On isomorphic representations of commutative automata with respect to α_i-products

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The purpose of this paper is to study the α_i-products (see [1]) from the point of view of isomorphic completeness for the class of all commutative automata. Namely, we give necessary and sufficient conditions for a system of automata to be isomorphically complete for the class of all commutative automata with respect to the α_i-products. It will turn out that if $i \geqq 1$ then such isomorphically complete systems coincide with each other with respect to different α_i-products. Furthermore they coincide with isomorphically complete systems of automata.

By an *automaton* we mean a finite automaton $\mathbf{A}=(X, A, \delta)$ without output. Moreover *isomorphism* and *subautomaton* will mean A-isomorphism and A-subautomaton.

Take an automaton $\mathbf{A}=(X, A, \delta)$ and let us denote by X^* the free monoid generated by X. The elements $p \in X^*$ are called *input words* of $\mathbf{A}$. The transition function δ can be extended to $A \times X^* \to A$ in a natural way: for any $p=p'x$ ($p' \in X^*$, $x \in X$) and $a \in A$ $\delta(a, p)=\delta(\delta(a, p'), x)$. Further on we shall use the more convenient notation $ap_{\mathbf{A}}$ for $\delta(a, p)$ and $A'p_{\mathbf{A}}$ for the set $\{ap_{\mathbf{A}}: a \in A'\}$ where $A' \subset A$ and $p \in X^*$. If there is no danger of confusion, then we omit the index $\mathbf{A}$ in $ap_{\mathbf{A}}$ and $A'p_{\mathbf{A}}$. Define a binary relation σ on X^* in the following manner: for two input words $p, q \in X^*$, $p \equiv q\ (\sigma)$ if and only if $ap=aq$ for all $a \in A$. The quotient semigroup X^*/σ is called the *characteristic semigroup* of $\mathbf{A}$, and it will be denoted by $S(\mathbf{A})$. We use the notation $[p]$ for the element of $S(\mathbf{A})$ containing $p \in X^*$.

An automaton $\mathbf{A}=(X, A, \delta)$ is *commutative* if $ax_1x_2=ax_2x_1$ for any $a \in A$ and $x_1, x_2 \in X$. Denote by $\mathfrak{K}$ the class of all commutative automata.

Take an automaton $\mathbf{A}=(X, A, \delta)$ and let ω be an equivalence relation of the set A. It is said that ω is a congruence relation of $\mathbf{A}$ if $a \equiv b(\omega)$ implies $ax \equiv bx(\omega)$ for all $a, b \in A$ and $x \in X$. The partition induced by the congruence relation ω is called *compatible partition* of $\mathbf{A}$.

Let $\mathbf{A}=(X, A, \delta)$ be an automaton. Define the relation C of A in the following way: $a \equiv b(C)$ if and only if there exist $p, q \in X^*$ such that $ap=b$ and $bq=a$. It is clear that C is a congruence relation of $\mathbf{A}$ if the automaton $\mathbf{A}$ is commutative. In the following we use the notation $C(a)$ for the block of the partition induced by C which contains a. On the set $A/C=\{C(a): a \in A\}$ we define a partial ordering in the following way: for any $a, b \in A$, $C(a) \leqq C(b)$ if there exists $p \in X^*$ such that $ap=b$. If $C(a) \leqq C(b)$ and $C(a) \neq C(b)$ then we write $C(a) < C(b)$.

The automaton $\mathbf{A}=(X, A, \delta)$ is called a *permutation automaton* if for any $a, b\in A$ and $p\in X^*$, $ap=bp$ implies $a=b$. The automaton $\mathbf{A}$ is *connected* if for any $a, b\in A$ there exist $p, q\in X^*$ such that $ap=bq$.

Let $\mathbf{A}_t=(X_t, A_t, \delta_t)$ $(t=1, \ldots, n)$ be a system of automata. Moreover, let X be a finite nonvoid set and φ a mapping of $A_1\times\ldots\times A_n\times X$ into $X_1\times\ldots\times X_n$ such that $\varphi(a_1, \ldots, a_n, x)=(\varphi_1(a_1, \ldots, a_n, x), \ldots, \varphi_n(a_1, \ldots, a_n, x))$, and each φ_j $(1\leqq j\leqq n)$ is independent of states having indices greater than or equal to $j+i$, where i is a fixed nonnegative integer. We say that the automaton $\mathbf{A}=(X, A, \delta)$ with $A=A_1\times\ldots\times A_n$ and $\delta((a_1, \ldots, a_n), x)=(\delta_1(a_1, \varphi_1(a_1, \ldots, a_n, x)), \ldots, \delta_n(a_n, \varphi_n(a_1, \ldots, a_n, x)))$ is the *α_i-product* of $\mathbf{A}_t$ $(t=1, \ldots, n)$ with respect to X and φ. For this product we use the notation $\prod_{t=1}^{n} \mathbf{A}_t(X, \varphi)$ and $\mathbf{A}_1\times\mathbf{A}_2(X, \varphi)$ for $n=2$. Moreover, if in α_i-product $\mathbf{A}$, $\mathbf{A}_t=\mathbf{B}$ for all t $(t=1, \ldots, n)$, then $\mathbf{A}$ is called an *α_i-power* of $\mathbf{B}$ and we use the notation $\mathbf{A}=\mathbf{B}^n(X, \varphi)$.

Let $\mathfrak{B}$ be an arbitrary class of automata. Further on let Σ be a system of automata. Σ is called *isomorphically complete for* $\mathfrak{B}$ with respect to the α_i-product if any automaton from $\mathfrak{B}$ can be embedded isomorphically into an α_i-product of automata from Σ. If $\mathfrak{B}$ is the class of all automata and Σ is isomorphically complete for $\mathfrak{B}$, then it is said that Σ is *isomorphically complete.*

Let us denote by $\mathbf{E}_2=(\{x, y\}, \{0, 1\}, \delta_{\mathrm{E}})$ the automaton for which $\delta_{\mathrm{E}}(0, y)=0$, $\delta_{\mathrm{E}}(0, x)=1$, $\delta_{\mathrm{E}}(1, x)=\delta_{\mathrm{E}}(1, y)=1$.

An automaton $\mathbf{A}=(X, A, \delta)$ is called *monotone* if there exists a partial ordering $\leqq$ on A such that $a\leqq\delta(a, x)$ holds for any $a\in A$ and $x\in X$.

For monotone automata the following result holds:

Lemma 1. Every connected monotone automaton can be embedded isomorphically into an α_0-power of $\mathbf{E}_2$.

