Functor state machines

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In the present paper we introduce a notion of a machine in an arbitrary category. A machine in a category is a computational device computing a morphism from a free algebra to another one. The computation is defined by means of homomorphic extension. We are dealing with two types of machines each of them having a functor as its state. These two families of machines are related to bottom-up and top-down tree transformations, respectively. The state functor of a machine working in top-down way is required to have a right adjoint. We show that every top-down computation can be carried out in bottom-up way.

A special type of machines, namely the generalized sequential machines in categories having binary products are investigated. A generalized sequential machine is a machine whose state funtor is a product functor and whose final state transformation is the corresponding projection. Morphisms can be computed by generalized sequential machines in a category are characterized. We show that the process transformations of Arbib and Manes, and the generalized sequential machines in a category have the same processing capacity. Results of the present paper have been announced in [6].

1. Preliminaries

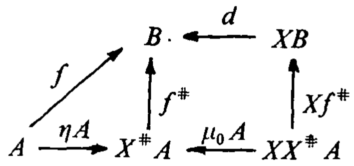
We assume the reader to be familiar with the elements of category theory such as the notion of category, functor and natural transformation. Now we will list some basic notions, definitions and results to be used in the sequel.

Definition 1.1. Let $\mathcal{K}$ be any category and let X: $\mathcal{K}\to\mathcal{K}$ be an endofunctor. An *X-algebra* is a pair (A, d) where A is an object and d: $XA\to A$ is a morphism in $\mathcal{K}$. Given two X-algebras (A, d), (A', d'), a morphism h: $A\to A'$ is an *X-homomorphism* if the diagram

$$\begin{array}{ccc} A' & \xleftarrow{\;d'\;} & XA' \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle Xh} \\ A & \xleftarrow{\;d\;} & XA \end{array} \qquad (1.1)$$

is commutative.

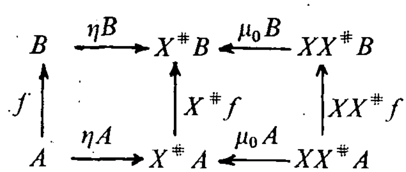
DEFINITION 1.2 (Arbib—Manes [3]). Let A be an object in $\mathcal{K}$. A *free X-algebra* over A is an X-algebra $(X^{\#}A, \mu_0 A)$ coupled with a morphism $\eta A\colon A \to X^{\#}A$ with the universal property that for every other X-algebra (B, d) and morphism $f\colon A \to B$ there exists a unique X-homomorphism $f^{\#}\colon (X^{\#}A, \mu_0 A) \to (B, d)$ such that $f^{\#} \cdot \eta A = f$. That is, given d and f there is a unique $f^{\#}$ such that (1.2) commutes.



(1.2)

The morphism $f^{\#}$ in (1.2) iscalled the *X-homomorphic extension* of f from the free X-algebra $(X^{\#}A, \mu_0 A)$ into the X-algebra (B, d).

Following Adámek and Trnková (see [1]) we say that a functor $X\colon \mathcal{K} \to \mathcal{K}$ is a *varietor* if there exists a free X-algebra over each object in $\mathcal{K}$. Arbib and Manes use the terms input process or recursion process [3, 4] depending on context. Let $X\colon \mathcal{K} \to \mathcal{K}$ be a varietor. If we fix a choice of $\eta A\colon A \to X^{\#}A$, $\mu_0 A\colon XX^{\#}A \to X^{\#}A$ in (1.2) for each object A in $\mathcal{K}$, and for every morphism $f\colon A \to B$ the morphism $X^{\#}f\colon X^{\#}A \to X^{\#}B$ is defined to be the X-homomorphic extension of $\eta B \cdot f$, i.e.

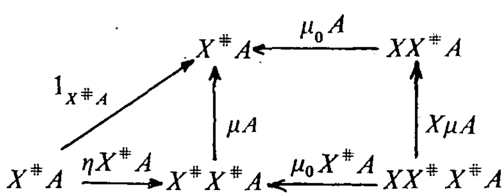


(1.3)

then we get a functor $X^{\#}\colon \mathcal{K} \to \mathcal{K}$. Moreover, we obtain a pair of natural transformations

$$\eta\colon I_{\mathcal{K}} \stackrel{.}{\to} X^{\#}, \quad \mu_0\colon XX^{\#} \stackrel{.}{\to} X^{\#},$$

