On the complexity of codes and pre-codes assigned to finite Moore automata

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§ 1.

The concepts of code (a table describing a Moore automaton such that each isomorphy family of automata contains precisely one automaton describable by a code), pre-code (an initial part of a code) and complexity (maximum of the distinguishability numbers for the state pairs of an automaton) were introduced in the earlier article [3]. In the present paper, the study of these notions and some related ones is continued.

In § 6 of [3] the following question was raised (Problem 4): *Is the set of complexities of all pre-codes fulfilling $s=0$ equal to the set of non-negative integers?* The main results of the present paper yield an affirmative answer to this question.

On one hand, we show that each pre-code with $s=0$ is of finite complexity. The proof of this theorem occupies Sections 3—5 of the paper.

The difficulties that arise in this proof follow from two motives. First, the continuation of a pre-code **D** with $s=0$ (till when we get a code) is permitted only in such a way that a certain distinguished role of **D** should be preserved in the whole code, too. Secondly, our basic idea gives a fundamental role to the rows of the code which satisfy $\gamma(i)=n$ (where n is the largest possible value of γ); since $\gamma(i)=n$ can be fulfilled already by some rows of the pre-code **D**, these rows must be handled very carefully during the procedure.

On the other hand, we obtain in § 6 (by a simple construction) that each non-negative integer is the complexity of an appropriate pre-code satisfying $s=0$. This construction enables us to derive in § 7 an interrelation between the complexity and the number of states of a Moore automaton.

The last section of the paper presents an example illustrating the constructions used in the proof of Theorem 1.

§ 2.

Most of the notions, to be defined in this section, were treated also in [3]. We denote by $\mathbf{N}_i^j$ the set

$$\{i, i+1, i+2, \ldots, j-1, j\}$$

of integers.

The (ordered) set $X=\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ (the set of input signs) is thought *to be fixed for the whole paper* $(n\geqq 1)$. $F(X)$ is the free monoid generated by X, the elements of $F(X)$ are often called *words*. The *length* $L(p)$ of a word $p=x_1x_2\ldots x_k$ is the number k (where $x_1\in X, x_2\in X, \ldots, x_k\in X$). We denote by $p_k^{(i)}$ the word consisting of k copies of $x^{(i)}$ $(1\leqq i\leqq n)$ (this notation will be used with $i=n$).

By a *pre-code* a sextuple $\mathbf{D}=(r, s, \beta, \gamma, \mu, \varphi)$ is meant such that the following eight postulates are satisfied:

(I) r, s are non-negative integers; $\beta, \gamma, \mu, \varphi$ are functions.

(II) The domains of $\beta, \gamma, \mu, \varphi$ are $\mathbf{N}_2^{r+s+1}, \mathbf{N}_2^{r+s+1}, \mathbf{N}_1^{r+1}, \mathbf{N}_{r+2}^{r+s+1}$, resp.

(III) The target of each of β, μ, φ is $\mathbf{N}_1^{r+1}$.

(IV) The target of γ is $\mathbf{N}_1^n$.

(V) $\beta(2)=1$. If $i\in\mathbf{N}_3^{r+1}$, then (a)&((b)$\vee$(c)) where

$$\text{(a)}\quad \beta(i-1)\leqq\beta(i)<i,$$

$$\text{(b)}\quad \beta(i-1)<\beta(i),$$

$$\text{(c)}\quad \gamma(i-1)<\gamma(i).$$

(VI) If $i\in\mathbf{N}_1^{r+1}$, then $\mu(i)-1\in\{0, \mu(1), \mu(2), \ldots, \mu(i-1)\}$.

(VII) If $i\in\mathbf{N}_{r+2}^{r+s+1}$, then $(\beta(i), \gamma(i))$ is the lexicographically smallest pair fulfilling

$$j\in\mathbf{N}_1^{i-1}\Rightarrow(\beta(i)\neq\beta(j)\vee\gamma(i)\neq\gamma(j)).$$

(VIII) If $i\in\mathbf{N}_{r+2}^{r+s+1}$, then either $\varphi(i)=1$ or (d)&((e)$\vee$(f)) where

$$\text{(d)}\quad \beta(\varphi(i))\leqq\beta(i),$$

$$\text{(e)}\quad \beta(\varphi(i))<\beta(i),$$

$$\text{(f)}\quad \gamma(\varphi(i))<\gamma(i).$$

The number $r+s+1$ is called the *size* of the pre-code $\mathbf{D}=(r, s, \beta, \gamma, \mu, \varphi)$. The quintuple $(i, \beta(i), \gamma(i), \mu(i), \varphi(i))$ is called the i^{th} row of the pre-code $\mathbf{D}$ $(i\in\mathbf{N}_1^{r+s+1})$. We use the notation $\mathbf{D}_1<\mathbf{D}_2$ if the pre-code $\mathbf{D}_2$ can be obtained from $\mathbf{D}_1$ by adding new rows (as last ones). We write $\mathbf{D}_1\prec\mathbf{D}_2$ when $\mathbf{D}_1<\mathbf{D}_2$ holds and $\mathbf{D}_2$ has one more row than $\mathbf{D}_1$. It can be shown that $s\leqq rn+n-r$ is valid for each pre-code.

If $\mathbf{D}_1$ is a pre-code and there exists no pre-code $\mathbf{D}_2$ satisfying $\mathbf{D}_1<\mathbf{D}_2$ (or, equivalently, if s takes its maximal possible value $rn+n-r$ in $\mathbf{D}_1$), then $\mathbf{D}_1$ is called a *code*.

The *first block* of a pre-code $\mathbf{D}$ consists of the first row only. The *second block* of $\mathbf{D}$ consists of the second, third, ..., $(r+1)^{\text{th}}$ rows. The *third block* consists of the $(r+2)^{\text{th}}, (r+3)^{\text{th}}, \ldots, (r+s+1)^{\text{th}}$ rows.

A pre-code $\mathbf{D}$ is called to be of *first type* if $r=0$. $\mathbf{D}$ is of *second type* if $s=0$. $\mathbf{D}$ is of *third type* if $r>0$ and[1] $s>0$. It is clear that each pre-code with at least two rows belongs to precisely one type, moreover, no code is of second type.

[1] These notions may be defined in terms of the emptiness of the second or third block, too. — We write out all the six components of a pre-code $\mathbf{D}=(r, s, \beta, \gamma, \mu, \varphi)$ even if some of the four functions does not exist really.

The iteration of the function β is defined by the recursion $\beta^0(i)=i$, $\beta^{k+1}(i)=$ $=\beta(\beta^k(i))$.

By an *automaton* we mean always an initially connected finite Moore automaton $\mathbf{A}=(A, X, Y, \delta, \lambda, a_1)$. To each code $\mathbf{C}$ we assign an automaton $\psi(\mathbf{C})$ constructed in the following manner:

$$A = \{a_1, a_2, \ldots, a_{r+1}\},$$

$$\delta(a_{\beta(i)}, x^{(\gamma(i))}) = \begin{cases} a_i & \text{if } i \leqq r+1, \\ a_{\varphi(i)} & \text{if } i \geqq r+2, \end{cases}$$

$$\lambda(a_i) = y_{\mu(i)}.$$

It is known that *to each standard automaton* $\mathbf{A}$ *there is exactly one code* $\mathbf{C}$ *such that* $\mathbf{A}$ *and* $\psi(\mathbf{C})$ *are isomorphic* (see §§ 3—4 of [3]).

