(p_4))\notin P$. Then, by (vii), it follows that $m_1, m_2, m_3>0$ and there is an integer $i\in[m_2]$ with $r_{3,i}(\mathbf{r}_4)\neq r_{4,\psi_2(i)}$. Without loss of generality we may assume that this integer i is in the range of ψ_1, i.e. there exist $j\in[m_1]$ satisfying

$\psi_1(j)=i$. Now suppose that neither $p_1(p_3(p_4))$ nor $p_1(p_4)$ is in P. Then $r_1(\mathbf{t}_1)=$ $=r_1(\mathbf{t}_2)=r_1(\mathbf{t}_3)$ $(=v)$. But this is impossible because $t_{1,j}\neq t_{3,j}$.

Note that Lemma 5 remains valid even if $A_2'\subseteqq A_3'$ and $B_2\subseteqq B_3$ are replaced by $A_2'\cup B_2\subseteqq A_3'\cup B_3$.

The proof of the next lemma is similar to the previous one.

Lemma 6. Let $p_1, p_2, p_3\in\hat{T}_{F,1}$, $p_4\in T_F$, $n_i, n_i', m_i\geqq 0$ $(i=1, 2, 3)$, $q_1\in\hat{T}_{G,n_1+1}$, $q_1'\in\hat{T}_{G,n_1'+1}$, $r_1\in\hat{T}_{G,m_1}$, $\mathbf{q}_2\in\hat{T}^{n_1}_{G,n_2}$, $\mathbf{q}_2'\in\hat{T}^{n_1'}_{G,n_2'}$, $\mathbf{r}_2\in\hat{T}^{m_1}_{G,m_2}$, $\mathbf{q}_3\in\hat{T}^{n_2}_{G,n_3}$, $\mathbf{q}_3'\in\hat{T}^{n_2'}_{G,n_3'}$, $\mathbf{r}_3\in\hat{T}^{m_2}_{G,m_3}$, $\mathbf{q}_4\in T_G^{n_3}$, $\mathbf{q}_4'\in T_G^{n_3'}$, $\mathbf{r}_4\in T_G^{m_3}$, $a_0, a_0'\in A_0$, $\mathbf{a}_i\in A^{n_i}$, $\mathbf{a}_i'\in A^{n_i'}$, $\mathbf{b}_i\in A^{m_i}$ $(i=1, 2, 3)$. Furthermore, let $v'\in T_G$ and $v=r_1(\mathbf{r}_2(\mathbf{r}_3(\mathbf{r}_4)))$. Denote by A_i, A_i' and B_i $(i=1, 2, 3)$ the sets of components of $\mathbf{a}_i$, $\mathbf{a}_i'$ and $\mathbf{b}_i$, respectively. Assume that

(i) $p_1(p_2(p_3(p_4)))\in T$,

(ii) $a_0 p_1 \overset{*}{\Rightarrow} q_1(r_1(\mathbf{b}_1\mathbf{x}_1^{m_1}),\ \mathbf{a}_1\mathbf{x}_1^{n_1})$, $\quad a_0' p_1 \overset{*}{\Rightarrow} q_1'(v',\ \mathbf{a}_1'\mathbf{x}_1^{n_1'})$,

(iii) $\mathbf{a}_1\mathbf{p}_2^{n_1} \overset{*}{\Rightarrow} \mathbf{q}_2(\mathbf{a}_2\mathbf{x}_1^{n_2})$; $\quad \mathbf{a}_1'\mathbf{p}_2^{n_1'} \overset{*}{\Rightarrow} \mathbf{q}_2'(\mathbf{a}_2'\mathbf{x}_1^{n_2'})$, $\quad \mathbf{b}_1\mathbf{p}_2^{m_1} \overset{*}{\Rightarrow} \mathbf{r}_2(\mathbf{b}_2\mathbf{x}_1^{m_2})$,

(iv) $\mathbf{a}_2\mathbf{p}_3^{n_2} \overset{*}{\Rightarrow} \mathbf{q}_3(\mathbf{a}_3\mathbf{x}_1^{n_3})$, $\quad \mathbf{a}_2'\mathbf{p}_3^{n_2'} \overset{*}{\Rightarrow} \mathbf{q}_3'(\mathbf{a}_3'\mathbf{x}_1^{n_3'})$, $\quad \mathbf{b}_2\mathbf{p}_3^{m_2} \overset{*}{\Rightarrow} \mathbf{r}_3(\mathbf{b}_3\mathbf{x}_1^{m_3})$,

(v) $\mathbf{a}_3\mathbf{p}_4^{n_3} \overset{*}{\Rightarrow} \mathbf{q}_4$, $\quad \mathbf{a}_3'\mathbf{p}_4^{n_3'} \overset{*}{\Rightarrow} \mathbf{q}_4'$, $\quad \mathbf{b}_3\mathbf{p}_4^{m_3} \overset{*}{\Rightarrow} \mathbf{r}_4$,

(vi) $(p_4)_{\mathbf{B}} = (p_3(p_4))_{\mathbf{B}} = (p_2(p_3(p_4)))_{\mathbf{B}}$,

$A_1 \subseteqq A_2 \subseteqq A_3$, $\quad A_1' \subseteqq A_2' \subseteqq A_3'$, $\quad B_1 = B_2 \subseteqq B_3$,

(vii) $v \neq v'$, $\quad \mathrm{path}_1(q_1) = \mathrm{path}_1(q_1')$.

Then at least one of the trees $p_1(p_2(p_4)), p_1(p_3(p_4)), p_1(p_4)$ is in P.

Our last lemma is stated as follows:

Lemma 7. Let $p_1, p_2\in\hat{T}_{F,1}$, $p_3\in T_F$, $k, l, m, k', l', m'\geqq 0$, $q_1\in\hat{T}_{G,k+1}$, $q_1'\in\hat{T}_{G,k'+1}$, $q_2\in\hat{T}_{G,l+1}$, $q_2'\in\hat{T}_{G,l'+1}$, $\mathbf{r}\in\hat{T}^k_{G,m}$, $\mathbf{r}'\in\hat{T}^{k'}_{G,m'}$, $q_3\in\hat{T}_{G,1}$, $q_3', v\in T_G$, $\mathbf{s}\in T_G^l$, $\mathbf{s}'\in T_G^{l'}$, $\mathbf{t}\in T_G^m$, $\mathbf{t}'\in T_G^{m'}$, $a_0, a_0'\in A_0$, $a, a'\in A$, $\mathbf{a}\in A^k$, $\mathbf{a}'\in A^{k'}$, $\mathbf{b}\in A^l$, $\mathbf{b}'\in A^{l'}$, $\mathbf{c}\in A^m$, $\mathbf{c}'\in A^{m'}$. Let A_1, B_1 and C_1 denote the sets of all components of **a**, **b** and **c**, respectively. Similarly, denote by A_1', B_1' and C_1' the sets of components of $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$. Suppose that the following conditions are satisfied:

(i) $p_1(p_2(p_3))\in T$,

(ii) $a_0 p_1 \overset{*}{\Rightarrow} q_1(ax_1, \mathbf{a}\mathbf{x}_1^k)$, $\quad a_0' p_1 \overset{*}{\Rightarrow} q_1'(a'x_1, \mathbf{a}'\mathbf{x}_1^{k'})$,

(iii) $ap_2 \overset{*}{\Rightarrow} q_2(ax_1, \mathbf{b}\mathbf{x}_1^l)$, $\quad a'p_2 \overset{*}{\Rightarrow} q_2'(a'x_1, \mathbf{b}'\mathbf{x}_1^{l'})$,

$\mathbf{a}\mathbf{p}_2^k \overset{*}{\Rightarrow} \mathbf{r}(\mathbf{c}\mathbf{x}_1^m)$, $\quad \mathbf{a}'\mathbf{p}_2^{k'} \overset{*}{\Rightarrow} \mathbf{r}'(\mathbf{c}'\mathbf{x}_1^{m'})$,

(iv) $ap_3 \overset{*}{\Rightarrow} q_3(v)$, $\quad a'p_3 \overset{*}{\Rightarrow} q_3'$, $\quad \mathbf{b}\mathbf{p}_3^l \overset{*}{\Rightarrow} \mathbf{s}$, $\quad \mathbf{b}'\mathbf{p}_3^{l'} \overset{*}{\Rightarrow} \mathbf{s}'$, $\quad \mathbf{c}\mathbf{p}_3^m \overset{*}{\Rightarrow} \mathbf{t}$; $\quad \mathbf{c}'\mathbf{p}_3^{m'} \overset{*}{\Rightarrow} t'$,

(v) $A_1 \subseteqq B_1\cup C_1$, $\quad A_1'\subseteqq B_1'\cup C_1'$, $\quad (p_3)_{\mathbf{B}} = (p_2(p_3))_{\mathbf{B}}$,

(vi) $\mathrm{path}_1(q_1') = \mathrm{path}_1(q_1)\,\mathrm{path}(q_3)$, $\mathrm{path}_1(q_2)\,\mathrm{path}(q_3) =$
$= \mathrm{path}(q_3)\,\mathrm{path}_1(q_2')$, $v \neq q_3'$.

Then $p_1(p_3) \in P$.