Proof. We proceed by induction on the number of states of the automaton. In the cases $n=1$ and $n=2$ our statement is trivial. Now let $n>2$ and suppose that the statement is valid for any natural number $m<n$. Denote by $\mathbf{A}=(X, A, \delta)$ an arbitrary connected monotone automaton with n states. Since $\mathbf{A}$ is connected thus among the blocks $C(a)$ $(a\in A)$ there exists exactly one maximal element under our partial ordering of blocks. On the other hand, since $\mathbf{A}$ is monotone thus the partition induced by C has one-element blocks only. Denote by a_n the element of the maximal block. Since $n>2$ thus there exists an $a\in A$ such that $C(a)<C(a_n)$. Denote by a_k an element of A for which $C(a_k)<C(a_n)$ and $C(a_k)<C(a)$ implies $a=a_n$ for any $a\in A$. Obviously there exists such an a_k. It is also obvious that $(X, H, \delta_{|H\times X})$ is a subautomaton of $\mathbf{A}$, where $H=\{a_k, a_n\}$ and the restriction to $H\times X$ of the function δ is denoted by $\delta_{|H\times X}$. Let us define the automata $\mathbf{A}_1=(X, (A\setminus H)\cup\{*\}, \delta_1)$ and $\mathbf{A}_2=(((A\setminus H)\cup\{*\})\times X, H\cup\{\square\}, \delta_2)$ in the following way:

$$\delta_1(a, x) = \begin{cases} \delta(a, x) & \text{if} \quad \delta(a, x)\notin H, \\ * & \text{otherwise,} \end{cases}$$

$$\delta_1(*, x) = *,$$

$$\delta_2(\square, (a, x)) = \begin{cases} \delta(a, x) & \text{if} \quad \delta(a, x)\in H, \\ \square & \text{otherwise,} \end{cases}$$

$$\delta_2(a', (a, x)) = a', \ \delta_2(a', (*, x)) = \delta(a', x), \ \delta_2(\square, (*, x)) = \square$$

for all $a\in A\setminus H, x\in X$ and $a'\in H$. Take the α_0-product $\mathbf{B}=\mathbf{A}_1\times\mathbf{A}_2(X,\varphi)$ where $\varphi_1(x)=x$, $\varphi_2(v,x)=(v,x)$ for all $x\in X$ and $v\in(A\setminus H)\cup\{*\}$. It is easy to prove that the correspondence

$$\nu(a)=\begin{cases}(a,\square) & \text{if } a\in A\setminus H,\\ (*,a) & \text{if } a\in H,\end{cases}$$

is an isomorphism of $\mathbf{A}$ into $\mathbf{B}$.

Now let us consider the automata $\mathbf{A}_1$ and $\mathbf{A}_2$. Since $\mathbf{A}_1$ is a connected monotone automaton with $n-1$ states thus, by our assumption, $\mathbf{A}_1$ can be embedded isomorphically into an α_0-power of $\mathbf{E}_2$. Denote by U the set of input signals of $\mathbf{A}_2$ and take the following partitions of U:

$$U_1=\{(a,x)\colon a\in A\setminus H,\ x\in X,\ \delta(a,x)\notin H\}\cup\{(*,x)\colon x\in X\},$$

$$U_2=\{(a,x)\colon a\in A\setminus H,\ x\in X,\ \delta(a,x)=a_k\},$$

$$U_3=\{(a,x)\colon a\in A\setminus H,\ x\in X,\ \delta(a,x)=a_n\},$$

$$V_1=\{(a,x)\colon a\in A\setminus H,\ x\in X\}\cup\{(*,x)\colon x\in X,\ \delta(a_k,x)=a_k\},$$

$$V_2=\{(*,x)\colon x\in X,\ \delta(a_k,x)=a_n\}.$$

Consider the α_0-product $\mathbf{E}^2(U,\varphi)$ where $\varphi_1(u_1)=y$, $\varphi_1(u_2)=\varphi_1(u_3)=x$, $\varphi_2(0,u_1)=\varphi_2(0,u_2)=y$, $\varphi_2(0,u_3)=x$, $\varphi_2(1,v_1)=y$ and $\varphi_2(1,v_2)=x$ for all $u_i\in U_i$ $(i=1,2,3)$ and $v_j\in V_j$ $(j=1,2)$. It can easily be seen that the correspondence $\square\to(0,0)$, $a_k\to(1,0)$ and $a_n\to(1,1)$ is an isomorphism of $\mathbf{A}_2$ into $\mathbf{E}^2(U,\varphi)$. Since the formation of the α_0-product is associative thus we have proved that $\mathbf{A}$ can be embedded isomorphically into an α_0-power of $\mathbf{E}_2$.

For any natural number $n\geqq 1$ let $\mathbf{M}_n=(\{x_0,\dots,x_{n-1}\},\{0,\dots,n-1\},\delta)$ denote the automaton for which $\delta(j,x_l)=j+l\,(\text{mod } n)$ for any $j\in\{0,\dots,n-1\}$ and $x_l\in\{x_0,\dots,x_{n-1}\}$, where $j+l\,(\text{mod } n)$ denotes the least nonnegative residue of $j+l$ modulo n. Moreover let $\mathfrak{M}$ denote the set of all $\mathbf{M}_n$ such that n is a prime number.

It holds the following

Lemma 2. If the number of states of a strongly connected commutative automaton $\mathbf{A}$ is a prime number, then there exists an automaton $\mathbf{M}\in\mathfrak{M}$ such that $\mathbf{A}$ is isomorphic to an α_0-product of $\mathbf{M}$ with a single factor.

Proof. First we prove that every strongly connected commutative automaton is a permutation automaton. Indeed, denote by $\mathbf{A}=(X,A,\delta)$ a strongly connected commutative automaton and assume that there exist $a,b\in A$ and $p\in X^*$ with $ap=bp$. Since $\mathbf{A}$ is strongly connected thus there exist input words $q,w\in X^*$ such $apq=a$ and $aw=b$. Using the commutativity of $\mathbf{A}$, we have $bpq=awpq=apqw=aw=b$. Therefore, $a=apq=bpq=b$, showing that $\mathbf{A}$ is a permutation automaton.

Now let us assume that the number of states of $\mathbf{A}$ is prime and denote it by r. Let $a\in A$ and $p\in X^*$ be arbitrary and consider the states $a, ap, ap^2, \dots$. Since $\mathbf{A}$ is a permutation automaton thus there exists a t $(1\leqq t\leqq r)$ such that $a=ap^t$. Denote by (a,p) the set $\{a,ap,\dots,ap^{t-1}\}$. Assume that $(a,p)\subset A$. Let $a'\in A\setminus(a,p)$ and consider the set (a',p), which is defined as above. Since $\mathbf{A}$ is a strongly connected automaton thus there exists a $q\in X^*$ such that $aq=a'$. Using the commutativity of $\mathbf{A}$ we have $ap^iq=aqp^i=a'p^i$ $(i=0, \dots, t-1)$. From this it follows that (a, p) and (a', p) have the same cardinality since $\mathbf{A}$ is a permutation automaton. On the other hand it can easily be seen that (a, p) and (a', p) are disjoint subsets of A. Therefore, the set $\varrho_p=\{(a, p): a\in A\}$ is a partition of A and the blocks of ϱ_p have the same cardinality. Since r is prime thus we get that ϱ_p has one-element blocks only, or it has one block only. Now we choose an $x\in X$ such that ϱ_x has one block only. The automaton $\mathbf{A}$ is strongly connected therefore such an $x\in X$ exists. Let $a\in A$ be a fixed state of $\mathbf{A}$ and write $a_0=a$, $a_i=a_0x^i$ $(i=1, \dots, r-1)$. Thus the mapping induced by x on A can be described in the form $a_ix=a_{i+1 \pmod r}$ $(i=0, \dots, r-1)$. Now let y be an arbitrary input signal of $\mathbf{A}$ and assume that $a_0y=a_j$ for some $j\in\{0, 1, \dots, r-1\}$. From the commutativity of $\mathbf{A}$ we have $a_iy=a_0x^iy=a_0yx^i=a_jx^i=a_{i+j \pmod r}$ for all $i\in\{0, 1, \dots, r-1\}$. Take the α_0-product $\mathbf{B}=\Pi\mathbf{M}_r(X, \varphi)$ with a single factor, where $\varphi(x)=x_k$ if $a_0x=a_k$ for all $x\in X$. It is easy to prove that $\mathbf{A}$ is isomorphic to $\mathbf{B}$, which completes the proof of Lemma 2.