We use extensively the well-known visualization of automata (or their parts) by directed graphs. This method can be transferred (by virtue of the assignment ψ) also for codes and pre-codes. If $\mathbf{C}$ is a code and $\mathbf{D}$ is the pre-code consisting of the first and second blocks of $\mathbf{C}$, then the graph of $\mathbf{D}$ is a spanning subtree of the graph of $\mathbf{C}$ (and any edge of $\mathbf{D}$ is directed outwards from a_1).

If a, b are states of an automaton $\mathbf{A}$, then we define $\omega(a, b)$ as the length $L(p)$ of a shortest word p such that

$$\lambda(\delta(a, p)) \neq \lambda(\delta(b, p)). \tag{2.1}$$

If (2.1) holds, then we say that p *distinguishes* a and b (for the automaton $\mathbf{A}$ or for the code $\psi^{-1}(\mathbf{A})$).

The *complexity* $\Omega_A(\mathbf{A})$ of $\mathbf{A}$ is the maximum of the values $\omega(a, b)$ where $a \neq b$. The complexity $\Omega_C(\mathbf{C})$ of a code $\mathbf{C}$ is defined by $\Omega_C(\mathbf{C})=\Omega_A(\psi(\mathbf{C}))$. Finally, the complexity $\Omega_C(\mathbf{D})$ of a pre-code $\mathbf{D}$ means the minimum of all complexities $\Omega_C(\mathbf{C})$ where $\mathbf{D} \leqq \mathbf{C}$.

The following two statements (exposed in [3] as Propositions 13, 19) will be used often in our further considerations (with or without an explicit reference):

Proposition A. *If* $i \in \mathbf{N}_2^{r+s+1}$, $j \in \mathbf{N}_2^{r+s+1}$, $\beta(i)=\beta(j)$, $\gamma(i)=\gamma(j)$ *are valid for a pre-code, then* $i=j$.

Proposition B. *If the pre-codes* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *satisfy* $\mathbf{D}_1 < \mathbf{D}_2$, *then* $\Omega_C(\mathbf{D}_1) \leqq$ $\leqq \Omega_C(\mathbf{D}_2)$.

§ 3.

In §§ 3—5 we prove the following result:

Theorem 1. *If* $\mathbf{D}$ *is a pre-code of second type, then its complexity* $\Omega_C(\mathbf{D})$ *is finite.*

In the proof of the theorem two constructions will have essential roles (each of them transforms a pre-code to another pre-code and augments the size by one).

CONSTRUCTION 1. Let $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ be an arbitrary pre-code of second type. Introduce the pre-code $\Gamma_1(\mathbf{D})=(r_1, s_1, \beta_1, \gamma_1, \mu_1, \varphi_1)$ by the following rules (i), (ii):

(i) $\Gamma_1(\mathbf{D})$ is of second type and $\mathbf{D} \prec \Gamma_1(\mathbf{D})$. (Hence $s_1=0$ and $r_1=r+1$.)
(ii) The function values at the place $r+2$ are:

$$\beta_1(r+2) = r+1,$$
$$\gamma_1(r+2) = n,$$
$$\mu_1(r+2) = \max\big(\mu(1), \mu(2), \ldots, \mu(r+1)\big)+1.$$

Proposition 1. *The pre-code $\Gamma_1(\mathbf{D})$ exists.*

Proof. The proposition asserts that $\Gamma_1(\mathbf{D})$, as it is determined by Construction 1, satisfies all the postulates (I)—(VIII). Most postulates are obviously fulfilled, except (V) in the particular case $i=r+2(=r_1+1)$.

(V) is completely satisfied since

$$\beta_1(r+2) = r+1 \begin{cases} > \beta(r+1) = \beta_1(r+1), \\ < r+2. \end{cases} \qquad \square$$

Before exposing Construction 2, we define some notions[2] for a pre-code $\mathbf{D}_0$. The set of numbers

$$\{r+1, \beta(r+1), \beta^2(r+1), \beta^3(r+1), \ldots, 1\}$$

is denoted by[3] H.

The set of all numbers $j(\in \mathbf{N}_2^{r+1})$ fulfilling at least one of the subsequent conditions (α), (β) is denoted by G:

(α) $\gamma(j)=n$,

(β) there is a number $h(\in \mathbf{N}_2^{r+1})$ such that $\beta(h)=j$ and $\gamma(h)=n$.

The set of numbers j which satisfy (α) but do not satisfy (β) are denoted by G_1. The set of numbers j which fulfil (β) but do not fulfil (α) are denoted by G_2. (Hence $G_1 \cap G_2 = \emptyset$ and $G_1 \cup G_2 \subseteq G$.)

Consider the subgraph induced by the vertex set G in the tree assigned to the pre-code consisting of the first and second blocks of $\mathbf{D}_0$. Each connected component of the induced subgraph is a path having at least two vertices. G_2 consists of the starting vertices of the connected components, G_1 consists of their end vertices.

We denote by G_h the set of numbers $i(\in G)$ such that the connected component (of G) containing i intersects H. Let G_g be the complementary set $G-G_h$. The intersection of H and a connected component C of G_h is a starting subpath of C. We define $G_{1,h}$, $G_{1,g}$ by $G_{1,h}=G_1 \cap G_h$ and $G_{1,g}=G_1 \cap G_g$.

If $j \in G_1$, then we denote by $\tau(j)$ the element of G_2 lying in the same connected component (of G) as j. Evidently, τ is a bijection of G_1 to G_2, and the containments $\tau(j) \in H$, $j \in G_{1,h}$ are equivalent. If $j \in G_{1,h}-H$, then we denote by $\tau'(j)$ the number $\beta^{w_0}(j)$ where w_0 is the smallest among the numbers w fulfilling $\beta^w(j) \in H$.

Construction 2. Let $\mathbf{D}_0=(r, s, \beta, \gamma, \mu, \varphi)$ be a pre-code of second or third type. We denote by $\mathbf{D}$ the pre-code consisting of the first and second blocks of $\mathbf{D}_0$. Let t mean the size $r+s+1$ of $\mathbf{D}_0$.

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[2] We do not specify the type of $\mathbf{D}_0$. The notions to be defined are independent of the third block of $\mathbf{D}_0$ (even if $\mathbf{D}_0$ belongs to the third type).

[3] The elements of H were enumerated here in decreasing order.

We introduce a pre-code $\Gamma_2(\mathbf{D}_0)=(r_2, s_2, \beta_2, \gamma_2, \mu_2, \varphi_2)$ by the subsequent two rules (iii), (iv):

(iii) $\Gamma_2(\mathbf{D}_0)$ is of third type and $\mathbf{D}_0 \prec \Gamma_2(\mathbf{D}_0)$. (Thus $r_2=r, s_2=s+1$ and the size r_2+s_2+1 of $\Gamma_2(\mathbf{D}_0)$ equals $t+1$.)

(iv) The value $\varphi_2(t+1)$ is prescribed[4] according to six cases (a)—(f) as follows:

(a) If $\gamma_2(t+1)<n$, then $\varphi_2(t+1)=1$.

(b) If $\gamma_2(t+1)=n$ and $\beta_2(t+1)=r+1$, then $\varphi_2(t+1)=r+1$.