Proof. Let us introduce the following notations: $\mathbf{d}=(\mathbf{b}, \mathbf{c})$, $\mathbf{d}'=(\mathbf{b}', \mathbf{c}')$, $\mathbf{u}=(\mathbf{s}, \mathbf{t})$, $\mathbf{u}'=(\mathbf{s}', \mathbf{t}')$. Choose the mappings $\varphi\colon [k] \to [l+m]$ and $\varphi'\colon [k'] \to [l'+m']$ in such a way that we have $a_i = d_{\varphi(i)}$ and $a_j' = d_{\varphi'(j)}$ for every $i \in [k]$ and $j \in [k']$. Obviously, $a_0 p_1(p_3) \overset{*}{\Rightarrow} q_1(q_3(v), u_{\varphi(1)}, \ldots, u_{\varphi(k)})$ and $a_0' p_1(p_3) \overset{*}{\Rightarrow} q_1'(q_3', u'_{\varphi'(1)}, \ldots, u'_{\varphi'(k')})$, furthermore, $p_1(p_3) \in T$. On the other hand $\mathrm{path}_1(q_1(q_3, u_{\varphi(1)}, \ldots, u_{\varphi(k)})) = \mathrm{path}_1(q_1'(x_1, u'_{\varphi(1)}, \ldots, u'_{\varphi'(k')}))$ and $q_3' \neq v$. Therefore, $q_1(q_3(v), u_{\varphi(1)}, \ldots, u_{\varphi(k)}) \neq q_1'(q_3', u'_{\varphi'(1)}, \ldots, u'_{\varphi'(k')})$, showing that $p_1(p_3) \in P$.

We are now able to prove our main result:

Theorem 8. The functionality of top-down as well as bottom-up tree transducers is decidable.

Proof. By Lemma 2 it suffices to prove our statement for regularly restricted top-down transducers. Hence take an arbitrary regularly restricted top-down transducer $\mathbf{A}=(F, T, A, G, A_0, \Sigma)$ with $T=T(\mathbf{B})$, where $\mathbf{B}$ is the tree automaton $\mathbf{B}=(F, B, B_0)$. Define the set P and integers $|A|$, $\|A\|$, $|B|$ and K as previously (cf. Lemma 4) and let L denote the number of nonempty strings over $[\nu(G)]$ with length not exceeding $\|A\|^2|B|K$. Furthermore, let $k=\|A\|^2|A|^2|B|(2L+1)$, $l=k+2\|A\|^3|A||B|(\|A\|^2|B|K+1)$ and finally, $m=l+2\|A\|^3|B|$.

We shall show that P is nonvoid if and only if it contains a tree of depth less than m. It is obvious if $K=0$. Therefore let $K \neq 0$ and assume that p is an element of P with minimal rank. Let q and q' be different images of p under $\tau_{\mathbf{A}}$.

Assume to the contrary $\mathrm{dp}(p) \geqq m$. Then there exist $a_0, a_0' \in A_0$, $p_0, \ldots, p_m \in \hat{T}_{F,1}$, $p_{m+1} \in T_F$, $n_i, n_i' \geqq 0$ $(i=0, \ldots, m)$, $q_0 \in \hat{T}_{G,n_0}$, $q_0' \in \hat{T}_{G,n_0'}$, $\mathbf{q}_i \in \hat{T}_{G,n_i}^{n_{i-1}}$, $\mathbf{q}_i' \in \hat{T}_{G,n_i'}^{n_{i-1}'}$ $(i=1, \ldots, m)$, $\mathbf{q}_{m+1} \in T_G^{n_m}$, $\mathbf{q}_{m+1}' \in T_G^{n_m'}$, $\mathbf{a}_i \in A^{n_i}$, $\mathbf{a}_i' \in A^{n_i'}$ $(i=0, \ldots, m)$ such that the following three conditions are satisfied:

(1) $p = p_0(p_1(\ldots(p_{m+1})\ldots))$, $p_i \neq x_1$ $(i = 1, \ldots, m)$,

(2) $q = q_0(\mathbf{q}_1(\ldots(\mathbf{q}_{m+1})\ldots))$, $q' = q_0'(\mathbf{q}_1'(\ldots(\mathbf{q}_{m+1}')\ldots))$,

(3) $a_0 p_0 \overset{*}{\Rightarrow} q_0(\mathbf{a}_0 \mathbf{x}_1^{n_0})$, $a_0' p_0 \overset{*}{\Rightarrow} q_0'(\mathbf{a}_0' \mathbf{x}_1^{n_0'})$,
$\mathbf{a}_i \mathbf{p}_{i+1}^{n_i} \overset{*}{\Rightarrow} \mathbf{q}_{i+1}(\mathbf{a}_{i+1} \mathbf{x}_1^{n_{i+1}})$, $\mathbf{a}_i' \mathbf{p}_{i+1}^{n_i'} \overset{*}{\Rightarrow} \mathbf{q}_{i+1}'(\mathbf{a}_{i+1}' \mathbf{x}_1^{n_{i+1}'})$ $(i = 0, \ldots, m-1)$,
$\mathbf{a}_m \mathbf{p}_{m+1}^{n_m} \overset{*}{\Rightarrow} \mathbf{q}_{m+1}$, $\mathbf{a}_m' \mathbf{p}_{m+1}^{n_m'} \overset{*}{\Rightarrow} \mathbf{q}_{m+1}'$.

Further on we shall often use the following notations. Let $i \in \{0, \ldots, m+1\}$, $j \in \{0, \ldots, m\}$. Then $\check{p}_i = p_0(p_1(\ldots(p_i)\ldots))$, $\check{q}_i = q_0(\mathbf{q}_1(\ldots(\mathbf{q}_i)\ldots))$, $\check{q}_i' = q_0'(\mathbf{q}_1'(\ldots(\mathbf{q}_i')\ldots))$. Similarly, $\hat{p}_j = \mathbf{p}_{j+1}(\ldots(\mathbf{p}_{m+1})\ldots)$ $\hat{\mathbf{q}}_j = \mathbf{q}_{j+1}(\ldots(\mathbf{q}_{m+1})\ldots)$, $\hat{\mathbf{q}}_j' = \mathbf{q}_{j+1}'(\ldots(\mathbf{q}_{m+1}')\ldots)$. Furthermore, for each $i=0, \ldots, m$, A_i and A_i' denotes the set of all components of $\mathbf{a}_i$ and $\mathbf{a}_i'$, respectively.

If for any $\mathbf{v} \in T_G^{n_l}$ and $\mathbf{v}' \in T_G^{n_l'}$ we have $\check{q}_l(\mathbf{v}) \neq \check{q}_l'(\mathbf{v}')$ then, by Lemma 3 and the fact that the cardinality of the set $\{l, \ldots, m\}$ is at least $\|A\|^2|B|+1$, we get that for some i, j $(l \leqq i < j \leqq m)$ $\check{p}_i(\hat{p}_j) \in P$. It is a contradiction.

Therefore we may assume that $n_l>0$ and the existence of an index $i_l\in[n_l]$ such that there are trees $u'\in\hat{T}_{G,1}, v'\in T_G$ with $q'=u'(v')$, $\mathrm{path}(u')=\mathrm{path}_{i_l}(\check{q}_l)$ and $v'\neq\hat{q}_{l,i_l}$. Obviously, $n_i>0$ holds for each $i<l$. Now let i_j $(0\leqq i<l, j\in[n_i])$ be those uniquely determined idices for which $\mathrm{path}_{i_j}(\check{q}_i)$ is a prefix of $\mathrm{path}_{i_l}(\check{q}_l)$. Of course we may assume that $i_0=\ldots=i_l=1$.

Suppose now that there is no $\alpha'\in\mathrm{path}(\check{q}_l')$ such that $\mathrm{path}_1(\check{q}_l)$ is a prefix of α' or conversely. In this case let

$B_i=\{a_{i,j}\mid \mathrm{path}_1(\check{q}_l)$ is a prefix of $\mathrm{path}_j(\check{q}_i)\}$,

$C_i=\{a_{i,j}\mid \mathrm{path}_1(\check{q}_l)$ is not a prefix of $\mathrm{path}_j(\check{q}_i)\}$

for each i $(l\leqq i\leqq m)$. Since the cardinality of the set $\{l,\ldots,m\}$ is exactly $2\|A\|^3|B|+1$ there exist indices i_1,i_2,i_3 $(l\leqq i_1<i_2<i_3\leqq m)$ satisfying the following conditions:

$$(\hat{p}_{i_1})_{\mathbf{B}}=(\hat{p}_{i_2})_{\mathbf{B}}=(\hat{p}_{i_3})_{\mathbf{B}},\quad B_{i_1}=B_{i_2}\subseteqq B_{i_3},\quad C_{i_1}\subseteqq C_{i_2}\subseteqq C_{i_3},\quad A'_{i_1}\subseteqq A'_{i_2}\subseteqq A'_{i_3}.$$

By Lemma 6 this yields that at least one of the trees $\check{p}_{i_1}(\hat{p}_{i_2}), \check{p}_{i_2}(\hat{p}_{i_3}), \check{p}_{i_1}(\hat{p}_{i_3})$ is in P, which is a contradiction.

We have shown that there exists an $\alpha'\in\mathrm{path}(\check{q}_l')$ such that $\mathrm{path}_1(\check{q}_l)$ is a prefix of α' or conversely. Consequently $n_i'>0$ holds for each i $(0\leqq i\leqq l)$ and there exist integers $i_0,\ldots,i_l$ with the property that $\mathrm{path}_{i_j}(\check{q}_j')$ is a prefix of $\mathrm{path}_1(\check{q}_l)$ or conversely $(j=0,\ldots,l)$. We may also assume that if $j_1<j_2$ then $\mathrm{path}_{i_{j_1}}(\check{q}_{j_1}')$ is a prefix of $\mathrm{path}_{i_{j_2}}(\check{q}_{j_2}')$, moreover, we may assume that $i_0=\ldots=i_l=1$. In this way either $\mathrm{path}_1(\check{q}_j)$ is a prefix of $\mathrm{path}_1(\check{q}_j')$ $(j=0,\ldots,l)$ or conversely.