Lemma 3. Every strongly connected commutative automaton can be embedded isomorphically into an α_0-product of automata from $\mathfrak{M}$.

Proof. We prove by induction on the number of states of the automaton. In case $n<4$, by Lemma 2, the statement holds. Now let $n\geqq 4$ and assume that our statement is valid for any natural number $m<n$. Denote by $\mathbf{A}=(X, A, \delta)$ an arbitrary strongly connected commutative automaton with n states. If n is prime then, by Lemma 2, the statement holds. Assume that n is not prime. Let $p\in X^*$ be arbitrary. Consider the partition ϱ_p. Since $\mathbf{A}$ is commutative thus ϱ_p is a compatible partition of $\mathbf{A}$. Denote by Ω the set of all partitions ϱ_p of $\mathbf{A}$ such that $[p]\in S(\mathbf{A})\setminus\{[e]\}$, where e denotes the empty word of X^*. Take the partition ϱ of $\mathbf{A}$ given by $\varrho=\bigcap_{\varrho_p\in\Omega}\varrho_p$. We distinguish two cases.

First assume that ϱ has one-element blocks only. In this case it can easily be seen that $\mathbf{A}$ can be embedded isomorphically into the direct product of the quotient automata $\mathbf{A}/\varrho_p$ $(\varrho_p\in\Omega)$. On the other hand, for any $\varrho_p\in\Omega$ the quotient automaton $\mathbf{A}/\varrho_p$ is a strongly connected commutative automaton with number of states less than n. Therefore, by our induction hypothesis the statement is valid.

Now assume, that there exist $a, b\in A$ such that $a\neq b$ and $a\equiv b(\varrho)$. Take an input signal x of $\mathbf{A}$ such that the mapping induced by it on A is not the identity. Then $\varrho_x\in\Omega$ and thus $\varrho_x\geqq\varrho$. Therefore, $a\equiv b(\varrho_x)$. This means that there exists a natural number $l>0$ such that $ax^l=b$. Since ϱ is compatible thus $ax^l\equiv bx^l(\varrho)$. From this, by the above equality, we get that the states $a, ax^l, ax^{2l}, \dots$ are in $\varrho(a)$. Therefore, $(a, x^l)\subseteqq\varrho(a)$. On the other hand $\varrho_{x^l}\geqq\varrho$ thus $(a, x^l)=\varrho(a)$, showing that $\varrho_{x^l}=\varrho$. Denote by p the word x^l and assume that $\varrho(a)=\{a, ap, \dots, ap^{k-1}\}$. We show that k is prime. Indeed, if $1<v<k$ and $_v|^k$ then $(a, p^v)\subset(a, p)$ which contradicts the relation $\varrho_{p^v}\geqq\varrho$. Denote by $\varrho(a_0), \varrho(a_1), \dots, \varrho(a_{s-1})$ the blocks of ϱ. From the equality $\varrho=\varrho_p$ it follows that $\varrho(a_i)=\{a_i, a_ip, \dots, a_ip^{k-1}\}$ $(i=0, 1, \dots, s-1)$. Thus $n=k\cdot s$. From this we get that $s\neq 1$ because k is prime. On the other hand, since $\mathbf{A}$ is strongly connected thus there exist words p_i, q_i $(i=0, \dots, s-1)$ such that $a_0p_i=a_i$ and $a_iq_i=a_0$ for all $i\in\{0, 1, \dots, s-1\}$. Using the commutativity of $\mathbf{A}$ we have $a_0p^jp_i=a_ip^j$ and $a_ip^jq_i=a_0p^j$ for any $j\in\{0, 1, \dots, k-1\}$ and $i\in\{0, 1, \dots, s-1\}$. Now define two automata $\mathbf{A}_1=(X, \varrho, \delta_1)$ and $\mathbf{A}_2=(\varrho\times X, \varrho(a_0), \delta_2)$ in the following way: $\delta_1(\varrho(a_i), x)=\varrho(\delta(a_i, x))$ for all $\varrho(a_i)\in\varrho$

and $x\in X$, $\delta_2(a_0p^j, (\varrho(a_i), x))=a_0p^jp_ixq_u$ if $\varrho(\delta(a_i, x))=\varrho(a_u)$ for all $a_0p^j\in\varrho(a_0)$ and $(\varrho(a_i), x)\in\varrho\times X$. Take the α_0-product $\mathbf{B}=\mathbf{A}_1\times\mathbf{A}_2(X, \varphi)$, where $\varphi_1(x)=x$ and $\varphi_2(\varrho(a_i), x)=(\varrho(a_i), x)$ for any $x\in X$ and $\varrho(a_i)\in\varrho$. It is not difficult to prove that the correspondence ν: $a_ip^j\to(\varrho(a_i), a_0p^j)$ $(i=0, 1, \ldots, s-1;\ j=0, 1, \ldots, k-1)$ is an isomorphism of $\mathbf{A}$ into $\mathbf{B}$. Now consider the automata $\mathbf{A}_1$ and $\mathbf{A}_2$. They are strongly connected commutative automata with number of states less then n. Therefore, by our assumption, the statement holds.

For any prime number r, let $\overline{\mathbf{M}}_r=(\{x_0, x_1, \ldots, x_r\}, \{0, \ldots, r\}, \delta)$ denote the automaton for which $\delta(l, x_j)=l+j \pmod r$, $\delta(r, x_j)=r$, $\delta(l, x_r)=r$ and $\delta(r, x_r)=r$ for any $l\in\{0, \ldots, r-1\}$ and $x_j\in\{x_0, \ldots, x_{r-1}\}$.

The next Theorem gives necessary and sufficient conditions for a system of automata to be isomorphically complete for $\mathfrak{K}$ with respect to the α_0-product.

Theorem 1. A system Σ of automata is isomorphically complete for $\mathfrak{K}$ with respect to the α_0-product if and only if the following conditions are satisfied:

(1) There exists $\mathbf{A}_0\in\Sigma$ such that the automaton $\mathbf{E}_2$ can be embedded isomorphically into an α_0-product of $\mathbf{A}_0$ with a single factor;

(2) For any prime number r there exists $\mathbf{A}\in\Sigma$ such that the automaton $\overline{\mathbf{M}}_r$ can be embedded isomorphically into an α_0-product of the automata $\mathbf{A}_0$ and $\mathbf{A}$.