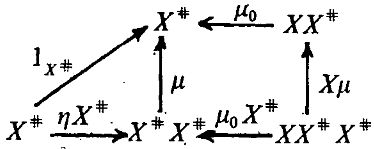
the *insertion of generators* and the *free operation* of X, respectively. We omit the subscript in the identity functor $I_{\mathcal{K}}\colon \mathcal{K} \to \mathcal{K}$ whenever $\mathcal{K}$ is understood. Note that each varietor X yields a family of morphisms $\mu A\colon X^{\#}X^{\#}A \to X^{\#}A$ defined by the diagram



(1.4)

where $1_{X^{\#}A}\colon X^{\#}A \to X^{\#}A$ is the identity morphism. One can show by an easy computation that μA is natural in A, i.e. we have a natural transformation

$\mu\colon X^\# X^\# \dot{\to} X^\#$, the *extended free operation* of X, rendering the diagram (1.5) commutative.



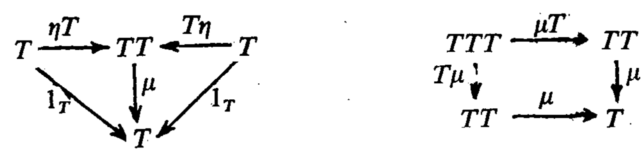
(1.5)

The basic algebraic structure in string processing is X_0^*, the free monoid over a set X_0 of generators. Monads, rather than monoids are fundamental in our development. Now we recall the definition of a monad.

DEFINITION 1.3. A *monad* (T, η, μ) in a category $\mathscr{K}$ consists of a functor $T\colon \mathscr{K} \to \mathscr{K}$ and two natural transformations

$$\eta\colon I \dot{\to} T, \quad \mu\colon TT \dot{\to} T$$

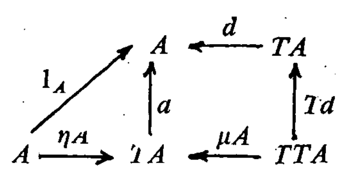
which make the following diagrams commute.



(1.6)

The diagrams in (1.6) are called unitary and associativity axioms, respectively. We state, without proof, the following well-known fact: for every varietor X the triple $(X^\#, \eta, \mu)$ is a monad in $\mathscr{K}$, where η is the insertion of the generators and μ is the extended free operation of X.

DEFINITION 1.4. Let (T, η, μ) be a monad in $\mathscr{K}$. A *T-monad algebra* is a pair (A, d) consisting of an object A of $\mathscr{K}$ and a $\mathscr{K}$-morphism $d\colon TA \to A$ such that



(1.7)

It is easy to prove that the pair $(X^\# A, \mu A)$ is an $X^\#$-monad algebra for every varietor X and object A.

CONVENTION 1.5. In the remaining of this paper if a varietor is referred to by the letter X, then the insertion of the generators, the free operation and the extended free operation of X are denoted by η, μ_0 and μ, respectively

$$\eta\colon I \dot{\to} X^\#, \quad \mu_0\colon XX^\# \dot{\to} X^\#, \quad \mu\colon X^\# X^\# \dot{\to} X^\#.$$

If we use the letter Y to denote another varietor then the items above are denoted by the same letters but with bar, i.e. $\bar{\eta}$, $\bar{\mu}_0$ and $\bar{\mu}$.

PROPOSITION 1.6. Let $X\colon \mathcal{K}\to\mathcal{K}$ be a varietor. Given functors $F, G\colon \mathcal{K}\to\mathcal{K}$ and natural transformations $\delta\colon XG \overset{\cdot}{\to} G$, $\varphi\colon F\overset{\cdot}{\to} G$ there is a unique natural transformation $\varphi^{\#}\colon X^{\#}F\overset{\cdot}{\to} G$ such that the following diagram is commutative.

$$\begin{array}{ccccc} & & G & \xleftarrow{\;\delta\;} & XG \\ & \nearrow^{\varphi} & \uparrow{\scriptstyle \varphi^{\#}} & & \uparrow{\scriptstyle X\varphi^{\#}} \\ F & \xrightarrow{\;\eta F\;} & X^{\#}F & \xleftarrow{\;\mu_0 F\;} & XX^{\#}F \end{array} \tag{1.8}$$

Proof is immediate.