Now there are two cases. First suppose that $\mathrm{path}_1(\check{q}_k')$ is a prefix of $\mathrm{path}_1(\check{q}_l)$. If, within this case, there exists an integer i $(0\leqq i\leqq k)$ such that $\|\mathrm{path}_1(\check{q}_i)|-|\mathrm{path}_1(\check{q}_i')\|>\|A\|^2|B|K$ then, by Lemma 4, there is a tree $r\in T_F$ satisfying both $\check{p}_i(r)\in P$ and $\mathrm{rn}(r)<\mathrm{rn}(\hat{p}_i)$. This is a contradiction because $\mathrm{rn}(r)<\mathrm{rn}(\hat{p}_i)$ implies $\mathrm{rn}(\check{p}_i(r))<\mathrm{rn}(p)$. Thus we have $\|\mathrm{path}_1(\check{q}_i)|-|\mathrm{path}_1(\check{q}_i')\|\leqq\|A\|^2|B|K$ for every i $(0\leqq i\leqq k)$. But this yields another contradiction. Indeed, the cardinality of the set $\{0,\ldots,k\}$ is equal to $\|A\|^2|A|^2|B|(2L+1)+1$, thus, there are at least two indices i,j $(0\leqq i<j\leqq k)$ such that — say — $\mathrm{path}_1(\check{q}_i)$ is a prefix of $\mathrm{path}_1(\check{q}_i')$, $\mathrm{path}_1(\check{q}_j)$ is a prefix of $\mathrm{path}_1(\check{q}_j')$, $\mathrm{path}_1(\check{q}_i')/\mathrm{path}_1(\check{q}_i)=\mathrm{path}_1(\check{q}_j')/\mathrm{path}_1(\check{q}_j)$, moreover, $(\hat{p}_i)_{\mathbf{B}}=(\hat{p}_j)_{\mathbf{B}}$, $a_{i,1}=a_{j,1}$, $a'_{i,1}=a'_{j,1}$, $B_i\subseteqq B_j$, $B_i'\subseteqq B_j'$ where $B_s=\{a_{s,t}\mid 2\leqq t\leqq n_s\}$, $B_s'=\{a'_{s,t}\mid 2\leqq t\leqq n_s'\}$ $(s=i,j)$. By an application of Lemma 7 this results that $\check{p}_i(\hat{p}_j)\in P$ — contrary to the minimality of p.

We have shown that $\mathrm{path}_1(\check{q}_k')$ can not be a prefix of $\mathrm{path}_1(\check{q}_l)$. Therefore $\mathrm{path}_1(\check{q}_l)$ is a prefix of $\mathrm{path}_1(\check{q}_k')$. If we prove that $|\mathrm{path}_1(\check{q}_l)|-|\mathrm{path}_1(\check{q}_k)|>\|A\|^2|B|K$ then also $|\mathrm{path}_1(\check{q}_k')|-|\mathrm{path}_1(\check{q}_k)|>\|A\|^2|B|K$. Again by Lemma 4, this yields a contradiction. Therefore it is enough to show that $|\mathrm{path}_1(\check{q}_l)|-|\mathrm{path}_1(\check{q}_k)|>\|A\|^2|B|K$.

Assume that this condition does not hold. The cardinality of the set $\{k+1,\ldots,l\}$ is exactly $2\|A\|^3|A||B|(\|A\|^2|B|K+1)$, therefore, there exist indices i_1,i_2 $(k\leqq i_1<i_2\leqq l)$ such that $i_2-i_1=2\|A\|^3|A||B|$ and $\mathrm{path}_1(\check{q}_{i_1})=\ldots=\mathrm{path}_1(\check{q}_{i_2})$, i.e. $q_{i_1+1,1}=\ldots=q_{i_2,1}=x_1$. Now let

$B_j=\{a'_{j,t}\mid 1\leqq t\leqq n_j'$, $\mathrm{path}_1(\check{q}_{i_1})$ is a prefix of $\mathrm{path}_t(\check{q}_j')\}$,

$C_j=\{a'_{j,t}\mid 1\leqq t\leqq n_j'$, $\mathrm{path}_1(\check{q}_{i_1})$ is not a prefix of $\mathrm{path}_t(\check{q}_j')\}$

for each $j(i_1\leqq j\leqq i_2)$. Since the cardinality of $\{i_1,\ldots,i_2\}$ is equal to $2\|A\|^3|A||B|+1$ there exist indices j_1,j_2,j_3 $(i_1\leqq j_1<j_2<j_3\leqq i_2)$ such that each of the following

conditions is satisfied: $(\hat{p}_{j_1})_B=(\hat{p}_{j_2})_B=(\hat{p}_{j_3})_B$, $\bar{A}_{j_1}\subseteqq\bar{A}_{j_2}\subseteqq\bar{A}_{j_3}$, $B_{j_1}=B_{j_2}\subseteqq B_{j_3}$, $C_{j_1}\subseteqq\subseteqq C_{j_2}\subseteqq C_{j_3}$, $a_{j_1,1}=a_{j_2,1}=a_{j_3,1}$, where $\bar{A}_{j_t}=\{a_{j_t,s}\mid 2\leqq s\leqq n_t\}$. Thus, applying Lemma 5, we get that one of the trees $\check{p}_{j_1}(\hat{p}_{j_2}), \check{p}_{j_2}(\hat{p}_{j_3}), \check{p}_{j_1}(\hat{p}_{j_3})$ is in P, contradicting to the minimality of p. This ends the proof of Theorem 8.

Observe that, by the decomposition result for top-down tree transducers with regular look-ahead in [6], the above theorem holds for this type of transducers as well. But Theorem 8 has some other important consequences, too.

Take two arbitrary top-down or bottom-up tree transducers $\mathbf{A}=(F, A, G, A_0, \Sigma)$ and $\mathbf{B}=(F, B, G, B_0, \Sigma')$. Assume that $\mathbf{A}$ is functional and A and B are disjoint. Then construct the sum of $\mathbf{A}$ and $\mathbf{B}$, i.e. take $\mathbf{C}=(F, A\cup B, G, A_0\cup B_0, \Sigma\cup\Sigma')$. For $\mathbf{C}$ we have the following equivalence: $\tau_{\mathbf{A}}=\tau_{\mathbf{B}}$ if and only if $\operatorname{dom}\tau_{\mathbf{A}}=\operatorname{dom}\tau_{\mathbf{B}}$ and $\mathbf{C}$ is functional. From this and by the fact that the equality of regular forests is decidable we get:

Theorem 9. There exists an algorithm to decide for an arbitrary tree transducer $\mathbf{A}$ and a functional transducer $\mathbf{B}$ whether they are equivalent, i.e. such that $\tau_{\mathbf{A}}=\tau_{\mathbf{B}}$.

COROLLARY. A similar argument shows that Theorem 9 holds even if $\tau_{\mathbf{A}}=\tau_{\mathbf{B}}$ is replaced by $\tau_{\mathbf{A}}\subseteqq\tau_{\mathbf{B}}$. On the other hand every deterministic transducer is functional. Thus, the equivalence problem for deterministic transducers is decidable.

Another consequence of Theorem 8 concerns with minimization of transducers. For any given tree transducer $\mathbf{A}$ one can compute a bound k with the following property: $\mathbf{A}$ has a corresponding tree transducer $\mathbf{B}$ which is minimal and satisfies that each tree in the right hand side of a rule of $\mathbf{B}$ has depth not exceeding k. This k can be obtained as $2K\|A\|$ in the top-down case and as $2K|A|$ in the bottom-up case. (Here $|A|$, $\|A\|$ and K are determined as in the proof of Theorem 8.) Therefore, if we assume that $\mathbf{A}$ is functional and we want to minimize $\mathbf{A}$, it is enough to check only for a finite number of transducers whether they are equivalent to $\mathbf{A}$ or not. This proves

Theorem 10. The minimization of functional tree transducers is effectively solvable.

COROLLARY. As every deterministic tree transducer is functional the same statement holds for deterministic transducers.

This corollary as well as the positive decidability result concerning the equivalence problem for deterministic bottom-up transducers and a restricted class of deterministic top-down transducers was independently achieved by Z. ZACHAR in [12] too.

3. Minimization of deterministic transducers

Let $\mathscr{K}$ be a class of tree transducers. A transducer $\mathbf{A}\in\mathscr{K}$ is said to be *minimal* in $\mathscr{K}$ if there is no transducer $\mathbf{B}\in\mathscr{K}$ which is equivalent to $\mathbf{A}$ and has fewer states than $\mathbf{A}$. In the preceding section we have shown that if $\mathscr{K}$ is the class of all functional top-down or all bottom-up transducers, or if $\mathscr{K}$ is the class of all deterministic top-down or all bottom-up transducers, then, for every given $\mathbf{A}\in\mathscr{K}$, one can effectively find a minimal equivalent transducer $\mathbf{B}\in\mathscr{K}$. However, these minimal realizations are not uniquely determined. In this section we investigate conditions assuring the uniqueness (up to isomorphism) of minimal realizations. Similar results are already known for Mealy-type automata (cf. [9]) and tree automata [1, 3, 10]. We point out that the minimizing process of Mealy-type automata can be generalized in a natural way for certain classes of deterministic tree transducers. For the sake of simplicity we shall consider *completely defined* deterministic tree transducers only. Therefore, from now on, by a tree transducer we shall always mean a completely defined deterministic transducer. Furthermore, all transducers will be taken with a fixed input type F and output type G. Since the case $F=F_0$ is trivial we assume that $F\neq F_0$.