(c) If $\gamma_2(t+1)=n, \beta_2(t+1)\leqq r$ and $\beta_2(t+1)\in H$, then $\varphi_2(t+1)$ is the smallest element of the set

$$\mathbf{N}^{r+1}_{\beta_2(t+1)+1}\cap H.$$

(d) If $\gamma_2(t+1)=n$ and $\beta_2(t+1)\in G_{1,h}-H$, then $\varphi_2(t+1)$ is the smallest element of the set

$$\mathbf{N}^{r+1}_{\tau'(\beta_2(t+1))+1}\cap H.$$

(e) If $\gamma_2(t+1)=n$ and $\beta_2(t+1)\in G_{1,g}$, then $\varphi_2(t+1)$ is the largest element of the set

$$(\mathbf{N}_2^{\tau(\beta_2(t+1))-1}-((G-G_2)\cup H))\cup\{1\}.$$

(f) If $\gamma_2(t+1)=n$ and $\beta_2(t+1)\notin G\cup H$, then $\varphi_2(t+1)$ is the largest element of the set

$$(\mathbf{N}_2^{\beta_2(t+1)-1}-((G-G_2)\cup H))\cup\{1\}.$$

The description of Construction 2 is completed.

REMARK. The reader may convince himself that $\varphi_2(t+1)$ has been defined correctly. On one hand, the conditions in (a)—(f) exclude each other.[5] On the other hand, we have defined $\varphi_2(t+1)$ in every possible case since the situation when $\gamma_2(t+1)=n$ and $\beta_2(t+1)\in G-G_1$ cannot occur.[6]

Next we assert two simple facts on the procedure of Construction 2.

Lemma 1. *If $\varphi_2(t+1)$ is determined by* (c), *then $\beta_2(\varphi_2(t+1))=\beta_2(t+1)$.*

Proof. The statement follows from (c) and the definition of H. □

Lemma 2. *If $\varphi_2(t+1)$ is determined by* (d) *,then $\beta_2(\varphi_2(t+1))=\beta_2^w(t+1)$ where w is the smallest number such that $\beta_2^w(t+1)\in H$.*

Proof. This is a consequence of (d) and the definition of τ'. □

Proposition 2. *The pre-code $\Gamma_2(\mathbf{D}_0)$ exists.*

Proof. Analogously to the proof of Proposition 1, it is clear that $\Gamma_2(\mathbf{D}_0)$ satisfies the postulates (I)—(VIII) almost completely. Only the fulfilment of (VIII) if $t+1$ plays the role of i is questionable. We show this dependingly on the cases (a)—(f).

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[4] By Postulate (VII), the values $\beta_2(t+1)$, $\gamma_2(t+1)$ are uniquely determined.

[5] This is mostly obvious. It holds for the pairs ((c), (e)) and ((d), (e)) since $G_{1,g}$ is disjoint to H and to $G_{1,h}$.

[6] Indeed, combine Proposition A with the fact that $j\in G-G_1$ is equivalent to the validity of (β).

(We can omit the subscripts in $\beta_2, \gamma_2, \varphi_2$ without the possibility of misunderstanding.)

(a) Trivially, $\varphi(t+1)=1$ guarantees (VIII).

(b) We have

$$\beta(\varphi(t+1)) = \beta(r+1) < r+1 = \beta(t+1).$$

(c) By Lemma 1, $\beta(\varphi(t+1))=\beta(t+1)$, consequently,

$$\gamma(\varphi(t+1)) \neq \gamma(t+1) = n$$

(by (VII)), hence $\gamma(\varphi(t+1))<\gamma(t+1)$ since n is the maximal possible value of γ.

(d) Lemma 2 and $\beta^w(t+1)\leqq r+1<t+1$ imply

$$\beta(\varphi(t+1)) = \beta^w(t+1)\leqq\beta(t+1).$$

Strict inequality must hold since $\beta^w(t+1)\in H$ and $\beta(t+1)\notin H$.

(e) Either $\varphi(t+1)=1$ or the deduction

$$\beta(\varphi(t+1)) \leqq \beta(\tau(\beta(t+1))) < \tau(\beta(t+1)) < \beta(t+1)$$

holds (by (V) and $\varphi(t+1)\leqq\tau(\beta(t+1))-1$).

(f) Either $\varphi(t+1)=1$ or

$$\beta(\varphi(t+1)) \leqq \beta(\beta(t+1)) < \beta(t+1). \quad \square$$

Lemma 3. *Let* **D** *be a pre-code of second type. The sequence*

$$\mathbf{D},\ \Gamma_2(\mathbf{D}),\ \Gamma_2(\Gamma_2(\mathbf{D})),\ \Gamma_2(\Gamma_2(\Gamma_2(\mathbf{D}))),\ \dots \tag{3.1}$$

breaks up after a finite number of steps. The last element of this sequence is a code.

Proof. On one hand, the first and second blocks are common for all the pre-codes in (3.1). Thus r is the same for them, and $rn+n-r$ is an upper bound for the lengths of the third blocks.

On the other hand, the sequence (3.1) can always be continued unless we reached a code. $\square$

DEFINITION. Let **D** be a pre-code of second type. The last element of the sequence (3.1) is denoted by $\Gamma^*(\mathbf{D})$.

In § 8 it will be shown by an example how $\Gamma^*(\mathbf{D})$ is formed.

§ 4.

Let the recursive definition

$$\Gamma_2^{(0)}(\mathbf{D}) = \mathbf{D}, \quad \Gamma_2^{(s)}(\mathbf{D}) = \Gamma_2(\Gamma_2^{(s-1)}(\mathbf{D}))$$

be introduced for a pre-code **D** of type 2.

Lemma 4. *Let* $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ *be a pre-code of second type. Suppose that the pre-code* $\Gamma_2^{(s)}(\mathbf{D})=(r, s, \beta, \gamma, \mu, \varphi)$ *exists*[7] *and* $\gamma(t)=n$ *holds where* $s\geqq 1$ *and*

[7] We can write the functions without subscripts.

t is the size $r+s+1$ of $\Gamma_2^{(s)}(\mathbf{D})$. The following statements (A), (B) *are true:*

(A) *If* $\gamma(\varphi(t))=n$, *then* $\beta(t)=\varphi(t)=r+1$.

(B) *If a number* i $(\in \mathbf{N}_{r+2}^{t-1})$ *satisfies the equalities* $\gamma(i)=n$ *and* $\varphi(i)=\varphi(t)$, *then the formulae* $\beta(i)=\beta(r+1)$ *and* $\beta(t)=\varphi(t)=r+1$ *hold.*

Before proving the exposed lemma, we note another statement which will be useful in the proof of Lemma 4.

Lemma 5. *If the premissa of the assertion* (B) *of Lemma* 4 *are valid, then* $\beta(i)<\beta(t)$.

Proof. The formula $\beta(i)\leqq\beta(t)$ follows from $r+2\leqq i<t$ by Postulate (VII). The equality $\beta(i)=\beta(t)$ leads to a contradiction to Proposition A because we have supposed $\gamma(i)=n=\gamma(t)$. □

Proof of Lemma 4. Since $\gamma(t)=n$ was assumed, the value $\varphi(t)$ has been determined by one of the cases (b)—(f) in Construction 2 (with t instead of $t+1$). An analogous statement holds for $\varphi(i)$ (in (B)). The proper proof splits to the verifications of (A) and (B).

(A) The assumption $\gamma(\varphi(t))=n$ implies $1<\varphi(t)\in G-G_2$. We distinguish five cases according to (b)—(f). In each case, we either show the conclusion of (A) or get a contradiction (indicating that the case cannot occur really).

(b) The conclusion of (A) is trivial.