DEFINITION 1.7. An *adjunction* $(F, U, \nu, \varepsilon)\colon \mathcal{K}\to\mathcal{L}$ consists of a pair of functors $F\colon \mathcal{K}\to\mathcal{L}$, $U\colon \mathcal{L}\to\mathcal{K}$ and natural transformations $\nu\colon I_{\mathcal{K}}\overset{\cdot}{\to} UF$, $\varepsilon\colon FU\overset{\cdot}{\to} I_{\mathcal{L}}$ (called *unit* and *counit*, respectively) subject to the so called "triangular identities":

$$\begin{array}{ccc} U & \xrightarrow{\;\nu U\;} & UFU \\ & {\scriptstyle 1_U}\searrow & \downarrow{\scriptstyle U\varepsilon} \\ & & U \end{array} \qquad\qquad \begin{array}{ccc} FUF & \xleftarrow{\;F\nu\;} & F \\ {\scriptstyle \varepsilon F}\downarrow & \swarrow{\scriptstyle 1_F} & \\ F & & \end{array} \tag{1.9}$$

F is said to be a *left adjoint* to U and U a *right adjoint* to F. We say that a functor F has right adjoint, if there is a functor U right adjoint to F.

2. Machines

In this section we introduce a notion of a machine in an arbitrary category. This is based on the notion of the free algebra. A machine is a computational device which computes a morphism of a free algebra into another one. The basic idea of our development — due to Alagić [2] — is to take a functor to be the state of a machine. Alagić offered in his paper [2] the general concept of a direct state transformation which took the form $XQ\overset{\cdot}{\to} QY^{\#}$, where X and Y are varietors and Q now is a functor. Arbib and Manes remarked in [4] that the Alagić approach has one flaw: because Q is a functor rather than an object, thus running the direct state transformation yields a natural transformation $X^{\#}Q\overset{\cdot}{\to} QY^{\#}$ instead of a morphism $X^{\#}A\to Y^{\#}B$ between free algebras. But, in spite of this note there is a general way in which we can extract from $X^{\#}Q\overset{\cdot}{\to} QY^{\#}$ a "state-free" input-output response of the form $X^{\#}A\to Y^{\#}B$. Thus, the benifits of the Alagić approach can be obtained in any category, not only those having binary products. Appart from the fact that we actually do not use the notion of the direct state transformation of Alagić in the definition of a machine and its response, there is a close relationship between them. We will show this relationship. There are several adventages of taking a functor to be the state of a machine. First of all this provides a uniform treatment of top-down and bottom-up computations which are well-known in the theory of tree transformations (see Engelfriet [5]).

DEFINITION 2.1. Let A, B be objects of a category $\mathscr{K}$, and let X, Y be varietors in $\mathscr{K}$. A *machine* $M: (A, X) \to (B, Y)$ in $\mathscr{K}$ is $M = (Q, i, \sigma, \beta)$, where

$Q: \mathscr{K} \to \mathscr{K}$ is a functor, the *state functor,*
$i: A \to QY^\# B$ is a morphism, the *initial state-output* morphism,
$\sigma: XQ \xrightarrow{\cdot} QY^\#$ is a natural transformation, the *transition,*
$\beta: Q \xrightarrow{\cdot} I$ is a natural transformation, the *final state* transformation.

DEFINITION 2.2. Let $M = (Q, i, \sigma, \beta): (A, X) \to (B, Y)$ be a machine in $\mathscr{K}$. The *response* of M is the morphism $f_M: X^\# A \to Y^\# B$ defined by the composite

$$f_M: X^\# A \xrightarrow{i^\#} QY^\# B \xrightarrow{\beta Y^\# B} Y^\# B, \tag{2.1}$$

where $i^\#$ is the *run map* of M, i.e. the X-homomorphic extension

$$\begin{array}{ccccccc} & & QY^\# B & \xleftarrow{Q\bar{\mu} B} & QY^\# Y^\# B & \xleftarrow{\sigma Y^\# B} & XQY^\# B \\ & {}^{i}\nearrow & \uparrow{\scriptstyle i^\#} & & & & \uparrow{\scriptstyle Xi^\#} \\ A & \xrightarrow{\eta A} & X^\# A & & \xleftarrow{\mu_0 A} & & XX^\# A \end{array} \tag{2.2}$$

of the initial state-output i.