First we treat top-down transducers. Let $\mathbf{A}=(F, A, G, \{a_0\}, \Sigma)$ be a top-down transducer. It is *completely defined,* i.e. for any $a\in A$ and $f\in F$ there is a rule in Σ with left side af. Let $\mathbf{B}=(F, B, G, \{b_0\}, \Sigma')$ be another top-down transducer and take a mapping $\varphi\colon A\to B$. If the following two conditions are satisfied for arbitrary $n, m\geqq 0$, $f\in F_n$, $p\in T_{G,m}$, $a, a_1, \ldots, a_m\in A$ and $i_1, \ldots, i_m\in[n]$ then φ is called a *homomorphism* of $\mathbf{A}$ into $\mathbf{B}$:

(i) if $af\to p(a_1x_{i_1}, \ldots, a_mx_{i_m})\in\Sigma$ then $bf\to p(b_1x_{i_1}, \ldots, b_mx_{i_m})\in\Sigma'$ where $b=\varphi(a)$, $b_j=\varphi(a_j)$ ($j\in[m]$),

(ii) $\varphi(a_0)=b_0$.

If, moreover, φ is surjective then $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$. If φ is bijective then we speak about *isomorphism,* written $\mathbf{A}\cong\mathbf{B}$. If $B\subseteq A$ and φ is the natural embedding of B into A then $\mathbf{B}$ is a *subtransducer* of $\mathbf{A}$. If $\mathbf{A}$ has not proper subtransducers then it is called *connected.*

The next statement is obvious:

Statement 11. If there is a homomorphism from $\mathbf{A}$ into $\mathbf{B}$ then $\tau_{\mathbf{A}}=\tau_{\mathbf{B}}$.

As in case of universal algebras there is a bijective correspondence between homomorphic images and congruence relations. Let $\mathbf{A}=(F, A, G, \{a_0\}, \Sigma)$ be an arbitrary top-down transducer and take an equivalence relation θ on $\mathbf{A}$. It is called a *congruence relation* if for any two rules $af\to p(a_1x_{i_1}, \ldots, a_mx_{i_m})$, $bf\to q(b_1x_{j_1}, \ldots, b_lx_{j_l})\in\Sigma$ ($n, m, l\geqq 0$, $f\in F_n$, $p\in\tilde{T}_{G,m}$, $q\in\tilde{T}_{G,l}$, $i_1, \ldots, i_m$, $j_1, \ldots, j_l\in[n]$, $a_1, \ldots, a_m, b_1, \ldots, b_l, a, b\in A$) $a\theta b$ implies $m=l$, $p=q$, $i_t=j_t$ and $a_t\theta b_t$ ($t=1, \ldots, m$). Here for any nonnegative integer n the notation $\tilde{T}_{G,n}$ is used to denote the set $\tilde{T}_{G,n}=\{p\in T_{G,n}\mid \mathrm{fr}\,(p)=x_1\ldots x_n\}$.

Assume that θ is a congruence relation of $\mathbf{A}$. Then we can define the *quotient of* $\mathbf{A}$ induced by θ. This is the top-down transducer $\mathbf{A}/\theta=(F, A/\theta, G, \{\theta(a_0)\}, \Sigma')$ where for every $n, m\geqq 0$, $f\in F_n$, $p\in T_{G,m}$, $a, a_1, \ldots, a_m\in A$

$$\theta(a)f\to p(\theta(a_1)x_{i_1}, \ldots, \theta(a_m)x_{i_m})\in\Sigma'$$

if and only if

$$af\to p(a_1x_{i_1}, \ldots, a_mx_{i_m})\in\Sigma.$$

Statement 12. $\mathbf{A}/\theta$ is a homomorphic image of A. If $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$ then there is a congruence relation θ of $\mathbf{A}$ such that $\mathbf{A}/\theta\cong\mathbf{B}$.

Take again the top-down tree transducer $\mathbf{A}=(F, A, G, \{a_0\}, \Sigma)$. Let us define an equivalence relation $\theta_{\mathbf{A}}$ on A: $a\theta_{\mathbf{A}}b$ if and only if $\tau_{\mathbf{A}(a)}=\tau_{\mathbf{A}(b)}$. Unfortunately, this will not always be a congruence relation. We need certain additional requirements on $\mathbf{A}$.

Let ϱ be any mapping of the set of nonnegative integers into itself, i.e. $\varrho: \omega \to \omega$. Then let $\mathscr{K}(\varrho)$ denote the class of all top-down tree transducers $\mathbf{A}=(F, A, G, \{a_0\}, \Sigma)$ which satisfy the condition $|\text{path}_j(p)|=\varrho(i_j)$ for every $n, m \geqq 0$, $f \in F_n$, $p \in \tilde{T}_{G,m}$, $a, a_1, \ldots, a_m \in A$, $x_{i_1}, \ldots, x_{i_m} \in X_n$, $j \in [m]$ and $af \to p(a_1 x_{i_1}, \ldots, a_m x_{i_m}) \in \Sigma$, as well as the condition $|\tau_{\mathbf{A}(a)}(T_F)|>1$ for arbitrary state a appearing in the right side of a rule in Σ.

Statement 13. If $\mathbf{A} \in \mathscr{K}(\varrho)$ then $\theta_{\mathbf{A}}$ is a congruence relation.

Proof. Let $\mathbf{A}=(F, A, G, \{a_0\}, \Sigma)$ and assume that $af \to p(a_1 x_{i_1}, \ldots, a_m x_{i_m})$ and $bf \to q(b_1 x_{j_1}, \ldots, b_l x_{j_l})$ are rules in Σ where $a, b \in A$, $a \theta_{\mathbf{A}} b$, $n, m, l \geqq 0$, $f \in F_n$, $p \in \tilde{T}_{G,m}$, $q \in \tilde{T}_{G,l}$, $a_1, \ldots, a_m, b_1, \ldots, b_l \in A$, $i_1, \ldots, i_m$, $j_1, \ldots, j_l \in [n]$. Assume that there is an integer $t \in [m]$ such that none of the strings in $\bigcup (\text{path}_s(q) | i_t = j_s, s \in [l])$ is a prefix of $\text{path}_t(p)$ or conversely. Then, by $|\tau_{\mathbf{A}(a_t)}(T_F)|>1$, it is easy to show the existence of a tree $r \in T_F$ with $\tau_{\mathbf{A}(a)}(r) \neq \tau_{\mathbf{A}(b)}(r)$. On the other hand if $i_t = j_s$ holds for some $t \in [m]$ and $s \in [l]$ then the equality $|\text{path}_t(p)|=|\text{path}_s(q)|$ is also valid. This proves that $m=l$, $i_t=j_t$, $\text{path}_t(p)=\text{path}_t(q)$ $(t=1, \ldots, m)$. But $\tau_{\mathbf{A}(a)}=\tau_{\mathbf{B}(b)}$, hence from this we get $p=q$, $a_t \theta_{\mathbf{A}} b_t$ $(t=1, \ldots, m)$.

Another class of top-down transducers in which $\theta_{\mathbf{A}}$ is always a congruence relation is the class $\mathscr{K}_d$, where d denotes an arbitrary nonnegative integer. A top-down transducer $\mathbf{A}=(F, A, G, \{a_0\}, \Sigma)$ is in $\mathscr{K}_d$ if and only if for every $a \in A$, $f \in F_0$ and $p \in T_G$ if $af \to p \in \Sigma$ then $\text{dp}(p)=d$, moreover, as in case of $\mathscr{K}(\varrho)$, $|\tau_{\mathbf{A}(a)}(T_F)|>1$ is satisfied for each $a \in A$ appearing in the right side of a rule in Σ.

Statement 14. If $\mathbf{A} \in \mathscr{K}_d$ then $\theta_{\mathbf{A}}$ is a congruence relation.

Proof. The proof of this statement is similar to that of Statement 13. Only use the conditions defining $\mathscr{K}_d$ to establish the bijective correspondence between the sets $\bigcup (\text{path}_t(p) | t \in [m])$ and $\bigcup (\text{path}_s(q) | s \in [l])$ for the rules $af \to p(a_1 x_{i_1}, \ldots, a_m x_{i_m})$ and $bf \to q(b_1 x_{j_1}, \ldots, b_l x_{j_l})$.

Note that for $\mathbf{A} \in \mathscr{K}(\varrho)$ or $\mathbf{A} \in \mathscr{K}_d$ the definition of $\theta_{\mathbf{A}}$ can be reformulated as follows. Let $a, b \in A$. Then $a \theta_{\mathbf{A}} b$ if and only if for every $n, m \geqq 0$, $p \in T_{F,n}$, $q \in \tilde{T}_{G,m}$ and $i_1, \ldots, i_m \in [n]$ the following equivalence holds:

$$\exists a_1, \ldots, a_m \in A \quad ap \overset{*}{\Rightarrow} q(a_1 x_{i_1}, \ldots, a_m x_{i_m})$$

if and only if

$$\exists b_1, \ldots, b_m \in A \quad bp \overset{*}{\Rightarrow} q(b_1 x_{i_1}, \ldots, b_m x_{i_m}).$$

This is an easy consequence of statements 13, 14. Observe that this new definition of $\theta_{\mathbf{A}}$ makes $\theta_{\mathbf{A}}$ a congruence relation without requiring $\mathbf{A} \in \mathscr{K}(\varrho)$ or $\mathbf{A} \in \mathscr{K}_d$.