(c) On one hand, $\gamma(t)=n=\gamma(\varphi(t))$ and $\varphi(t)\leqq r+1<t$; on the other hand, $\beta(t)=\beta(\varphi(t))$ by Lemma 1. Contradiction to Proposition A.

(d) Let w be as in Lemma 2. On one hand, $\gamma(\beta^{w-1}(t))=n=\gamma(\varphi(t))$ and $\beta^{w-1}(t)\neq\varphi(t)$ (since $\beta^{w-1}(t)\notin H$ and $\varphi(t)\in H$); on the other hand, $\beta(\varphi(t))= =\beta^{w}(t)=\beta(\beta^{w-1}(t))$ by Lemma 2. Again a contradiction to Proposition A.

(e), (f). These cases are contradictory because $\varphi(t)\in G-G_2$ cannot be true and false simultaneusly.

(B) We can again distinguish five cases according to how $\varphi(t)$ has been defined, and an analogous distinction is made with respect to $\varphi(i)$. Combining these distinctions, twenty-five cases can be separated. We are going to show that the conclusion of (B) holds in one case and all the remaining twenty-four cases are contradictory.

We begin the discussion with the single consistent case. Suppose that $\varphi(i)$ has been determined by (c), and $\varphi(t)$ has been defined by (b). (This is called case (c_i)—(b_t) briefly.) Then $\beta(t)=\varphi(t)=r+1$ by (b) (applied for t). Furthermore,

$$\beta(i)=\beta(\varphi(i))=\beta(\varphi(t))=\beta(r+1)$$

(where Lemma 1 was used for i).

Now we turn to the other 24 cases that are imaginable. We do not discuss them separately but divide them into seven groups as indicated in Table 1. (E.g., the case (e_i)—(c_t) belongs to the second group.)

First group. In case (b_i)—(e_t) we have

$$r+1=\varphi(i)=\varphi(t)<\tau(\beta(t))<\beta(t),$$

Table 1.

$i \backslash t$	(b)	(c)	(d)	(e)	(f)
(b)	4	5	6	1	1
(c)	—	4	7	2	2
(d)	6	7	4	2	2
(e)	1	2	2	4	3
(f)	1	2	2	3	4

this is impossible since the value of β cannot exceed $r+1$ (by Postulate (III)). In the other three cases (belonging to this group) a similar inference holds, possibly with interchanging i and t, or with dropping $\tau(\beta(t))$.

Second group. We get that exactly one of $\varphi(i)$ and $\varphi(t)$ belongs to $H-\{1\}$, this contradicts the assumption $\varphi(i)=\varphi(t)$.

Third group. Denote the set

$$\mathbf{N}_2^{r+1}-((G-G_2)\cup H)$$

by J. We partition J to the classes J_1 and J_2 in the following manner: $j(\in J)$ belongs to J_1 or to J_2 according as the smallest element of $\mathbf{N}_{j+1}^{r+1}\cap J$ is contained in $J-G_2$ or in G_2, respectively. (If $\mathbf{N}_{j+1}^{r+1}\cap J=\emptyset$, then $j\in J_1$.) It is clear that $\varphi(t)\in J_1$ if $\varphi(t)$ is defined by (e), and $\varphi(t)\in J_2$ if $\varphi(t)$ is defined by (f).

One of $\varphi(i), \varphi(t)$ belongs to J_1 and the other of them belongs to J_2. This excludes $\varphi(i)=\varphi(t)$.

Fourth group. We try to deduce the equality $\beta(i)=\beta(t)$ in each case belonging to the present group; this equality is impossible by Lemma 5.

In the case (b_i)—(b_t), $\beta(i)=\beta(t)$ follows clearly. In the further considered cases, we have to keep in mind the situation of H, G, G_2 (in the tree assigned to **D**). $\varphi(i)=\varphi(t)$ implies $\beta(i)=\beta(t)$ in the cases (c_i)—(c_t) and (f_i)—(f_t) immediately. $\varphi(i)=\varphi(t)$ implies $\beta(i)=\beta(t)$ through the equalities $\tau'(\beta(i))=\tau'(\beta(t))$ and $\tau(\beta(i))=\tau(\beta(t))$ in the cases (d_i)—(d_t) and (e_i)—(e_t), respectively.

Fifth group. We can obtain the deduction

$$\beta(t) = \beta(\varphi(t)) = \beta(\varphi(i)) = \beta(r+1) < r+1 = \beta(i)$$

(the first step follows from Lemma 1), this contradicts Lemma 5.

Sixth group. We discuss the case (b_i)—(d_t) only (the other case belonging to this group can be treated analogously, by interchanging i and t). The deduction

$$\beta(\beta^{w-1}(t)) = \beta^{w}(t) = \beta(\varphi(t)) = \beta(\varphi(i)) = \beta(r+1) \tag{4.1}$$

is valid (in the second step we used Lemma 2). The structure of G, H and the containment $\beta(t)\in G_{1,h}-H$ imply

$$\gamma(\beta^{w-1}(t)) = n. \tag{4.2}$$

Clearly,

$$\gamma(r+1) \leqq n. \tag{4.3}$$

The formulae (4.1), (4.2), (4.3) are consistent with Postulate (V) only if

$$r+1 \leqq \beta^{w-1}(t). \tag{4.4}$$

The obvious formula $\beta^{w-1}(t) \notin H$ and (4.4) imply $\beta^{w-1}(t) > r+1$, contradicting Postulate (III).

Seventh group. It suffices to deal with the case (c_i)—(d_t) (by a similar reason as in the sixth group). Lemmas 1 and 2 imply

$$\beta(i) = \beta(\varphi(i)) = \beta(\varphi(t)) = \beta^w(t) = \beta(\beta^{w-1}(t)), \tag{4.5}$$

and (4.2) holds also in the considered case. Comparing (4.5), (4.2) and $\gamma(i)=n$, we get $i=\beta^{w-1}(t)$. This is impossible since $\beta^{w-1}(t) \leqq r+1 < i$.

The proof of Lemma 4 is completed. □

§ 5.

Recall how the automaton $\psi(\mathbf{C})$ (assigned to a code $\mathbf{C}$) and the word $p_k^{(i)}$ have been defined in § 2.

In the following considerations — yielding the completion of the proof of Theorem 1 — we shall deal chiefly with automata given in form $\psi(\Gamma^*(\mathbf{D}))$ from such a point of view that only the effect of the input sign $x^{(n)}$ (with largest possible superscript) is taken into account.[8]

Lemma 6. *Let* $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ *be a pre-code of second type. Consider the automaton*

$$\psi(\Gamma^*(\mathbf{D})) = \mathbf{A} = (A, X, Y, \delta, \lambda, a_1).$$

If $i \in \mathbf{N}_1^r - (H \cup G_h)$, *then there are two numbers* j, k *such that* $1 \leqq j < i$ *and* $a_j = \delta(a_i, p_k^{(n)})$ *(where* $a_i \in A$, $a_j \in A$*).*

Proof. Case 1: $i \notin G - G_1$. Define the number i' by the conditions $\beta(i')=i$, $\gamma(i')=n$. Then $\varphi(i')$ is defined by the rule (f) (in Construction 2) and the conclusion of the lemma is obviously fulfilled with $k=1$.

Case 2: $i \in G_g - G_1$. There is a $k'(>0)$ and a $j(\in G_1)$ such that $\beta^{k'}(j)=i$ and i, j are in the same connected component of G. It is clear that

$$n = \gamma(j) = \gamma(\beta(j)) = \gamma(\beta^2(j)) = \ldots = \gamma(\beta^{k'}(j)).$$

Consider the number j' satisfying $\beta(j')=j$ and $\gamma(j')=n$. Obviously, $j \geqq r+2$ and $\varphi(j')$ is defined by the rule (e).