Proof. In order to prove the necessity assume that Σ is isomorphically complete for $\mathfrak{K}$ with respect to the α_0-product. Then $\mathbf{E}_2$ can be embedded isomorphically into an α_0-product $\prod_{i=1}^{k}\mathbf{A}_i(\{x, y\}, \varphi)$ of automata from Σ. Assume that $k>1$ and let μ denote a suitable isomorphism. For any $j\in\{0, 1\}$ denote by $(a_{j1}, \ldots, a_{jk})$ the image of j under μ. Among the sets $\{a_{0t}, a_{1t}\}$ $(t=1, \ldots, k)$ there should be at least one which has more than one element. Let l be the least index for which $a_{0l}\neq a_{1l}$. It is obvious that the automaton $\mathbf{A}_l\in\Sigma$ satisfies condition (1).

Now take an arbitrary prime number r and consider the automaton $\overline{\mathbf{M}}_r$. By our assumption $\overline{\mathbf{M}}_r$ can be embedded isomorphically into an α_0-product $\prod_{i=1}^{k}\mathbf{A}_i(\{x_0, \ldots, x_r\}, \varphi)$ of automata from Σ. Assume that $k>1$ and let μ denote a suitable isomorphism. For any $t\in\{0, \ldots, r\}$ denote by $(a_{t1}, \ldots, a_{tk})$ the image of t under μ. Define compatible partitions π_j $(j=1, \ldots, k)$ of $\overline{\mathbf{M}}_r$ in the following way: for any $u, v\in\{0, \ldots, r\}$, $u\equiv v(\pi_j)$ if and only if $a_{u1}=a_{v1}, \ldots, a_{uj}=a_{vj}$. It is obvious that $\pi_1\geqq\pi_2\geqq\ldots\geqq\pi_k$ and π_k has one-element blocks only. On the other hand $\overline{\mathbf{M}}_r$ has only one nontrivial compatible partition: $\sigma=\{\{0, \ldots, r-1\}, \{r\}\}$. Denote by s the least index for which $\sigma>\pi_s$. It is not difficult to prove that the automaton $\mathbf{A}_s\in\Sigma$ satisfies condition (2).

To prove the sufficiency of the conditions of Theorem 1 we shall show that arbitrary commutative automaton can be embedded isomorphically into an α_0-product of automata from $\mathfrak{N}$ where $\mathfrak{N}=\{\mathbf{E}_2\}\cup\{\overline{\mathbf{M}}_r\colon r \text{ is a prime number}\}$.

We prove by induction on the number of states of the automaton. In the case $n\leqq2$ our statement is trivial. Now let $n>2$ and assume that for any $m<n$ the statement is valid. Denote by $\mathbf{A}=(X, A, \delta)$ an arbitrary commutative automaton with n states.

If $\mathbf{A}$ is not connected then it can be given as a direct sum of its connected subautomata. Denote by $\mathbf{A}_t=(X, A_t, \delta_t)$ $(t=1, \ldots, k)$ these subautomata of $\mathbf{A}$. Take an arbitrary symbol z such that $z\notin X$. Define the automata $\bar{\mathbf{A}}_i=(X\cup\{z\}, A_i, \bar{\delta}_i)$ $(i=1, \ldots, k)$ in the following way: $\bar{\delta}_i(a_i, x)=\delta_i(a_i, x)$ and $\bar{\delta}_i(a_i, z)=a_i$, for all $a_i\in A_i$ and $x\in X$ $(i=1, \ldots, k)$. Take the α_0-products $\mathbf{B}_i=\mathbf{E}_2\times\bar{\mathbf{A}}_i(X, \varphi^{(i)})$ $(i=1, \ldots, k)$ where $\varphi_1^{(i)}(x)=y$, $\varphi_2^{(i)}(0, x)=z$ and $\varphi_2^{(i)}(1, x)=x$ for all $x\in X$. It is clear that $\mathbf{A}$ can be embedded isomorphically into the direct product $\prod_{i=1}^{k}\mathbf{B}_i$. On the other hand, for any index i $(1\leq i\leq k)$ the automaton $\bar{\mathbf{A}}_i$ is commutative with number of states less than n. Therefore, by our induction hypothesis the statement holds.

Now assume that $\mathbf{A}$ is connected. Consider the partition $\{C(a)\colon a\in A\}$ and the partial ordering of blocks introduced on page 1. Since $\mathbf{A}$ is connected thus among the blocks there exists one maximal only. Let $C(\bar{a})$ denote this block. We distinguish two cases.

(I) Assume that the cardinality of $C(\bar{a})$ is greater than one. In this case $(X, C(\bar{a}), \delta_{|C(\bar{a})\times X})$ is a strongly connected subautomaton of $\mathbf{A}$. If $C(\bar{a})=A$ then, by Lemma 2 and Lemma 3, the statement holds. If $C(\bar{a})\subset A$ then we distinguish three cases.

(a) Assume that the cardinality of $C(\bar{a})$ is prime and denote it by r. Let us define the automata $\mathbf{A}_1=(X, (A\setminus C(\bar{a}))\cup\{*\}, \delta_1)$ and $A_2=(((A\setminus C(\bar{a}))\cup\{*\})\times X, C(\bar{a})\cup\{\square\}, \delta_2)$ in the following way:

$$\delta_1(a, x)=\begin{cases}\delta(a, x) & \text{if } \delta(a, x)\notin C(\bar{a}),\\ * & \text{otherwise,}\end{cases}$$

$$\delta_1(*, x)=*,$$

$$\delta_2(a', (a, x))=a',\quad \delta_2(a', (*, x))=\delta(a', x),\quad \delta_2(\square, (*, x))=\square,$$

$$\delta_2(\square, (a, x))=\begin{cases}\delta(a, x) & \text{if } \delta(a, x)\in C(\bar{a}),\\ \square & \text{otherwise,}\end{cases}$$

for all $x\in X$, $a\in A\setminus C(\bar{a})$ and $a'\in C(\bar{a})$. Take the α_0-product $\mathbf{B}=\mathbf{A}_1\times\mathbf{A}_2(X, \varphi)$ where $\varphi_1(x)=x$ and $\varphi_2(v, x)=(v, x)$ for any $x\in X$, $v\in(A\setminus C(\bar{a}))\cup\{*\}$. It can be proved easily that the correspondence

$$\nu(a)=\begin{cases}(a, \square) & \text{if } a\in A\setminus C(\bar{a}),\\ (*, a) & \text{if } a\in C(\bar{a}),\end{cases}$$

is an isomorphism of $\mathbf{A}$ into $\mathbf{B}$. Consider the automata $\mathbf{A}_1$ and $\mathbf{A}_2$. $\mathbf{A}_1$ is a commutative automaton with number of states less than n. Therefore, by our induction assuption, it can be decomposed in the form required. For investigating $\mathbf{A}_2$ we need the automaton $\mathbf{C}=(\{x_0, \ldots, x_r\}, \{0, \ldots, r\}, \delta_{\mathbf{C}})$ where $\delta_{\mathbf{C}}(l, x_i)=l+i \pmod r$, $\delta_{\mathbf{C}}(l, x_r)=l$, $\delta_{\mathbf{C}}(r, x_i)=i$ and $\delta_{\mathbf{C}}(r, x_r)=r$ for any $l\in\{0, \ldots, r-1\}$, $x_i\in\{x_0, \ldots, x_{r-1}\}$. Now denote by U the set of the input signals of $\mathbf{A}_2$ and consider the following partitions of U:

$$U_1=\{(*, x)\colon x\in X\}\cup\{(a, x)\colon a\in A\setminus C(\bar{a}),\ x\in X,\ \delta(a, x)\notin C(\bar{a})\},$$

$$U_2=\{(a, x)\colon a\in A\setminus C(\bar{a}),\ x\in X,\ \delta(a, x)\in C(\bar{a})\},$$

$$V_1=\{(a, x)\colon a\in A\setminus C(\bar{a}),\ x\in X\},$$

$$V_2=\{(*, x)\colon x\in X\}.$$

By Lemma 2, we have that $(X, C(\bar{a}), \delta_{|C(\bar{a})\times X})$ is isomorphic to an α_0-product of $\mathbf{M}_r$ with a single factor. Denote by μ this isomorphism. We write $a=a_i$ if $\mu(i)=a$ $(i=0, 1, \dots, r-1)$. Now take the α_0-product $\mathbf{E}_2\times\mathbf{C}(U, \varphi)$ where for any $u_1\in U_1$, $u_2\in U_2$ and $v_1\in V_1$, $v_2\in V_2$, $\varphi_1(u_1)=y$, $\varphi_1(u_2)=x$, $\varphi_2(0, u_1)=x_r$, $\varphi_2(0, u_2)=x_i$ if $\delta_2(\square, u_2)=a_i$, $\varphi_2(1, v_1)=x_r$ and $\varphi_2(1, v_2)=x_j$ if $\delta_2(a_0, v_2)=a_j$. It is clear that the correspondence ν given by $\nu(\square)=(0, r)$ and $\nu(a_i)=(1, i)$ $(i=0, \dots, r-1)$ is an isomorphism of $\mathbf{A}_2$ into $\mathbf{E}_2\times\mathbf{C}(U, \varphi)$. On the other hand, it is not difficult to prove that $\mathbf{C}$ can be embedded isomorphically into an α_0-product of $\mathbf{E}_2$ and $\mathbf{M}_r$. Thus $\mathbf{A}_2$ can be embedded isomorphically into an α_0-product of $\mathbf{E}_2$ and $\mathbf{M}_r$. Taking into consideration the above decomposition of $\mathbf{A}_1$, this ends the discussion of (a) in case (I).

(b) Assume that the cardinality of $C(\bar{a})$ is not prime and the partition ϱ of $(X, C(\bar{a}), \delta_{|C(\bar{a})\times X})$ has one-element blocks only where ϱ is defined for $(X, C(\bar{a}), \delta_{|C(\bar{a})\times X})$ in the same way as in the proof of Lemma 3. Now for any $\varrho_p\in\Omega$, define the partition $\bar{\varrho}_p$ of $\mathbf{A}$ in the following way:

$$\bar{\varrho}_p(a) = \begin{cases} \{a\} & \text{if} \quad a\in A\setminus C(\bar{a}), \\ \varrho_p(a) & \text{otherwise.} \end{cases}$$

Now let $\bar{\Omega}$ denote the set of all such $\bar{\varrho}_p$. It can easily be seen that $\mathbf{A}$ can be embedded isomorphically into the direct product $\prod_{\bar{\varrho}_p\in\bar{\Omega}} \mathbf{A}/\bar{\varrho}_p$. On the other hand for any $\bar{\varrho}_p\in\bar{\Omega}$ the quotient automaton $\mathbf{A}/\bar{\varrho}_p$ is commutative with number of states less than n. Thus, by our induction assumption, we have a required decomposition of $\mathbf{A}$.

(c) Assume that the cardinality of $C(\bar{a})$ is not prime and the partition ϱ of $(X, C(\bar{a}), \delta_{|C(\bar{a})\times X})$ has at least one block whose cardinality is greater than one. Then, by the proof of Lemma 3, $(X, C(\bar{a}), \delta_{|C(\bar{a})\times X})$ can be embedded isomorphically into an α_0-product of automata $\bar{\mathbf{A}}_1=(X, \varrho, \bar{\delta}_1)$ and $\bar{\mathbf{A}}_2=(\varrho\times X, \varrho(a_0), \bar{\delta}_2)$ where $\bar{\mathbf{A}}_2$ is isomorphic to an α_0-product of $\mathbf{M}_r$ with a single factor for some prime $r<n$. Define the automata $\mathbf{A}_1=(X, (A\setminus C(\bar{a}))\cup\varrho, \delta_1)$ and $\mathbf{A}_2=(((A\setminus C(\bar{a}))\cup\varrho)\times X, \varrho(a_0)\cup\{\square\}, \delta_2)$ in the following way: for any $a\in A\setminus C(\bar{a})$, $\varrho(a_i)\in\varrho$, $x\in X$ and $a_0p^j\in\varrho(a_0)$

$$\delta_1(\varrho(a_i), x) = \bar{\delta}_1(\varrho(a_i), x),$$

$$\delta_1(a, x) = \begin{cases} \delta(a, x) & \text{if} \quad \delta(a, x)\in A\setminus C(\bar{a}) \\ \varrho(a_i) & \text{if} \quad \delta(a, x)\in C(\bar{a}) \quad \text{and} \quad \delta(a, x)\in\varrho(a_i), \end{cases}$$

$$\delta_2(a_0p^j, (a, x)) = a_0p^j, \delta_2(a_0p^j, (\varrho(a_i), x)) = \bar{\delta}_2(a_0p^j, (\varrho(a_i), x)),$$

$$\delta_2(\square, (\varrho(a_i), x)) = \square,$$

$$\delta_2(\square, (a, x)) = \begin{cases} \delta(a, x)q_s & \text{if} \quad \delta(a, x)\in\varrho(a_s), \\ \square & \text{if} \quad \delta(a, x)\notin C(\bar{a}). \end{cases}$$

Notations used in the above definition coincide with those used in the proof of Lemma 3. Take the α_0-product $\mathbf{A}_1\times\mathbf{A}_2(X, \varphi)$ where $\varphi_1(x)=x$ and $\varphi_2(v, x)=(v, x)$ for any $x\in X$ and $v\in(A\setminus C(\bar{a}))\cup\varrho$. It can easily be seen that the correspondence

$$\nu(a) = \begin{cases} (a, \square) & \text{if} \quad a\in A\setminus C(\bar{a}), \\ (\varrho(a_i), a_0p^j) & \text{if} \quad a\in\varrho(a_i) \quad \text{and} \quad a = a_ip^j, \end{cases}$$

is an isomorphism of **A** into $\mathbf{A}_1\times\mathbf{A}_2(X,\varphi)$. Consider the automata $\mathbf{A}_1$ and $\mathbf{A}_2$. The automaton $\mathbf{A}_1$ is commutative with number of states less than n. Therefore, by our induction hypothesis, it can be decomposed in the form required. The automaton $\mathbf{A}_2$ can be embedded isomorphically into an α_0-product of automata $\mathbf{E}_2$ and $\mathbf{M}_r$. This can be proved in a similar way as in the case (a). Thus we get a required decomposition of **A**.