A transducer $\mathbf{A} \in \mathscr{K}(\varrho)$ or $\mathbf{A} \in \mathscr{K}_d$ is called *reduced* if $\theta_{\mathbf{A}}$ is the equality relation. As both $\mathscr{K}(\varrho)$ and $\mathscr{K}_d$ are closed under homomorphic images the transducer $\mathbf{A}/\theta_{\mathbf{A}}$ is reduced for any $\mathbf{A} \in \mathscr{K}(\varrho)$ or $\mathbf{A} \in \mathscr{K}_d$. The following statement is the basic step to show that minimal transducers in $\mathscr{K}(\varrho)$ and $\mathscr{K}_d$ are exactly the connected and reduced transducers.

Theorem 15. Let $\mathbf{A}, \mathbf{B} \in \mathscr{K}(\varrho)$ be connected top-down transducers. Then $\mathbf{A}$ and $\mathbf{B}$ are equivalent if and only if $\mathbf{A}/\theta_{\mathbf{A}} \cong \mathbf{B}/\theta_{\mathbf{B}}$. The same holds for $\mathscr{K}_d$.

Proof. Sufficiency follows by statements 11—14. In order to prove necessity first observe that if $\mathbf{A}=(F, A, G, \{a_0\}, \Sigma)$ and $\mathbf{B}=(F, B, G, \{b_0\}, \Sigma')$, moreover,

$a_0p\overset{*}{\Rightarrow}_{\mathbf{A}}q(a_1x_{i_1}, \ldots, a_mx_{i_m})$ — where $p\in T_{F,n}$, $n\geqq 0$, $q\in \hat{T}_{G,m}$, $m\geqq 0$, $a_1, \ldots, a_m\in A$, $i_1, \ldots, i_m\in[n]$ — then there exist states $b_1, \ldots, b_m\in B$ with $b_0p\overset{*}{\Rightarrow}_{\mathbf{B}}q(b_1x_{i_1}, \ldots, b_mx_{i_m})$. Furthermore, for these states b_i $(i=1, \ldots, m)$ we have $\tau_{\mathbf{A}(a_i)}=\tau_{\mathbf{B}(b_i)}$. This is a consequence of the assumption $\tau_{\mathbf{A}}=\tau_{\mathbf{B}}$ and the definitions of $\mathcal{K}(\varrho)$ and $\mathcal{K}_d$. Using the above mentioned facts it is easy to prove that the correspondence $\varphi\colon A/\theta_{\mathbf{A}}\to B/\theta_{\mathbf{B}}$ defined by $\varphi(\theta_{\mathbf{A}}(a))=\theta_{\mathbf{B}}(b)$ if and only if there exist $p\in\hat{T}_{F,1}$, $q\in\hat{T}_{G,m+1}$ $(m\geqq 0)$, $a_1, \ldots, a_m\in A$, $b_1, \ldots, b_m\in B$ such that $a_0p\overset{*}{\Rightarrow}_{\mathbf{A}}q(ax_1, a_1x_1, \ldots, a_mx_1)$ and $b_0p\overset{*}{\Rightarrow}_{\mathbf{B}}q(bx_1, b_1x_1, \ldots, b_mx_1)$ forms an isomorphism of $\mathbf{A}/\theta_{\mathbf{A}}$ into $\mathbf{B}/\theta_{\mathbf{B}}$.

The next theorem is an immediate consequence of Theorem 15 and the fact that $\mathcal{K}(\varrho)$ and $\mathcal{K}_d$ are closed under the formation of subtransducers and homomorphic images.

Theorem 16. A transducer is minimal in $\mathcal{K}(\varrho)$ if and only if it is connected and reduced. If both $\mathbf{A}$ and $\mathbf{B}$ are minimal in $\mathcal{K}(\varrho)$ and they are equivalent then $\mathbf{A}\cong\mathbf{B}$, i.e. the minimal realization of a transducer in $\mathcal{K}(\varrho)$ is unique up to isomorphism. The same holds for the class $\mathcal{K}_d$.

Of course Theorem 16 holds for every class $\mathcal{K}\subseteq\mathcal{K}(\varrho)$ or $\mathcal{K}\subseteq\mathcal{K}_d$ provided $\mathcal{K}$ is closed under the formation of subtransducers and homomorphic images. The most important example for a class of this type is the class of all *top-down relabelings* (cf. [5]).

It is natural to raise the question whether the minimal transducers in $\mathcal{K}(\varrho)$ or $\mathcal{K}_d$ are minimal in the class of all top-down transducers. The following examples prove that the answer is negative in general. In these examples the adjectives "linear", "nondeleting" are used in the sense of [5]. Furthermore, a top-down tree transducer $\mathbf{A}=(F, A, G, \{a_0\}, \Sigma)$ will be called uniform if each rule $af\to p$ $(a\in A, f\in F_n\ (n\geqq 0), p\in T_{G, A\times X_n})$ can be written as $af\to q(a_1x_1, \ldots, a_nx_n)$ for a tree $q\in T_{G,n}$ and states $a_1, \ldots, a_n\in A$.

Example 17. This example shows that there is a linear nondeleting top-down tree transducer $\mathbf{A}\in\mathcal{K}_1\cap\mathcal{K}(\varrho)$ which is connected and reduced — i.e. minimal in both $\mathcal{K}_1$ and $\mathcal{K}(\varrho)$ — but which is not minimal in the class of all linear nondeleting top-down tree transducers. Here $\varrho\colon\omega\to\omega$ is the mapping defined by $\varrho(n)=1$ $(n\geqq 0)$. Indeed, let $\mathbf{A}=(F, [5], F, [1], \Sigma)$ where F is the type determined by the conditions $F_0=\{\#\}$, $F_1=\{f, g\}$, $F_n=\emptyset$ if $n>1$ and Σ consists of the rules (1)—(5) listed below:

(1) $1\#\to f(\#)$, $1f(x_1)\to f(2x_1)$, $1g(x_1)\to g(3x_1)$,

(2) $2\#\to f(\#)$, $2f(x_1)\to f(4x_1)$, $2g(x_1)\to f(4x_1)$,

(3) $3\#\to g(\#)$, $3f(x_1)\to g(4x_1)$, $3g(x_1)\to g(4x_1)$,

(4) $4\#\to f(\#)$, $4f(x_1)\to f(5x_1)$, $4g(x_1)\to g(5x_1)$,

(5) $5\#\to f(\#)$, $5f(x_1)\to f(1x_1)$, $5g(x_1)\to g(1x_1)$.

However, $\mathbf{A}$ is equivalent to $\mathbf{A}'=(F, [4], F, [1], \Sigma')$ where Σ' contains the following rules (1)—(4):

(1) $1\#\to f(\#)$, $1f(x_1)\to f(f(2x_1))$, $1g(x_1)\to g(g(2x_1))$,

(2) $2\# \to \#$, $2f(x_1) \to 3x_1$, $2g(x_1) \to 3x_1$,

(3) $3\# \to f(\#)$, $3f(x_1) \to f(4x_1)$, $3g(x_1) \to g(4x_1)$,

(4) $4\# \to f(\#)$, $4f(x_1) \to f(1x_1)$, $4g(x_1) \to g(1x_1)$.

Example 18. This example proves that there is a top-down tree transducer $\mathbf{A}\in\mathcal{K}_0$ which is minimal in $\mathcal{K}_0$ but not minimal in the class of all top-down transducers.

Let us define the types F and G by $F_0=\{\#\}$, $F_1=\{f\}$, $F_n=\emptyset$ if $n>1$ and $G_0=\{\#, \#_1, \#_2\}$, $G_1=\{f\}$, $G_2=\{g\}$, $G_n=\emptyset$ $(n>2)$, respectively. Then put $\mathbf{A}=(F, [4], G, [1], \Sigma)$ where Σ consists of the following rules:

(1) $1\# \to \#$, $1f(x_1) \to g(2x_1, 3x_1)$,

(2) $2\# \to \#_1$, $2f(x_1) \to f(4x_1)$,

(3) $3\# \to \#_2$, $3f(x_1) \to f(4x_1)$,

(4) $4\# \to \#$, $4f(x_1) \to f(4x_1)$.

It is easy to check that $\mathbf{A}$ is minimal in $\mathcal{K}_0$. On the other hand $\mathbf{A}$ is equivalent to $\mathbf{A}'=(F, [3], G, [1], \Sigma')$ with Σ' containing the following rules:

(1) $1\# \to \#$, $1f(x_1) \to 2x_1$,

(2) $2\# \to g(\#_1, \#_2)$, $2f(x_1) \to g(f(3x_1), f(3x_1))$,

(3) $3\# \to \#$, $3f(x_1) \to f(3x_1)$.

Observe that $\mathbf{A}$ was not uniform.

In spite of Example 18 we have

Theorem 19. If a uniform transducer is minimal in $\mathcal{K}_0$ then it is minimal in the class of all top-down tree transducers.