We are going to show that the conclusion of the lemma holds if $k'+1$ is chosen for k. The definition of $\psi(\Gamma^*(\mathbf{D}))$ implies the equalities

$$\delta(a_i, p_{k'}^{(n)}) = a_j$$

[8] Automata having a single input sign are often called *autonomous*. The possible structures of finite autonomous automata follow from a graph-theoretical result of Ore ([5], § 4.4; see also [2], Chapter I). Although we do not use Ore's theorem explicitly, its knowledge makes perhaps easier to understand the considerations of the present §.

and

$$\delta(a_i, p_{k'+1}^{(n)}) = \delta(a_j, x^{(n)}) = a_{\varphi(j')}.$$

Since $\varphi(j')$ was defined by the rule (e), $\varphi(j')<\tau(\beta(j))\leqq i$. □

Lemma 7. *Let* **D**, **A** *be as in Lemma* 6. *Suppose* $i \in G_h$. *There are two numbers* j, k *such that* $j \in H$, $a_j=\delta(a_i, p_k^{(n)})$ *are true and one of the formulae* $i \notin H$, $j>i$ *holds.*

Proof.[9] Let us consider the numbers $k'(\geqq 0)$, j and j' with the same properties as in the preceding proof. $j' \geqq r+2$ is again true and $\varphi(j')$ is defined by one of the rules (c), (d). By use of Lemmas 1, 2 we obtain that

$$\varphi(j') > \beta(\varphi(j')) = \begin{cases} \beta(j') = j > i & \text{if (c) is applied,} \\ \tau'(\beta(j')) = \tau'(j) & \text{if (d) is applied.} \end{cases}$$

$i \in H$ implies $i \leqq \tau'(j)$, hence the lemma is valid with $k'+1$ (as k) in both cases. □

Lemma 8. *Let* **D** *and* **A** *be as in Lemma* 6. *If* $i \in H-\{r+1\}$, *then there are two numbers* j, k *such that* $i<j\leqq r+1$ *and* $a_j=\delta(a_i, p_k^{(n)})$ *(where* $a_i \in A$, $a_j \in A$).

Proof. If $i \notin G-G_1$, then the conclusion of the lemma is evidently fulfilled such that $k=1$ and j is the smallest element of $\mathbf{N}_{i+1}^{r+1} \cap H$. If $i \in G-G_1$, then Lemma 7 implies the present assertion. □

Lemma 9. *Let* **D** *and* **A** *be as in Lemma* 6. *For each number* $i(\in H)$ *there is a number* $k(\geqq 0)$ *such that* $\delta(a_i, p_k^{(n)})=a_{r+1}$.

Proof. Apply Lemma 8 repeatedly till it is possible. □

Lemma 10. *Let* **D** *and* **A** *be as in Lemma* 6. *For each number* $i(\in \mathbf{N}_1^{r+1})$ *there is a number* $k(\geqq 0)$ *such that* $\delta(a_i, p_k^{(n)})=a_{r+1}$.

Proof. Case 1*:* $i \in H$. Then Lemma 9 guarantees the statement.

Case 2*:* $i \in G_h-H$. Lemma 7 assures the existence of a k' such that $\delta(a_i, p_{k'}^{(n)}) \in H$. By Lemma 8, also the equality

$$\delta(\delta(a_i, p_{k'}^{(n)}), p_{k''}^{(n)}) = a_{r+1}$$

is valid with a suitable k''. The left-hand side of this equality is clearly $\delta(a_i, p_{k'+k''}^{(n)})$.

Case 3*:* $i \notin G_h \cup H$. By a successive application of Lemma 6, there exists a k' such that $\delta(a_i, p_{k'}^{(n)})=a_1$. Since a_1 belongs to H, the further inference is the same as in Case 2. □

Lemma 11. *Let* **D** *and* **A** *be as in Lemma* 6. *Suppose that* i *and* j *are distinct numbers in* $\mathbf{N}_1^{r+1}$. *If*

$$\delta(a_i, x^{(n)}) = \delta(a_j, x^{(n)}) = a_m,$$

then

$$\max(i, j) = m = r+1.$$

[9] In the proof we consider an i chosen arbitrarily. It is easy to see that the lemma is satisfied with $k=1$, too, if, particularly, $i \in H$ and i does not belong to the range of τ'.

Proof. Case 1: one of i and j equals $\beta(m)$. We can assume (without loss of the generality) that $\beta(m)=i$. Then, by the connection of **D** and **A**, we have $\gamma(m)=n$ and there exists a number $w(\in \mathbf{N}_{r+2}^{r+s+1})$ such that $\beta(w)=j, \gamma(w)=n$ and $\varphi(w)=m$ hold in $\Gamma^*(\mathbf{D})$. By applying the assertion (A) of Lemma 4 (for w) we get that $j=\beta(w)=r+1>i$ and $m=\varphi(w)=r+1$.

Case 2: $\beta(m)$ coincides neither with i nor with j. There exist two numbers v, w in $\mathbf{N}_{r+2}^{r+s+1}$ such that $\beta(v)=i, \beta(w)=j, \gamma(v)=\gamma(w)=n$ and $\varphi(v)=\varphi(w)=m$. We can suppose $v<w$. Apply the statement (B) of Lemma 4 for v, w (instead of i, t, resp.). We obtain $i=\beta(v)=\beta(r+1)$ and $j=\beta(w)=r+1=\varphi(w)=m$. □

Lemma 12. *Let* **D** *and* **A** *be as in Lemma* 6. *Consider two different states a_i, a_j of* **A**. *Denote by k_i the smallest number fulfilling $\delta(a_i, p_{k_i}^{(n)})=a_{r+1}$; let k_j be defined analogously. Then $k_i\neq k_j$.*

Proof. The existence of k_i and k_j follows from Lemma 10. Let z_j be the smallest number such that $\delta(a_j, p_{k_j}^{(n)})$ belongs to the set

$$\{a_i, \delta(a_i, x^{(n)}), \delta(a_i, p_2^{(n)}), \delta(a_i, p_3^{(n)}), \ldots, \delta(a_i, p_{k_i}^{(n)})\},$$

let z_i be the smallest number such that $\delta(a_i, p_{z_i}^{(n)})=\delta(a_j, p_{z_j}^{(n)})$. Evidently, $0\leqq z_i\leqq k_i$ and $0\leqq z_j\leqq k_j$. (The situation is illustrated in Fig. 1.) We can distinguish four cases (two of them will be contradictory).