(II) Now assume that the cardinality of $C(\bar{a})$ is equal to one. Denote by R' the set of all $a\in A$ for which the cardinality of $C(a)$ is equal to one and $C(a)<C(b)$ implies $b=\bar{a}$ for all $b\in A$. Let R be the set $R'\cup\{\bar{a}\}$. We distinguish two cases:

(a) First assume that R' is nonvoid. Then $(X, R, \delta_{|R\times X})$ is a connected monotone subautomaton of **A**. Define the automata $\mathbf{A}_1=(X,(A\setminus R)\cup\{*\},\delta_1)$ and $A_2=(((A\setminus R)\cup\{*\})\times X, R\cup\{\square\},\delta_2)$ in the following way: for any $a\in A\setminus R$, $a'\in R$ and $x\in X$

$$\delta_1(a,x)=\begin{cases}\delta(a,x) & \text{if } \delta(a,x)\notin R,\\ * & \text{otherwise,}\end{cases}$$

$$\delta_1(*,x)=*,$$

$$\delta_2(a',(a,x))=a',\ \delta_2(a',(*,x))=\delta(a',x),\ \delta_2(\square,(*,x))=\square,$$

$$\delta_2(\square,(a,x))=\begin{cases}\delta(a,x) & \text{if } \delta(a,x)\in R,\\ \square & \text{otherwise.}\end{cases}$$

Take the α_0-product $\mathbf{A}_1\times\mathbf{A}_2(X,\varphi)$ where $\varphi_1(x)=x$, $\varphi_2(v,x)=(v,x)$ for any $x\in X$ and $v\in(A\setminus R)\cup\{*\}$. It is obvious that the correspondence

$$v(a)=\begin{cases}(a,\square) & \text{if } a\in A\setminus R,\\ (*,a) & \text{if } a\in R,\end{cases}$$

is an isomorphism of **A** into $\mathbf{A}_1\times\mathbf{A}_2(X,\varphi)$. Consider $\mathbf{A}_1$ and $\mathbf{A}_2$. $\mathbf{A}_1$ is commutative with number of states less than n. Thus by our induction assumption, it can be decomposed in the form required. On the other hand $\mathbf{A}_2$ is a connected monotone automaton thus, by Lemma 1, it can be embedded isomorphically into an α_0-power of $\mathbf{E}_2$. Therefore, we get a required decomposition of **A**.

(b) Now assume that R' is empty. Denote by Q the set of all blocks $C(a)$ for which the cardinality of $C(a)$ is greater than one, and $C(a)<C(b)$ implies $b=\bar{a}$ for all $b\in A$. Since **A** is connected and R' is empty thus the set Q contains at least one block. We distinguish two cases.

(1) First assume that Q contains the bloks $C(a_1),\ldots,C(a_k)$ where $k>1$. Define compatible partitions ϱ_i $(i=1,\ldots,k)$ of **A** in the following way:

$$\varrho_i(a)=\begin{cases}\{a\} & \text{if } a\notin C(a_i)\cup\{\bar{a}\},\\ C(a_i)\cup\{\bar{a}\} & \text{otherwise.}\end{cases}$$

It is not difficult to prove that $\bigcap_{1\le i\le k}\varrho_i=\{\{a\}: a\in A\}$. From this we get that **A** can be embedded isomorphically into the direct product $\prod_{i=1}^{k}\mathbf{A}/\varrho_i$. On the other hand, for any $i\in\{1,\ldots,k\}$ the quotient automaton $\mathbf{A}/\varrho_i$ is commutative with number of states less than n. Therefore, by our induction assumption, we have a required decomposition of **A**.

(2) Now assume that Q contains one block only and denote it by $C(b)$. Since C is a compatible partition of **A** thus $\{X_1, X_2\}$ is a partition of X where $X_1 = \{x: x \in X, C(b)x \subseteq C(b)\}$ and $X_2 = \{x: x \in X, C(b)x = \bar{a}\}$. It is clear that X_1 and X_2 are nonvoid sets and $\mathbf{B} = (X_1, C(b), \delta_{|C(b) \times X_1})$ is a strongly connected commutative automaton. Now we distinguish three cases according to Lemma 3.

(i) Assume that the number of states of **B** is prime and denote it by r. Define the automata $\mathbf{A}_1 = (X, (A \setminus (C(b) \cup \{\bar{a}\})) \cup \{*\}, \delta_1)$ and $\mathbf{A}_2 = (((A \setminus (C(b) \cup \{\bar{a}\})) \cup \{*\}) \times X, C(b) \cup \{\bar{a}, \square\}, \delta_2)$ in the following way: for any $x \in X$, $a \in A \setminus (C(b) \cup \{\bar{a}\})$ and $a' \in C(b) \cup \{\bar{a}\}$

$$\delta_1(a, x) = \begin{cases} \delta(a, x) & \text{if} \quad \delta(a, x) \notin C(b) \cup \{\bar{a}\}, \\ * & \text{otherwise,} \end{cases}$$

$$\delta_1(*, x) = *,$$

$$\delta_2(\square, (a, x)) = \begin{cases} \delta(a, x) & \text{if} \quad \delta(a, x) \in C(b) \cup \{\bar{a}\}, \\ \square & \text{otherwise,} \end{cases}$$

$$\delta_2(a', (a, x)) = a', \quad \delta_2(a', (*, x)) = \delta(a', x), \quad \delta_2(\square, (*, x)) = \square.$$

Take the α_0-product $\mathbf{A}_1 \times \mathbf{A}_2(X, \varphi)$ where $\varphi_1(x) = x$ and $\varphi_2(v, x) = (v, x)$ for any $x \in X$, $v \in (A \setminus (C(b) \cup \{\bar{a}\})) \cup \{*\}$. It is clear that the correspondence

$$\nu(a) = \begin{cases} (a, \square) & \text{if} \quad a \notin C(b) \cup \{\bar{a}\}, \\ (*, a) & \text{if} \quad a \in C(b) \cup \{\bar{a}\} \end{cases}$$

is an isomorphism of **A** into $\mathbf{A}_1 \times A_2(X, \varphi)$. Consider the factors of the previous α_0-product. $\mathbf{A}_1$ is commutative with number of states less than n. Thus, by our induction assumption it can be decomposed in the required form. For investigating $\mathbf{A}_2$, we need the following automaton. Denote by $\mathbf{W} = (\{x_0, \ldots, x_r, \bar{x}\}, \{0, \ldots, r, \bar{r}\}, \delta_\mathrm{W})$ the automaton where $\delta_\mathrm{W}(l, x_i) = l + i \pmod r$, $\delta_\mathrm{W}(\bar{r}, x_i) = i$, $\delta_\mathrm{W}(l, x_r) = r$, $\delta_\mathrm{W}(l, \bar{x}) = l$, $\delta_\mathrm{W}(r, x_i) = r$ for any $l \in \{0, \ldots, r-1\}$ and $x_i \in \{x_0, \ldots, x_{r-1}\}$, and $\delta_\mathrm{W}(r, x_r) = \delta_\mathrm{W}(r, \bar{x}) = \delta_\mathrm{W}(\bar{r}, x_r) = r$, $\delta_\mathrm{W}(\bar{r}, \bar{x}) = \bar{r}$. Now denote by U the set of the input signals of $\mathbf{A}_2$ and take the following partitions of U.