Proof. Let $\mathbf{A}=(F, A, G, \{a_0\}, \Sigma)\in\mathcal{K}_0$ be uniform and minimal in $\mathcal{K}_0$. Assume that the top-down tree transducer $\mathbf{B}=(F, B, G, \{b_0\}, \Sigma')$ is equivalent to $\mathbf{A}$ and has fewer states than $\mathbf{A}$, i.e. $|B|<|A|$.

Take an arbitrary state $a\in A$. We shall correspond to this state a state $\varphi(a)\in B$ as follows. First let us choose the trees $p\in\tilde{T}_{F,1}$ and $q\in\tilde{T}_{G,n}$ $(n>0)$ in such a way that we have $a_0p\overset{*}{\Rightarrow}_{\mathbf{A}}q(\mathbf{a}^n\mathbf{x}_1^n)$. If $a=a_0$ choose $p=q=x_1$. This can be done since $\mathbf{A}$ is connected. Let $r\in\tilde{T}_{G,m}$ $(m\geqq 0)$ and $b_1, \ldots, b_m\in B$ be determined by $b_0p\overset{*}{\Rightarrow}_{\mathbf{B}}r(b_1x_1, \ldots, b_mx_1)$. As $|\tau_{\mathbf{A}(c)}(T_F)|>1$ is satisfied for each $c\in A$ occuring in the right side of a rule in Σ we must have $m>0$. Or even, there must be an index $j_i\in[m]$ for each $i\in[n]$ with the property that either $\text{path}_{j_i}(r)$ is a prefix of $\text{path}_i(q)$ or conversely. But, by the definition of $\mathcal{K}_0$, it is impossible that $\text{path}_i(q)$ is a proper prefix of $\text{path}_j(r)$. Therefore j_i is uniquely determined for each $i\in[n]$ and $\text{path}_{j_i}(r)$ is a prefix of $\text{path}_i(q)$. As $\mathbf{A}$ and $\mathbf{B}$ are equivalent this implies that there exist trees $r_1, \ldots, r_m\in T_{G,1}$ with $r(r_1, \ldots, r_m)=q$. Let $\varphi(a)=b_1$ and $r_a=r_{j_1}$. We must have $r_a(\tau_{\mathbf{A}(a)}(t))=\tau_{\mathbf{B}(\varphi(a))}(t)$ for each $t\in T_F$, i.e. $r_a(\tau_{\mathbf{A}(a)})=\tau_{\mathbf{B}(\varphi(a))}$.

As $|B|<|A|$ there exist states $a_1\neq a_2\in A$ with $\varphi(a_1)=\varphi(a_2)$. Consequently, $r_{a_1}(\tau_{\mathbf{A}(a_1)})=r_{a_2}(\tau_{\mathbf{A}(a_2)})$. But, again by the definition of $\mathcal{K}_0$, this is possible only if $r_{a_1}=r_{a_2}$ and $\tau_{\mathbf{A}(a_1)}=\tau_{\mathbf{A}(a_2)}$ yielding a contradiction.

We will now turn our attention to the bottom-up case. A deterministic bottom-pu tree transducer $\mathbf{A}=(F, A, G, A_0, \Sigma)$ is called *completely defined* if there is a rule in Σ with left hand side $f(a_1x_1, \ldots, a_nx_n)$ for every $n\geqq 0$, $f\in F_n$ and $a_1, \ldots, a_n\in A$. First of all we have to define homomorphisms, congruence relations etc.

Let $\mathbf{A}=(F, A, G, A_0, \Sigma)$ and $\mathbf{B}=(F, B, G, B_0, \Sigma')$ be bottom-up transducers. By a *homomorphism* of $\mathbf{A}$ into $\mathbf{B}$ we mean a mapping $\varphi\colon A\to B$ which satisfies the following two conditions:

(i) $f(b_1x_1, \ldots, b_nx_n)\to bp\in\Sigma'$ if $f(a_1x_1, \ldots, a_nx_n)\to ap\in\Sigma$, $b_i=\varphi(a_i)$ $(i=1, \ldots, n)$, $b=\varphi(a)$ $(n\geqq 0,\ f\in F_n,\ a_1, \ldots, a_n,\ a\in A,\ p\in T_{G,n})$,

(ii) $\varphi(A_0)\subseteqq B_0$, $\varphi^{-1}(B_0)\subseteqq A_0$.

Again, if φ is surjective then $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$ and bijective homomorphisms are called *isomorphisms*. If $B\subseteqq A$ and φ is the natural embedding of B into A then $\mathbf{B}$ is a *substransducer* of $\mathbf{A}$.

We now define *congruence relations*. A congruence relation of $\mathbf{A}$ is an equivalence relation θ on A with the following property: for any $n\geqq 0$, $f\in F_n$, $a_i, b_i\in A$ $(i=1, \ldots, n)$, $a, b\in A$ and $p, q\in T_{G,n}$ if both $f(a_1x_1, \ldots, a_nx_n)\to ap$ and $f(b_1x_1, \ldots, b_nx_n)\to bq$ are in Σ and $a_i\theta b_i$ $(i=1, \ldots, n)$ are satisfied then $p=q$ and $a\theta b$ hold too. Furthermore, A_0 is required to be equal to the union of certain blocks of the partition induced by θ: $A_0=\bigcup(\theta(a)\,|\,a\in A_0)$. The *quotient transducer* determined by θ is the transducer $\mathbf{A}/\theta=(F, A/\theta, G, A_0/\theta, \Sigma')$ where

$$\Sigma'=\{f(\theta(a_1)x_1, \ldots, \theta(a_n)x_n)\to\theta(a)p\,|\,f(a_1x_1, \ldots, a_nx_n)\to ap\in\Sigma\}.$$

With the above definitions in mind one can easily prove the analogues of statements 11 and 12.

For a bottom-up transducer $\mathbf{A}=(F, A, G, A_0, \Sigma)$ the relation $\theta_{\mathbf{A}}$ is defined as follows. Let $a, b\in A$. Then $a\theta_{\mathbf{A}}b$ if and only if the equivalence $\exists a_0\in A_0$ $p(a_1x_1, \ldots, a_{i-1}x_{i-1}, ax_i, a_{i+1}x_{i+1}, \ldots, a_nx_n)\overset{*}{\Rightarrow}a_0q\Leftrightarrow\exists b_0\in A_0$ $p(a_1x_1, \ldots, a_{i-1}x_{i-1}, bx_i, a_{i+1}x_{i+1}, \ldots, a_nx_n)\overset{*}{\Rightarrow}b_0q$ holds for all $n>0$, $i\in[n]$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\in A$, $p\in T_{F,n}$ (or equivalently $p\in\hat{T}_{F,n}$ or $p\in\tilde{T}_{F,n}$) and $q\in T_{G,n}$.

Likewise in the top-down case, $\theta_{\mathbf{A}}$ will not always be a congruence relation, but it will be a congruence relation if we require $\mathbf{A}$ to be in $\mathcal{K}(\varrho)$ for a mapping ϱ of the set of nonnegative integers into itself. A bottom-up transducer $\mathbf{A}=(F, A, G, A_0, \Sigma)$ belongs to $\mathcal{K}(\varrho)$ provided it satisfies the following three conditions:

(i) if $f(a_1x_1, \ldots, a_nx_n)\to ap\in\Sigma$ $(n>0,\ f\in F_n,\ a, a_1, \ldots, a_n\in A,\ p\in T_{G,n})$ then $|w|=\varrho(i)$ holds for each $i\in[n]$ and $w\in\mathrm{path}_i(p)$,

(ii) $\mathbf{A}$ is nondeleting, i.e. for all $n>0$, $f\in F_n$, $a, a_1, \ldots, a_n\in A$ and $p\in T_{G,n}$ if $f(a_1x_1, \ldots, a_nx_n)\to ap\in\Sigma$ then each of the variables $x_1, \ldots, x_n$ occurs in $\mathrm{fr}(p)$,

(iii) for any $a\in A$ there exist $p\in\hat{T}_{F,n+1}$, $q\in T_{G,n+1}$ $(n\geqq 0)$, $a_0\in A_0$, $a_1, \ldots, a_n\in A$ such that $p(ax_1, a_1x_2, \ldots, a_nx_{n+1})\overset{*}{\Rightarrow}a_0q$.

Statement 20. If $\mathbf{A}\in\mathcal{K}(\varrho)$ then $\theta_{\mathbf{A}}$ is a congruence relation.

Proof. Let $\mathbf{A}=(F, A, G, A_0, \Sigma)$, $a, b \in A$. Assume that $a \theta_{\mathbf{A}} b$ and let

$$f(a_1 x_1, \ldots, a_{i-1} x_{i-1}, a x_i, a_{i+1} x_{i+1}, \ldots, a_n x_n) \to c p,$$
$$f(a_1 x_1, \ldots, a_{i-1} x_{i-1}, b x_i, a_{i+1} x_{i+1}, \ldots, a_n x_n) \to d q$$

be arbitrary rules in Σ. Here $n>0$, $i \in [n]$, $f \in F_n$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, c, d \in A$, $p, q \in T_{G,n}$. We have to show that $p=q$ and $c \theta_{\mathbf{A}} d$.