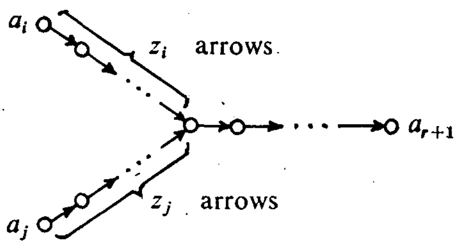

Fig. 1

If $z_i=z_j=0$, then we get $a_i=a_j$. Contradiction.
If $z_i=0<z_j$, then $k_j=k_i+z_j>k_i$.
If $z_j=0<z_i$, then $k_i=k_j+z_i>k_j$.
If $z_i>0$ and $z_j>0$, then

$$\delta(\delta(a_i, p_{z_i-1}^{(n)}), x^{(n)}) = \delta(a_i, p_{z_i}^{(n)}) = \delta(a_j, p_{z_j}^{(n)}) = \delta(\delta(a_j, p_{z_j-1}^{(n)}), x^{(n)}).$$

Apply Lemma 11 for $\delta(a_i, p_{z_i-1}^{(n)})$ and $\delta(a_j, p_{z_j-1}^{(n)})$. The conclusion of Lemma 11 implies that one of this states equals a_{r+1}, this is impossible by the definition of k_i and k_j. □

Proof of Theorem 1. Consider a pre-code $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ of second type. Let **A** be the automaton $\psi(\Gamma^*(\Gamma_1(\mathbf{D})))=(A, X, Y, \delta, \lambda, a_1)$. Clearly, $|A|=r+2$. It is obvious by Construction 1 that $\lambda(a_i)\neq\lambda(a_{r+2})$ if $i\in\mathbf{N}_1^{r+1}$.

Consider two different states a_i, a_j of **A**. Introduce k_i, k_j as the smallest numbers fulfilling $\delta(a_i, p_{k_i}^{(n)})=a_{r+2}, \delta(a_j, p_{k_j}^{(n)})=a_{r+2}$, respectively. Lemma 12 (applied for $\Gamma_1(\mathbf{D})$ instead of $\mathbf{D}$) assures $k_i \neq k_j$. We can suppose (without loss of generality) $k_i < k_j$. We obtain

$$\delta(a_i, p_{k_i}^{(n)}) = a_{r+2} \neq \delta(a_j, p_{k_i}^{(n)})$$

from the previous considerations, hence

$$\lambda(\delta(a_i, p_{k_i}^{(n)})) = \lambda(a_{r+2}) \neq \lambda(\delta(a_j, p_{k_i}^{(n)})),$$

thus $\omega(a_i, a_j) \leqq k_i < \infty$.

Since the above inference holds for each pair (a_i, a_j) of states of the finite automaton $\mathbf{A}$, the complexity $\Omega_A(\mathbf{A})$ is finite. Consequently,

$$\Omega_C(\mathbf{D}) \leqq \Omega_C(\Gamma^*(\Gamma_1(\mathbf{D}))) = \Omega_A(\mathbf{A}) < \infty$$

by $\mathbf{D} < \Gamma^*(\Gamma_1(\mathbf{D}))$ and Proposition B. □

The next result follows from Lemmas 10 and 11 immediately:

Corollary 1. *Let* **D** *and* **A** *be as in Lemma* 6. *There exists a permutation* π *of the set* $\{1, 2, \dots, r\}$ *such that*

$$\delta(a_{\pi(i)}, x^{(n)}) = \begin{cases} a_{\pi(i+1)} & \text{if } 1 \leqq i < r, \\ a_{r+1} & \text{if } i = r, \end{cases}$$

and moreover, $\delta(a_{r+1}, x^{(n)}) = a_{r+1}$. □

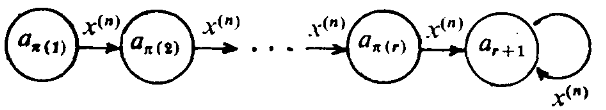

Fig. 2

Corollary 2. *If* $\mathbf{D} = (r, 0, \beta, \gamma, \mu, \varphi)$ *is a pre-code of second type, then* $\Omega_C(\mathbf{D}) \leqq r$.

Proof. Analyze the proof of Theorem 1, let π have the same sense (for $\Gamma_1(\mathbf{D})$) as in Corollary 1. It is clear that $a_{\pi(i)}$ and $a_{\pi(j)}$ can be distinguished by the word $p_{r+2-\pi(j)}^{(n)}$ if $\pi(i) < \pi(j)$, hence

$$\omega(a_{\pi(i)}, a_{\pi(j)}) \leqq r + 2 - \pi(j) \leqq r$$

(the second inequality holds because $\pi(i) + 1 = \pi(j) = 2$ is the worst choice). Thus $\Omega_A(\mathbf{A}) \leqq r$. □

§ 6.

The assertion (iii) of the next result is a conversion of Theorem 1.

Theorem 2. *Let* k *be an arbitrary non-negative integer. Then*

(i) *there is a code* $\mathbf{C}_k$ *such that* $\Omega_C(\mathbf{C}_k) = k$,
(ii) *there is an automaton* $\mathbf{A}_k$ *such that* $\Omega_A(\mathbf{A}_k) = k$,
(iii) *there is a pre-code* $\mathbf{D}_k$ *such that* $\Omega_C(\mathbf{D}_k) = k$ *and* $\mathbf{D}_k$ *is of second type.*

Proof. We define $\mathbf{C}_k=(r, s, \beta, \gamma, \mu, \varphi)$ in the following manner:

$$r = k+1 \quad (\text{hence } s(= rn+n-r) = kn+2n-k+1),$$
$$\beta(i) = i-1 \quad \text{if} \quad i\in \mathbf{N}_2^{r+1},$$
$$\gamma(i) = n \quad \text{if} \quad i\in \mathbf{N}_2^{r+1},$$
$$\mu(i) = 1 \quad \text{if} \quad i\in \mathbf{N}_1^{r},$$
$$\mu(r+1) = 2,$$
$$\varphi(i) = 1 \quad \text{if} \quad i\in \mathbf{N}_{r+2}^{r+s+1}.$$

$\beta(i)$ and $\gamma(i)$ are defined, of course, by virtue of Postulate (VII) if $i\in \mathbf{N}_{r+2}^{r+s+1}$.

Fig. 3 shows a part of $\mathbf{A}_k=\psi(\mathbf{C}_k)$. (In the full graph of $\mathbf{A}_k$ every edge which is not indicated in this figure goes into a_1.)

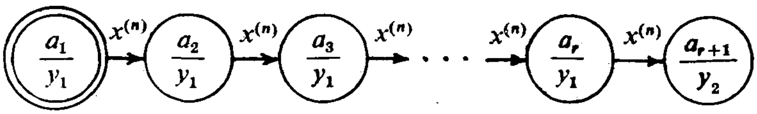

Fig. 3

It can be seen easily that $\mathbf{C}_k$ satisfies all the postulates (I)—(VIII). Thus $\mathbf{C}_k$ is a pre-code; it is a code since s equals the maximal possible value $rn+n-r$ (see the remark in § 4.3 of [3]).

We can verify easily that $\omega(a_i, a_j)=r-j+1$ is valid in $\mathbf{A}_k$ if $i<j$. (Indeed, on one hand,

$$\delta(a_i, p_{r-j+1}^{(n)}) = a_{(r+1)-(j-i)} \neq a_{r+1} = \delta(a_j, p_{r-j+1}^{(n)});$$

on the other hand, the relations $\delta(a_i, p)\in\{a_1, a_2, \dots, a_r\}$ and $\delta(a_j, p)\in\{a_1, a_2, \dots, a_r\}$ are true if $i<j$ and $L(p)\leq r-j$.) The value of $\omega(a_i, a_j)$ reaches its maximum when $i=1$ and $j=2$, namely,

$$\omega(a_1, a_2) = r-1 = k.$$

Hence $\Omega_C(\mathbf{C}_k)=\Omega_A(\mathbf{A}_k)=k$. The proof of (i) and (ii) is completed.