$$U_1 = \{(*, x): x \in X\} \cup \{(a, x): a \in A \setminus (C(b) \cup \{\bar{a}\}),\ x \in X,\ \delta(a, x) \notin C(b) \cup \{\bar{a}\}\},$$

$$U_2 = \{(a, x): a \in A \setminus (C(b) \cup \{\bar{a}\}),\ x \in X,\ \delta(a, x) \in C(b)\},$$

$$U_3 = \{(a, x): a \in A \setminus (C(b) \cup \{\bar{a}\}),\ x \in X,\ \delta(a, x) = \bar{a}\},$$

$$V_1 = \{(a, x): a \in A \setminus (C(b) \cup \{\bar{a}\}),\ x \in X\},$$

$$V_2 = \{(*, x): x \in X_1\} \quad \text{and} \quad V_3 = \{(*, x): x \in X_2\}.$$

By definitions, we have that $(V_1 \cup V_2, C(b), \delta_{2|C(b) \times (V_1 \cup V_2)})$ is a strongly connected commutative automaton with r states. Thus, by Lemma 2, it is isomorphic to an α_0-product of $\mathbf{M}_r$ with a single factor. Denote by μ a suitable isomorphism, and for any $t \in \{0, 1, \ldots, r-1\}$ denote by b_t the image of t under μ. Now take the α_0-product $\mathbf{E}_2 \times \mathbf{W}(U, \varphi)$ where $\varphi_1(u_1) = y$, $\varphi_1(u_2) = \varphi_1(u_3) = x$, $\varphi_2(0, u_1) = \bar{x}$,

$\varphi_2(0, u_2)=x_i$ if $\delta_2(\Box, u_2)=b_i$, $\varphi_2(0, u_3)=x_r$, $\varphi_2(1, v_1)=\bar{x}$, $\varphi_2(1, v_2)=x_s$ if $\delta_2(b_0, v_2)=b_s$, $\varphi_2(1, v_3)=x_r$ for any $u_t\in U_t$ $(t=1, 2, 3)$, $v_j\in V_j$ $(j=1, 2, 3)$. It is obvious that the correspondence ν given by $\nu(\Box)=(0, \bar{r})$, $\nu(\bar{a})=(1, r)$, $\nu(b_i)=(1, i)$ $(i=0, \dots, r-1)$ is an isomorphism of $\mathbf{A}_2$ into $\mathbf{E}_2\times\mathbf{W}(U, \varphi)$. On the other hand, it is not difficult to prove that the automaton $\mathbf{W}$ can be embedded isomorphically into an α_0-product of $\mathbf{E}_2$ and $\overline{\mathbf{M}}_r$. Thus we get a required decomposition of $\mathbf{A}$.

(ii) Assume that the number of states of $\mathbf{B}$ is not prime and the partition ϱ of $\mathbf{B}$ has one-element blocks only where ϱ is defined for $\mathbf{B}$ in the same way as above. Now for any $\varrho_p\in\Omega$ define a partition $\bar{\varrho}_p$ of $\mathbf{A}$ in the following way:

$$\bar{\varrho}_p(a) = \begin{cases} \{a\} & \text{if} \quad a\in A\setminus C(\bar{a}), \\ \varrho_p(a) & \text{otherwise.} \end{cases}$$

Let $\bar{\Omega}$ denote the set of all such $\bar{\varrho}_p$. It is clear that $\mathbf{A}$ can be embedded isomorphically into the direct product $\prod_{\bar{\varrho}_p\in\bar{\Omega}} \mathbf{A}/\bar{\varrho}_p$. The quotient automaton $\mathbf{A}/\bar{\varrho}_p$ is commutative and its number of states is less than n for any $\bar{\varrho}_p\in\bar{\Omega}$. Thus, by our induction assumption we have a required decomposition of $\mathbf{A}$.

(iii) Assume that the number of states of $\mathbf{B}$ is not prime and the partition ϱ of $\mathbf{B}$ has at least one block whose cardinality is greater than one. Then, by Lemma 3, $\mathbf{B}$ can be embedded isomorphically into an α_0-product of the automata $\mathbf{B}_1=(X_1, \varrho, \bar{\delta}_1)$ and $\mathbf{B}_2=(\varrho\times X_1, \varrho(b_0), \bar{\delta}_2)$ where $\mathbf{B}_2$ is isomorphic to an α_0-product of $\mathbf{M}_r$ with a single factor for some prime r. Define the automata $\mathbf{A}_1=(X, (A\setminus C(b))\cup\varrho, \delta_1)$ and $\mathbf{A}_2=(((A\setminus C(b))\cup\varrho)\times X, \varrho(b_0)\cup\{*, \Box\}, \delta_2)$ in the following way: for any $a\in A\setminus C(b)$, $\varrho(b_i)\in\varrho$, $x\in X$ and $b_0p^j\in\varrho(b_0)$

$$\delta_1(a, x) = \begin{cases} \delta(a, x) & \text{if} \quad \delta(a, x)\notin C(b), \\ \varrho(\delta(a, x)) & \text{otherwise,} \end{cases}$$

$$\delta_1(\varrho(b_i), x) = \begin{cases} \bar{\delta}_1(\varrho(b_i), x) & \text{if} \quad x\in X_1, \\ \bar{a} & \text{if} \quad x\in X_2, \end{cases}$$

$$\delta_2(b_0p^j, (\varrho(b_i), x)) = \begin{cases} \bar{\delta}_2(b_0p^j, (\varrho(b_i), x)) & \text{if} \quad x\in X_1, \\ * & \text{if} \quad x\in X_2, \end{cases}$$

$$\delta_2(\Box, (a, x)) = \begin{cases} \Box & \text{if} \quad \delta(a, x)\in A\setminus(C(b)\cup\{\bar{a}\}), \\ \delta(a, x)q_s & \text{if} \quad \delta(a, x)\in\varrho(b_s), \\ * & \text{if} \quad \delta(a, x)=\bar{a}, \end{cases}$$

$$\delta_2(b_0p^j, (a, x)) = b_0p^j, \quad \delta_2(*, (a, x)) = \delta_2(*, (\varrho(b_i), x)) = *,$$

$$\delta_2(\Box, (\varrho(b_i), x)) = \Box.$$

(The notations coincide with those used in the proof of the Lemma 3.) Take the α_0-product $\mathbf{A}_1\times\mathbf{A}_2(X, \varphi)$ where $\varphi_1(x)=x$ and $\varphi_2(v, x)=(v, x)$ for any $x\in X$ and $v\in(A\setminus C(b))\cup\varrho$. It is not difficult to prove that the correspondence

$$\nu(a) = \begin{cases} (a, \Box) & \text{if} \quad a\in A\setminus(C(b)\cup\{\bar{a}\}), \\ (\varrho(b_i), b_0p^j) & \text{if} \quad a\in C(b) \ \text{and} \quad a=b_ip^j, \\ (\bar{a}, *) & \text{if} \quad a=\bar{a}, \end{cases}$$

is an isomorphism of **A** into $\mathbf{A}_1 \times \mathbf{A}_2(X, \varphi)$. Consider the automata $\mathbf{A}_1$ and $\mathbf{A}_2$. The automaton $\mathbf{A}_1$ is commutative with number of states less than n. Thus, by our induction assumption, it can be decomposed in the required form. The automaton $\mathbf{A}_2$ can be embedded isomorphically into an α_0-product of $\mathbf{E}_2$ and $\overline{\mathbf{M}}_r$. This can be proved in a similar way as in the case (i). Thus we get a required decomposition of **A**.