As $\mathbf{A} \in \mathscr{K}(\varrho)$, there exist $m \geqq 0, c_1, \ldots, c_m \in A$, $a_0 \in A_0$ $r \in \hat{T}_{F,m+1}$ and $s \in T_{G,m+1}$ such that

$$r(c x_1, c_1 x_2, \ldots, c_m x_{m+1}) \overset{*}{\Rightarrow} a_0 s.$$

Let $r_1 = r(f(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{n+m})$, $s_1 = s(p, x_{n+1}, \ldots, x_{n+m})$. Of course we have

$$r_1(a_1 x_1, \ldots, a_{i-1} x_{i-1}, a x_i, a_{i+1} x_{i+1}, \ldots, a_n x_n, c_1 x_{n+1}, \ldots, c_m x_{n+m}) \overset{*}{\Rightarrow} a_0 s_1.$$

Since $a \theta_{\mathbf{A}} b$, this implies

$$r_1(a_1 x_1, \ldots, a_{i-1} x_{i-1}, b x_i, a_{i+1} x_{i+1}, \ldots, a_n x_n, c_1 x_{n+1}, \ldots, c_m x_{n+m}) \overset{*}{\Rightarrow} b_0 s_1$$

for a state $b_0 \in A_0$. But this is possible only if s_1 is of form $s_1 = t(q, x_{n+1}, \ldots, x_{n+m})$ where $t \in T_{G,m+1}$ and $r(d x_1, c_1 x_2, \ldots, c_m x_{m+1}) \overset{*}{\Rightarrow} b_0 t$.

We know that $s(p, x_{n+1}, \ldots, x_{n+m}) = t(q, x_{n+1}, \ldots, x_{n+m})$. By (i) and (ii) in the definition of $\mathscr{K}(\varrho)$ this results that $s=t$ and $p=q$. Essentially the same argument shows that $c \theta_{\mathbf{A}} d$.

Observe that for a bottom-up transducer $\mathbf{A}=(F, A, G, A_0, \Sigma) \in \mathscr{K}(\varrho)$ the relation $\theta_{\mathbf{A}}$ can be redefined as follows. Let $a, b \in A$. Then $a \theta_{\mathbf{A}} b$ if and only if the following two equivalences are satisfied for arbitrary $p \in T_{F,n}$, $q \in T_{G,n}$ $(n \geqq 0)$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$ and $i \in [n]$:

(i) $\exists a_0 \in A\ p(a_1 x_1, \ldots, a_{i-1} x_{i-1}, a x_i a_{i+1} x_{i+1}, \ldots, a_n x_n) \overset{*}{\Rightarrow} a_0 q$

if and only if

$$\exists b_0 \in A\ p(a_1 x_1, \ldots, a_{i-1} x_{i-1}, b x_i, a_{i+1} x_{i+1}, \ldots, a_n x_n) \overset{*}{\Rightarrow} b_0 q,$$

(ii) for a_0 and b_0 of (i) it holds that $a_0 \in A_0$ if and only if $b_0 \in A_0$.

A transducer $\mathbf{A} \in \mathscr{K}(\varrho)$ is called *reduced* if $\theta_{\mathbf{A}}$ is the equality relation on A. $\mathbf{A}/\theta_{\mathbf{A}}$ is always reduced.

In contrast with the top-down case there are nonisomorphic but equivalent minimal transducers in $\mathscr{K}(\varrho)$. However, if a bottom-up transducer is minimal in $\mathscr{K}(\varrho)$ then it is both reduced and connected (i.e. it has not proper subtransducers). The converse is not true in general.

According to the above discussion we need some further restrictions to guarantee the uniqueness of minimal realizations. For this purpose we introduce the subclass $\mathscr{K}'(\varrho)$ of $\mathscr{K}(\varrho)$. A bottom-up transducer $\mathbf{A}=(F, A, G, A_0, \Sigma) \in \mathscr{K}(\varrho)$ belongs to $\mathscr{K}'(\varrho)$ if and only if it satisfies the condition:

if $f(a_1 x_1, \ldots, a_n x_n) \to a p \in \Sigma$ where $n>0$, $f \in F_n$, $a_1, \ldots, a_n$, $a \in A$ and $p \in T_{G,n}$ then $p \in \hat{T}_{G,n}$ and none of the operational symbols in G_0 occurs in p.

Now we are able to state an analogue of Theorem 15 for bottom-up transducers.

Theorem 21. Let $\mathbf{A}, \mathbf{B} \in \mathscr{K}'(\varrho)$ be connected. Then they are equivalent if and only if $\mathbf{A}/\theta_{\mathbf{A}} \cong \mathbf{B}/\theta_{\mathbf{B}}$.

Proof. The sufficiency follows in the same way as in Theorem 14. In order to prove the necessity of our statement, first observe that if $\mathbf{A}=(F, A, G, A_0, \Sigma)$ and $\mathbf{B}=(F, B, G, B_0, \Sigma')$, moreover, $\tau_{\mathbf{A}(a)}(p)=q$ where $p\in T_F$, $q\in T_G$ and $a\in A$, then there is a state $b\in B$ with $\tau_{\mathbf{B}(b)}(p)=q$. In fact, if $a_i\in A$, $b_i\in B$ $(i=1, \dots, n,\ n>0)$ are such that $\operatorname{dom}\tau_{\mathbf{A}(a_i)}\cap\operatorname{dom}\tau_{\mathbf{B}(b_i)}\neq\emptyset$ $(i=1, \dots, n)$ and $p(a_1x_1, \dots, a_nx_n)\overset{*}{\Rightarrow}_{\mathbf{A}} aq$ where $p\in T_{F,n}$, $q\in T_{G,n}$ and $a\in A$ then there is a state $b\in B$ satisfying $p(b_1x_1, \dots, b_nx_n)\overset{*}{\Rightarrow}_{\mathbf{B}} bq$. The same assertions holds if we change the role of **A** and **B**. By these observations it is easy to verify that the correspondence φ defined by $\varphi(\theta_{\mathbf{A}}(a))=\theta_{\mathbf{B}}(b)$ if and only if $\operatorname{dom}\tau_{\mathbf{A}(a)}\cap\operatorname{dom}\tau_{\mathbf{B}(b)}\neq\emptyset$ is an isomorphism of $\mathbf{A}/\theta_{\mathbf{A}}$ into $\mathbf{B}/\theta_{\mathbf{B}}$.

Theorem 22. A bottom-up transducer is minimal in $\mathcal{K}'(\varrho)$ if and only if it is both reduced and connected. The minimal realization of a bottom-up transducer in $\mathcal{K}'(\varrho)$ is unique up to isomorphism.

Proof. Immediate by Theorem 21.

Observe that Theorem 22 holds for every class $\mathcal{K}\subseteq\mathcal{K}'(\varrho)$ provided it is closed under the formation of subtransducers and homomorphic images. An example of a class of this sort is the class of all bottom-up relabelings satisfying condition (iii) in the definition of $\mathcal{K}(\varrho)$. A tree transducer $\mathbf{A}=(F, A, G, A_0, \Sigma)$ is called a bottom-up relabeling if each rule in Σ is of form

$$f(a_1x_1, \dots, a_nx_n)\to ag(x_1, \dots, x_n)$$

where $n\geqq 0$, $f\in F_n$, $g\in G_n$, $a_1, \dots, a_n, a\in A$.

The following example shows that there is a transducer which is minimal in $\mathcal{K}'(\varrho)$ but which is not minimal in the class of all bottom-up transducers. Let $F_0=\{\#\}$, $F_1=\{f, g\}$ and $F_i=\emptyset$ if $i>1$. Take the bottom-up transducer $\mathbf{A}=(F, [5], F, [1], \Sigma)$ where Σ consists of the following rules:

(1) $\#\to 1\#$,

(2) $f(1x_1)\to 2f(x_1)$, $g(1x_1)\to 3g(x_1)$,

(3) $f(2x_1)\to 4f(x_1)$, $g(2x_1)\to 4f(x_1)$,

(4) $f(3x_1)\to 4g(x_1)$, $g(3x_1)\to 4g(x_1)$,

(5) $f(4x_1)\to 5f(x_1)$, $g(4x_1)\to 4g(x_1)$,

(6) $f(5x_1)\to 1f(x_1)$, $g(5x_1)\to 1g(x_1)$.

It is easy to see that **A** is minimal in $\mathcal{K}'(\varrho)$ where ϱ is a constant mapping: $\varrho(n)=1$ for all $n\geqq 0$. On the other hand $\tau_{\mathbf{A}}$ can be induced by a four state transducer $\mathbf{B}=(F, [4], F, [1], \Sigma')$ where Σ' consists of the rules (1)—(5) listed below:

(1) $\#\to 1\#$,

(2) $f(1x_1)\to 2f(f(x_1))$, $g(1x_1)\to 2g(g(x_1))$,

(3) $f(2x_1)\to 3x_1$, $g(2x_1)\to 3x_1$,

(4) $f(3x_1)\to 4f(x_1)$, $g(3x_1)\to 4g(x_1)$,

(5) $f(4x_1)\to 1f(x_1)$, $g(4x_1)\to 1g(x_1)$.

In spite of the preceding example the following theorem is valid.

Theorem 23. Let $\mathbf{A}=(F, A, G, A_0, \Sigma)$ be minimal in $\mathcal{K}'(\varrho)$. Assume that $A=A_0$. Then $\mathbf{A}$ is minimal in the class of all bottom-up transducers.