Denote by $\mathbf{D}_k$ the pre-code satisfying $\mathbf{D}_k<\mathbf{C}_k$ and having the size $r+1$. (In other words, $\mathbf{D}_k$ consists of the first and second blocks of $\mathbf{C}_k$.) The estimate

$$\Omega_C(\mathbf{D}_k) \leqq \Omega_C(\mathbf{C}_k) = k \tag{6.1}$$

is obvious. Before verifying the converse inequality, we interrupt the proof by stating a lemma.

Lemma 13. *Consider an arbitrary code* $\mathbf{C}$ *such that* $\mathbf{D}_k<\mathbf{C}$. *Let the automaton* $\psi(\mathbf{C})=\mathbf{A}=(A, X, Y, \delta, \lambda, a_1)$ *be studied. If* $a_i\in A$, $i\leqq r$ *(where r is understood in* $\mathbf{D}_k$*) and a state* $a_j(\in A)$ *is representable in form* $a_j=\delta(a_i, x^{(h)})$ *(where* $x^{(h)}$ *is an arbitrary element of* X*), then* $j\leqq i+1$.

Proof. Case 1: $h=n$. The transition $\delta(a_i, x^{(h)})$ is determined by a row of the pre-code $\mathbf{D}_k$, hence $a_j=a_{i+1}$.

Case 2: $h \neq n$. Since $n=|X|$, we have $h<n$. The transition $\delta(a_i, x^{(h)})$ is determined by a row being in the third block[10] of $\mathbf{C}$; say, by the m^{th} row. Then $\beta(m)=i, \gamma(m)=h$ and $\varphi(m)=j$. We have $\beta(\varphi(m)) \leqq \beta(m)$ by Postulate (VIII), this implies

$$j = \varphi(m) \leqq \beta(m)+1 = i+1$$

by $\beta(m)=i \leqq r$ and the construction of $\mathbf{D}_k$. □

Proof of Theorem 2 (final part). If $\mathbf{C}$ is an arbitrary code fulfilling $\mathbf{D}_k < \mathbf{C}$, then the equality

$$\lambda(\delta(a_1, p)) = y_1 = \lambda(\delta(a_2, p))$$

holds in $\psi(\mathbf{C})$ for every word p whose length does not exceed $r-2$ (by an iterated application of Lemma 13). Hence $\omega(a_1, a_2) \geqq r-1$ holds in $\psi(\mathbf{C})$, consequently

$$\Omega_A(\psi(\mathbf{C})) \geqq r-1 = k$$

and

$$\Omega_C(\mathbf{C}) \geqq k, \tag{6.2}$$

thus

$$\Omega_C(\mathbf{D}_k) \geqq k, \tag{6.3}$$

since (6.2) holds for each $\mathbf{C}$ satisfying $\mathbf{D}_k < \mathbf{C}$.

The inequalities (6.1) and (6.3) give together the assertion (iii) of the theorem. □

§ 7.

By use of Corollary 2 and slight modifications of the idea of the proof of Theorem 2, we can infer the following assertions concerning the complexity and the first component r of codes and pre-codes:

Proposition 3. *Let two non-negative integers k, r be given. The inequality $k \leqq r$ is a necessary and sufficient condition of the existence of a pre-code $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ such that $\mathbf{D}$ is of second type and $\Omega_C(\mathbf{D})=k$.*

Proposition 4. *If the non-negative integers k and r satisfy $k<r$, then there exists a code $\mathbf{C}=(r, s, \beta, \gamma, \mu, \varphi)$ such that $\Omega_C(\mathbf{C})=k$.*

Proof of Propositions 3 and 4. The proof will consist of three parts. In (A) we verify Proposition 4 and we show that $k<r$ is sufficient in Proposition 3. In (B) we make some preparations for proving the sufficiency of $k=r$. In (C) we verify the necessity part of Proposition 3 and we complete the proof of the sufficiency of the equality $k=r$.

(A) Consider k and r $(k<r)$. Recall the procedure proving Theorem 2, let us start with the code $\mathbf{C}_{r-1}$ (i.e., with $\mathbf{C}_k$ such that $r-1$ is taken for k). Alter $\mathbf{C}_{r-1}$ by putting

$$\mu(i) = \begin{cases} 1 & \text{if } \; i \in \mathbf{N}_2^{k+1}, \\ i-k & \text{if } \; i \in \mathbf{N}_{k+2}^{r+1}; \end{cases}$$

[10] This row cannot be in the second block of $\mathbf{C}$ (by Postulate (V)) even if the second block has $>r$ rows.

denote the originating code by $\mathbf{C}'_{k,r}$ (of course, $\mathbf{C}'_{r-1,r}=\mathbf{C}_{r-1}$) and the pre-code consisting of the first and second blocks of $\mathbf{C}'_{k,r}$ by $\mathbf{D}'_{k,r}$. The first component of $\mathbf{C}'_{k,r}$ and of $\mathbf{D}'_{k,r}$ is clearly r.

The whole proof of Theorem 2 remains valid for $\mathbf{C}'_{k,r}$, $\mathbf{D}'_{k,r}$ with certain numerical changes. In fact, $\omega(a_i, a_j)=\max(0, k-j+2)$ (where $i<j$), especially,

$$k = \omega(a_1, a_2) = \Omega_A(\psi(\mathbf{C}'_{k,r})) = \Omega_C(\mathbf{C}'_{k,r}).$$

Thus Proposition 4 is proved.

No word whose length is smaller than k can distinguish a_1 and a_2 for an arbitrary code $\mathbf{C}(>\mathbf{D}'_{k,r})$, consequently, $\Omega_C(\mathbf{D}'_{k,r})=k$.

(B) We start again with the code $\mathbf{C}_k$ occurring in the proof of Theorem 2. We modify it by putting $\mu(r+1)=1$; we denote the resulting code by $\mathbf{C}^*_k$ and the pre-code of its first $r+1$ rows by $\mathbf{D}^*_k$. Although the considerations of the proof of Theorem 2 do not remain valid in general, Lemma 13 holds in the present case, too, hence no word whose length is $<r$ can distinguish a_1 and a_2 for an arbitrary code $\mathbf{C}(>\mathbf{D}^*_k)$, thus $\Omega_C(\mathbf{D}^*_k)\geqq r$.

(C) Corollary 2 states that $\Omega_C(\mathbf{D})\leqq r$ holds for each pre-code $\mathbf{D}=(r, 0, \beta, \gamma, \mu, \varphi)$ of second type. The necessity of the condition in Proposition 3 is proved.

Especially, $\Omega_C(\mathbf{D}^*_k)\leqq r$. This inequality and the conclusion of (B) mean that $k=r$ is sufficient in Proposition 3. □□

Since the automaton $\psi(\mathbf{C})$ has $r+1$ states, Proposition 4 can be formulated in the following (equivalent) form:

Corollary 3. *If the non-negative integers k and v satisfy $k\leqq v-2$, then there exists a Moore automaton* $\mathbf{A}$ *such that $\Omega_A(\mathbf{A})=k$ and the number of states of* $\mathbf{A}$ *is v.* □

I conjecture that the conversion of Corollary 3 is also true, see [4].

§ 8.

In the last section of the paper, an example will be studied how $\Gamma_1(\mathbf{D})$ and $\Gamma^*(\Gamma_1(\mathbf{D}))$ are built up if a pre-code $\mathbf{D}$ of second type is given concretely.

Suppose $X=\{x^{(1)}, x^{(2)}\}$. Let $\mathbf{D}$ be the pre-code given by Table 2/a. (r equals 24. The tree assigned to $\mathbf{D}$ can be seen in Fig. 4. For the sake of simplicity, the vertices are labelled by i and the edges are by j instead of a_i and $x^{(j)}$, resp.)