The following statement is obvious for arbitrary natural number $i \geqq 0$.

Lemma 4. If **A** can be embedded isomorphically into an α_i-product of **B** with a single factor and **B** can be embedded isomorphically into an α_i-product of **C** with a single factor, then **A** can be embedded isomorphically into an α_i-product of **C** with a single factor.

The next Theorem holds for α_i-products with $i \geqq 1$.

Theorem 2. A system Σ of automata is isomorphically complete for $\mathfrak{K}$ with respect to the α_i-product ($i \geqq 1$) if and only if for any prime number r there exists an automaton $\mathbf{A} \in \Sigma$ such that $\mathbf{M}_r$ can be embedded isomorphically into an α_i-product of **A** with a single factor.

Proof. To prove the sufficiency, by Lemma 4, it is enough to show that arbitrary automaton with n states can be embedded isomorphically into an α_1-product of $\mathbf{M}_r$ with a single factor for some prime $r > n$. This is trivial.

To prove the necessity take a prime r. First we prove that $\mathbf{M}_r$ can be embedded isomorphically into an α_i-product of automata from Σ with at most i factors if $\mathbf{M}_r$ can be embedded isomorphically into an α_i-product of automata from Σ. Indeed, assume that $\mathbf{M}_r$ can be embedded isomorphically into the α_i-product $\mathbf{B} = \prod_{j=1}^{k} \mathbf{A}_j(\{x_0, \ldots, x_{r-1}\}, \varphi)$ of automata from Σ with $k > i$ and denote by μ such an isomorphism. For any $l \in \{0, \ldots, r-1\}$ denote by $(a_{l1}, \ldots, a_{lk})$ the image of l under μ. We may suppose that there exist natural numbers $s \neq t$ $(0 \leqq s, t \leqq r-1)$ such that $a_{s1} \neq a_{t1}$ since in the opposite case $\mathbf{M}_r$ can be embedded isomorphically into an α_i-product of automata from Σ with $k-1$ factors. Now assume that there exist natural numbers $u \neq v$ $(0 \leqq u, v \leqq r-1)$ such that $a_{ul} = a_{vl}$ $(l = 1, \ldots, i)$. Then $\varphi_1(a_{u1}, \ldots, a_{ui}, x_j) = \varphi_1(a_{v1}, \ldots, a_{vi}, x_j)$ for any $x_j \in \{x_0, \ldots, x_{r-1}\}$. Thus in the α_i-product **B** the automaton $\mathbf{A}_1$ obtains the same input signal in the states a_{u1} and a_{v1} for any $x_j \in \{x_0, \ldots, x_{r-1}\}$. Since μ is an isomorphism thus we have that $a_{u+j(\mathrm{mod}\, r)1} = a_{v+j(\mathrm{mod}\, r)1}$ for any $j \in \{0, \ldots, r-1\}$. On the other hand, r is prime thus from the above equations we get that $a_{u1} = a_{l1}$ for any $l \in \{0, \ldots, r-1\}$ which contradicts our assumption. Therefore, we have that the elements $(a_{l1}, \ldots, a_{li})$ $(0 \leqq l \leqq r-1)$ are pairwise different. Take the following α_i-product $\mathbf{C} = \prod_{t=1}^{i} \mathbf{A}_t(\{x_0, \ldots, x_{r-1}\}, \psi)$ where for any $j \in \{1, \ldots, i\}$, $(a_1, \ldots, a_i) \in A_1 \times \ldots \times A_i$ and $x_s \in \{x_0, \ldots, x_{r-1}\}$

$$\psi_j(a_1, \ldots, a_i, x_s) = \begin{cases} \varphi_j(a_{l1}, \ldots, a_{lj+i-1}, x_s) & \text{if } j+i-1 \leqq k \text{ and there exists} \\ \quad 0 \leqq l \leqq r-1 \text{ such that } a_u = a_{lu} \quad (u = 1, \ldots, i), \\ \varphi_j(a_{l1}, \ldots, a_{lk}, x_s) & \text{if } j+i-1 > k \text{ and there exists} \\ \quad 0 \leqq l \leqq r-1 \text{ such that } a_u = a_{lu} \quad (u = 1, \ldots, i), \\ \text{arbitrary input signal from } X_j \text{ otherwise.} \end{cases}$$

It is not difficult to prove that the correspondence $\nu(l)=(a_{l1}, \ldots, a_{li})$ $(l=0, \ldots, r-1)$ is an isomorphism of $\mathbf{M}_r$ into $\mathbf{C}$.

Now we prove that if $\mathbf{M}_r$ can be embedded isomorphically into an α_i-product $\prod_{j=1}^{k} \mathbf{A}_j(\{x_0, \ldots, x_{r-1}\}, \varphi)$ of automata from Σ with $k \leqq i$, then there exists an automaton $\mathbf{A} \in \Sigma$ such that $\mathbf{M}_{\mathrm{prime}[\sqrt[i]{r}]}$ can be embedded isomorphically into an α_i-product of $\mathbf{A}$ with a single factor, where prime $[\sqrt[i]{r}]$ denotes the largest prime less than $\sqrt[i]{r}$. Denote by μ such an isomorphism. For any $l \in \{0, \ldots, r-1\}$ denote by $(a_{l1}, \ldots, a_{lk})$ the image of l under μ. Since μ is a $1-1$ mapping thus the elements $(a_{l1}, \ldots, a_{lk})$ $(l=0, \ldots, r-1)$ are pairwise different. Therefore, there exists an s $(1 \leqq s \leqq k)$ such that the number of pairwise different elements among $a_{0s}, a_{1s}, \ldots, a_{r-1s}$ is greater than or equal to prime $[\sqrt[i]{r}]$. Let $a_{j_0 s}, \ldots, a_{j_{u-1} s}$ denote pairwise different elements, where $u=$ prime $[\sqrt[i]{r}]$, and denote by $\bar{X}$ the set $\{x_0, \ldots, x_{u-1}\}$. Take the α_0-product $\mathbf{C}=\Pi \mathbf{A}_s(\bar{X}, \psi)$ with a single factor, where for any $a_{j_t s} \in \{a_{j_0 s}, \ldots, a_{j_{u-1} s}\}$ and $x_v \in \bar{X}$, $\psi(a_{j_t s}, x_v)=\varphi_s(a_{j_t 1}, \ldots, a_{j_t k}, x_d)$ if $\delta_{\mathbf{M}_r}(\mu^{-1}(a_{j_t 1}, \ldots, a_{j_t k}), x_d)= =\mu^{-1}(a_{j_{t+v(\mathrm{mod}\, u)} 1}, \ldots, a_{j_{t+v(\mathrm{mod}\, u)} k})$. It is not difficult to prove that $\mathbf{M}_u$ can be embedded isomorphically into $\mathbf{C}$ which ends the proof of Theorem 2.

From Theorem 2 we get the following.

Corollary. A system Σ of automata is isomorphically complete for $\mathfrak{K}$ with respect to the α_i-product if and only if it is isomorphically complete with respect to the α_i-product $(i \geqq 1)$.

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