Proof. Let us correspond to each $a\in A$ a tree $p_a\in \operatorname{dom} \tau_{\mathbf{A}(a)}$. This can be done because $\mathbf{A}$ is connected. Assume that $\mathbf{B}=(F, B, G, B_0, \Sigma')$ is equivalent to $\mathbf{A}$ and has fewer states than $\mathbf{A}$, i.e. $|B|<|A|$. Of course $B=B_0$. Define the mapping $\varphi\colon A\to B$ by $\varphi(a)=b$ if and only if $p_a\in \operatorname{dom} \tau_{\mathbf{B}(b)}$. Since $|B|<|A|$ there are distinct states $a_1, a_2\in A$ with $\varphi(a_1)=\varphi(a_2)$. Denote this state $\varphi(a_1)$ by b. As $\mathbf{A}$ is reduced, there exist $p\in \tilde{T}_{F,n}$, $q_1\neq q_2\in T_{G,n}$ $(n>0)$ and $i_0\in[n]$, as well as states $c_1, \dots, c_{i_0-1}, c_{i_0+1}, \dots, c_n, d_1, d_2\in A$ such that

$$p(c_1x_1, \dots, c_{i_0-1}x_{i_0-1}, a_1x_{i_0}, c_{i_0+1}x_{i_0+1}, \dots, c_nx_n) \overset{*}{\Rightarrow}_{\mathbf{A}} d_1q_1,$$

$$p(c_1x_1, \dots, c_{i_0-1}x_{i_0-1}, a_2x_{i_0}, c_{i_0+1}x_{i_0+1}, \dots, c_nx_n) \overset{*}{\Rightarrow}_{\mathbf{A}} d_2q_2.$$

Of course $q_1, q_2\in \tilde{T}_{G,n}$.

As $A\in\mathcal{K}'(\varrho)$ we may assume that $p=f(x_1, \dots, x_n)$ for an operational symbol $f\in F_n$. It can be seen, by $q_1\neq q_2$ and $\mathbf{A}\in\mathcal{K}'(\varrho)$, that q_1 and q_2 are of form $q_1=q_0(r_1, \dots, r_m)$ and $q_2=q_0(r'_1, \dots, r'_m)$, respectively, where $q_0\in\tilde{T}_{G,m}$ $(m>0)$, $r_j, r'_j\in T_{G,n}$, furthermore, there is at least one index $j_0\in[m]$ such that $r_{j_0}\neq r'_{j_0}$, r_{j_0}, $r'_{j_0}\notin X_n$. More exactly, we may choose q_0 in such a way that $r_{j_0}=g_1(\mathbf{s}_1)$ and $r'_{j_0}=g_2(\mathbf{s}_2)$ hold for some vectors $\mathbf{s}_1, \mathbf{s}_2$ and different operational symbols $g_1, g_2\in G$. This implies that

$$\tau_{\mathbf{A}}\big(f(p_{c_1}, \dots, p_{c_{i_0-1}}, p_{a_1}, p_{c_{i_0+1}}, \dots, p_{c_n})\big) \neq \tau_{\mathbf{A}}\big(f(p_{c_1}, \dots, p_{c_{i_0-1}}, p_{a_2}, p_{c_{i_0+1}}, \dots, p_{c_n})\big).$$

Now let $b_i=\varphi(c_i)$ $(i=1, \dots, n,\ i\neq i_0)$. There is a state $e\in B$ and a tree $q\in T_{G,n}$ with $f(b_1x_1, \dots, b_{i_0-1}x_{i_0-1}, bx_{i_0}, b_{i_0+1}x_{i_0+1}, \dots, b_nx_n)\to eq\in\Sigma'$. Since $\mathbf{A}$ and $\mathbf{B}$ are equivalent we have $\tau_{\mathbf{A}}(p_{c_i})=\tau_{\mathbf{B}}(p_{b_i})$ $(i=1, \dots, n,\ i\neq i_0)$, $\tau_{\mathbf{A}}(p_{a_i})=\tau_{\mathbf{B}}(p_{a_i})$ $(i=1, 2)$, $q_i(\tau_{\mathbf{A}}(p_{c_1}), \dots, \tau_{\mathbf{A}}(p_{c_{i_0-1}}), \tau_{\mathbf{A}}(p_{a_i}), \tau_{\mathbf{A}}(p_{c_{i_0+1}}), \dots, \tau_{\mathbf{A}}(p_{c_n}))=q(\tau_{\mathbf{B}}(p_{c_1}), \dots, \tau_{\mathbf{B}}(p_{c_{i_0-1}}), \tau_{\mathbf{B}}(p_{a_i}), \tau_{\mathbf{B}}(p_{c_{i_0+1}}), \dots, \tau_{\mathbf{B}}(p_{c_n}))$ $(i=1, 2)$.

But $\tau_{\mathbf{A}}\big(f(p_{c_1}, \dots, p_{c_{i_0-1}}, p_{a_1}, p_{c_{i_0+1}}, \dots, p_{c_n})\big)\neq\tau_{\mathbf{A}}\big(f(p_{c_1}, \dots, p_{c_{i_0-1}}, p_{a_2}, p_{c_{i_0+1}}, \dots, p_{c_n})\big)$. Thus $\tau_{\mathbf{B}}(p_{a_1})\neq\tau_{\mathbf{B}}(p_{a_2})$ and $\operatorname{path}_{i_0}(q)\neq\emptyset$. Even more, by $r_{j_0}\neq r'_{j_0}$, there is a string $w\in\operatorname{path}_{i_0}(q)$ which is a prefix of $\operatorname{path}_{j_0}(q_0)$. Now there are two cases.

First suppose that $\operatorname{path}_{j_0}(q_0)$ is a prefix of $\operatorname{path}_{i_0}(q_1)$ and let $p_i=f(p_{c_1}, \dots, p_{c_{i_0-1}}, p_{a_i}, p_{c_{i_0+1}}, \dots, p_{c_n})$ $(i=1, 2)$. Then $\tau_{\mathbf{A}}(p_1)=u(\tau_{\mathbf{A}}(p_{a_1}))$ and $\tau_{\mathbf{B}}(p_1)=u'(\tau_{\mathbf{B}}(p_{a_1}))$ where $u, u'\in\hat{T}_{F,1}$ satisfy $\operatorname{path}(u)=\operatorname{path}_{i_0}(q_1)$ and $\operatorname{path}(u')=w$, respectively. As w is a proper prefix of $\operatorname{path}_{i_0}(q_1)$ and $\tau_{\mathbf{A}}(p_{a_1})=\tau_{\mathbf{B}}(p_{a_1})$ this results that $\tau_{\mathbf{A}}(p_1)\neq\tau_{\mathbf{B}}(p_1)$, contrary to our assumption $\tau_{\mathbf{A}}=\tau_{\mathbf{B}}$. A similar argument yields a contradiction if $\operatorname{path}_{j_0}(q_0)$ is assumed to be a prefix of $\operatorname{path}_{i_0}(q_2)$.

Thus none of the strings $\operatorname{path}_{i_0}(q_1)$ and $\operatorname{path}_{i_0}(q_2)$ is a postfix of $\operatorname{path}_{j_0}(q_0)$. This implies that $\tau_{\mathbf{A}}(p_1)=u(v)$, $\tau_{\mathbf{A}}(p_2)=u'(v)$, $\tau_{\mathbf{B}}(p_1)=u(v)$ and $\tau_{\mathbf{B}}(p_2)=u'(v')$ where $u, u'\in\hat{T}_{F,1}$, $v, v'\in T_G$ satisfy the conditions $\operatorname{path}(u)=\operatorname{path}(u')=w$ and $v\neq v'$. Indeed, $v=\tau_{\mathbf{B}}(p_{a_1})$, and $v'=\tau_{\mathbf{B}}(p_{a_2})$. It is again a contradiction.

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References

[1] ARBIB, M. A. and Y. GIVE'ON, Algebra automata I: Parallel programming as a prolegomena to the categorical approach, *Inform. and Control*, v. 12, 1968, pp. 331—345.
[2] BLATTNER, M. and T. HEAD, The decidability of equivalence for deterministic finite transducers, *J. Comput. System Sci.*, v. 19, 1979, pp. 45—49.
[3] BRAINERD, W. S., The minimization of tree-automata, *Inform. and Control*, v. 13, 1968, pp. 484—491.
[4] CULIC II, K. and A. SALOMAA, On the decidability of homomorphism equivalence for languages, *J. Comput. System Sci.*, v. 17, 1978, pp. 163—175.
[5] ENGELFRIET, J., Bottom-up and top-down tree transformations, A comparison, *Math. Systems Theory*, v. 9, 1975, pp. 198—231.
[6] ENGELFRIET, J., Top-down tree transducers with regular look-ahead, *Math. Systems Theory*, v. 10, 1977, pp. 289—303.
[7] ENGELFRIET, J., On tree transducers for partial functions, *Inform. Process. Lett.*, v. 7, 1978, pp. 170—172.
[8] ÉSIK, Z., On functional tree transducers, in *Proceedings, Conference on Fundamentals of Compatation Theory*, ed. Budach, L., Akademie-Verlag, Berlin, 1979, pp. 121—127.
[9] GÉCSEG, F. and I. PEÁK, *Algebraic theory of automata*, Akadémia Kiadó, Budapest, 1972.
[10] GÉCSEG, F. and M. STEINBY, Minimal ascending tree automata, *Acta Cybernet.*, v. 4, 1978, pp. 37—44.
[11] GRIFFITHS, T. V., The unsolvability of the equivalence problem for λ-free nondeterministic generalized machines, *J. Assoc. Comput. Mach.*, v. 15, 1968, pp. 409—413.
[12] ZACHAR, Z., The solvability of the equivalence problem for deterministic frontier-to-root tree transducers, *Acta Cybernet.*, v. 4, 1978, pp. 167—177.

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