We get $\Gamma_1(\mathbf{D})$ if we supplement $\mathbf{D}$ by a 26[th] row given by Table 2/b. The sets $H, G, G_1, G_2, G_h, G_g, G_{1,h}, G_{1,g}$ are (for $\Gamma_1(\mathbf{D})$) the following:

$$H = \{1, 2, 4, 7, 11, 15, 17, 20, 22, 25, 26\},$$
$$G = \{2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 24, 25, 26\},$$
$$G_1 = \{7, 13, 14, 19, 24, 26\},$$
$$G_2 = \{2, 3, 5, 11, 16, 22\},$$
$$G_h = \{2, 4, 7, 11, 15, 17, 21, 22, 24, 25, 26\},$$
$$G_g = \{3, 5, 6, 9, 10, 13, 14, 16, 19\},$$
$$G_{1,h} = \{7, 24, 26\},$$
$$G_{1,g} = \{13, 14, 19\}.$$

Table 2.

i	$\beta(i)$	$\gamma(i)$	$\mu(i)$	$\varphi(i)$
1	—	—	1	—
2	1	1	1	—
3	2	1	1	—
4	2	2	1	—
5	3	1	1	—
6	3	2	1	—
7	4	2	1	—
8	5	1	1	—
9	5	2	1	—
10	6	2	1	—
11	7	1	1	—
12	9	1	1	—
13	9	2	1	—
14	10	2	1	—
15	11	2	1	—
16	14	1	1	—
17	15	2	1	—
18	16	1	1	—
19	16	2	1	—
20	17	1	1	—
21	17	2	1	—
22	20	1	1	—
23	21	1	1	—
24	21	2	1	—
25	22	2	1	—
		(a)		
26	25	2	2	—
		(b)		

Table 3.

i	$\beta(i)$	$\gamma(i)$	$\mu(i)$	$\varphi(i)$
27	1	2	—	2
28	4	1	—	1
29	6	1	—	1
30	7	2	—	11
31	8	1	—	1
32	8	2	—	5
33	10	1	—	1
34	11	1	—	1
35	12	1	—	1
36	12	2	—	8
37	13	1	—	1
38	13	2	—	3
39	14	2	—	1
40	15	1	—	1
41	18	1	—	1
42	18	2	—	16
43	19	1	—	1
44	19	2	—	12
45	20	2	—	22
46	22	1	—	1
47	23	1	—	1
48	23	2	—	18
49	24	1	—	1
50	24	2	—	20
51	25	1	—	1
52	26	1	—	1
53	26	2	—	26

The functions τ and τ' are indicated in Table 4.

Table 4.

i	$\tau(i)$	$\tau'(i)$
7	2	—
13	5	—
14	3	—
19	16	—
24	11	17
26	22	—

Now we are able to obtain $\Gamma^*(\Gamma_1(\mathbf{D}))$ by applying Construction 2 as many times as possible (beginning with $\Gamma_1(\mathbf{D})$). We get that the 26 rows (seen in Table 2) are supplemented by 27 rows (as a third block) which are given in Table 3.

In course of forming Table 3, the values $\varphi(27)$, $\varphi(30)$, $\varphi(45)$ are determined in sense of case (c) of rule (iv) of Construction 2. The values $\varphi(32)$, $\varphi(36)$, $\varphi(42)$, $\varphi(48)$ are determined by case (f). The values $\varphi(38)$, $\varphi(39)$, $\varphi(44)$ are determined by case (e). $\varphi(50)$ and $\varphi(53)$ are determined by cases (d) and (b), respectively. (The remaining 15 values are by case (a).)

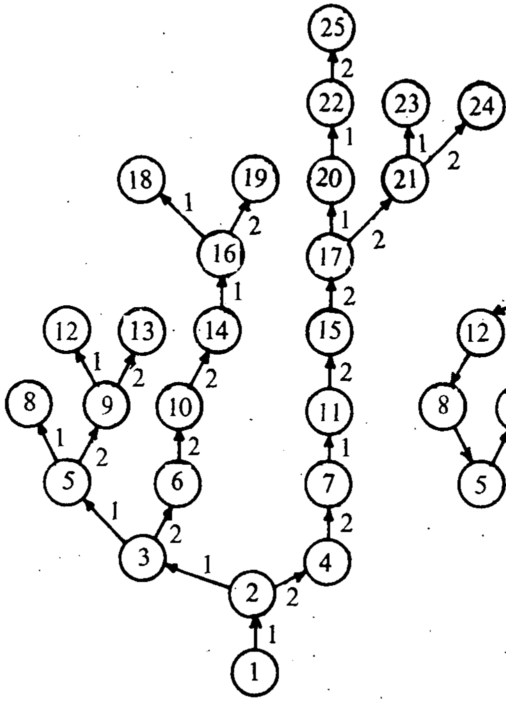

Fig. 4

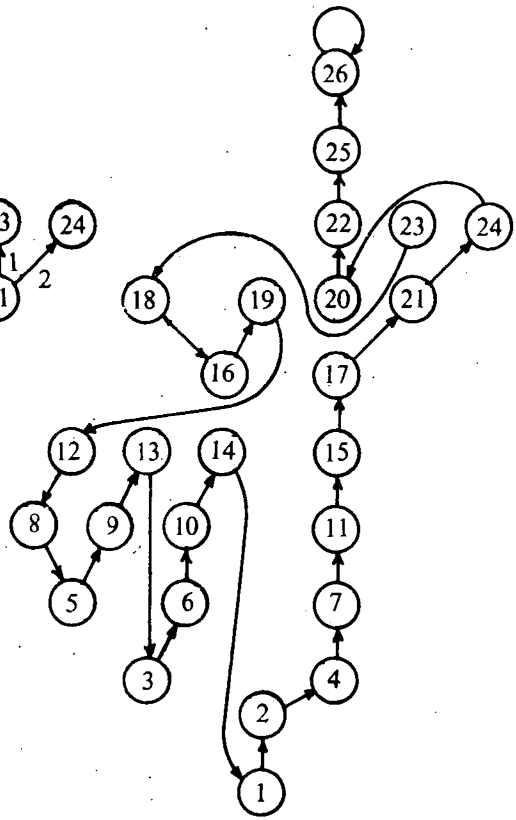

Fig. 5

Fig. 5 shows the (autonomous) automaton that is obtained from $\Gamma^*(\Gamma_1(\mathbf{D}))$ if solely the input sign $x^{(2)}$ is considered. It is evident that Corollary 1 (in § 5) is fulfilled by a suitable permutation π (for which $\pi(1)=23, \pi(2)=18, \pi(3)=16$, and so on).

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References

[1] ÁDÁM, A., Automata-leképezések, félcsoportok, automaták (Automaton mappings, semigroups, automata), *Mat. Lapok*, v. 19, 1968, pp. 327—343.
[2] ÁDÁM, A., Gráfok és ciklusok (Graphs and cycles), *Mat. Lapok*, v. 22, 1971, pp. 269—282.
[3] ÁDÁM, A., On the question of description of the behaviour of finite automata, *Studia Sci. Math. Hungar.*, v. 13, 1978 pp. 105—124.
[4] ÁDÁM, A., Research problem 29 (The connection of the state number and the complexity of finite Moore automata), *Period. Math. Hungar.*, v. 12, 1981, pp. 229—230.
[5] ORE, O., *Theory of graphs*, Amer. Math. Soc., Providence, 1962.

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