By Proposition 1.6 the transition $\sigma: XQ \xrightarrow{\cdot} QY^\#$ has a unique extension $\sigma^\#: X^\# Q \xrightarrow{\cdot} QY^\#$ defined by

$$\begin{array}{ccccccc} & & QY^\# & \xleftarrow{Q\bar{\mu}} & QY^\# Y^\# & \xleftarrow{\sigma Y^\#} & XQY^\# \\ & {}^{Q\bar{\eta}}\nearrow & \uparrow{\scriptstyle \sigma^\#} & & & & \uparrow{\scriptstyle X\sigma^\#} \\ Q & \xrightarrow{\eta Q} & X^\# Q & & \xleftarrow{\mu_0 Q} & & XX^\# Q \end{array} \tag{2.3}$$

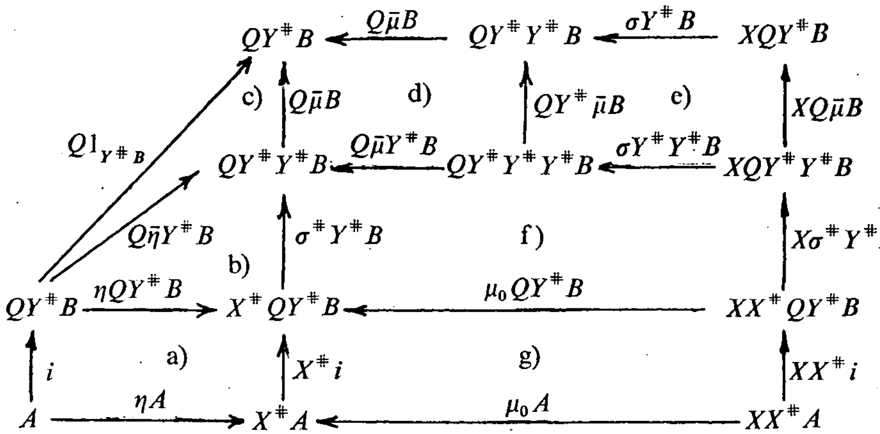
$\sigma^\#$ is called the *extended transition* of the machine M. Natural transformations like $\sigma^\#$ in (2.3) were studied by Alagić in [2] under the name "direct state transformation".

We show that the response of a machine M can be expressed in terms of the extended transition of M.

STATEMENT 2.3. Let $M = (Q, i, \sigma, \beta): (A, X) \to (B, Y)$ be a machine in $\mathscr{K}$. Then the response of M is

$$f_M = \beta Y^\# B \cdot Q\bar{\mu} B \cdot \sigma^\# Y^\# B \cdot X^\# i. \tag{2.4}$$

Proof. Consider the following diagram.



(2.5)

The parts a), e) and g) are naturality squares for η, σ, and μ_0, respectively. Commutativity of b) and f) directly follow from the definition of $\sigma^{\#}$ (2.3). The monad identities (1.6) for the monad $(Y^{\#}, \bar{\eta}, \bar{\mu})$ imply c) and d), thus, (2.5) is completely commutative. Since the homomorphic extension is unique, putting thogether (2.2) and (2.5) we have $i^{\#}=Q\bar{\mu}B\cdot\sigma^{\#}Y^{\#}B\cdot X^{\#}i$. Hence by (2.1) $f_M=\beta Y^{\#}B\cdot i^{\#}=\beta Y^{\#}B\cdot \cdot Q\bar{\mu}B\cdot\sigma^{\#}Y^{\#}B\cdot X^{\#}i$. □

Now we introduce a definition of a machine working in such a way that elementary input produces an elementary output.

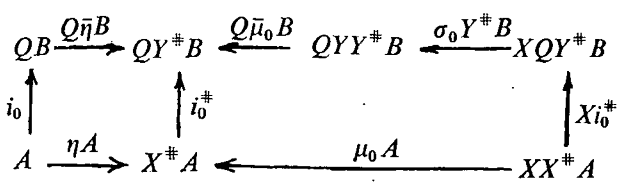
Definition 2.4. Let X and Y be varietors in $\mathcal{K}$ and let A, B be objects of $\mathcal{K}$. A *simple machine* in $\mathcal{K}$ is a system $M=(Q, i_0, \sigma_0, \beta)\colon (A, X)\to(B, Y)$, where

$Q\colon \mathcal{K}\to\mathcal{K}$ is a functor, the state functor,
$i_0\colon A\to QB$ is a $\mathcal{K}$-morphism, the initial state-output,
$\sigma_0\colon XQ\dot{\to}QY$ is a natural transformation, the transition,
$\beta\colon Q\dot{\to}I$ is a natural transformation, the final state transformation.

The response of a simple machine $M=(Q, i_0, \sigma_0, \beta)$ is the composite morphism

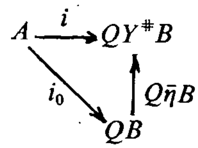
$$f_M\colon X^{\#}A\xrightarrow{i_0^{\#}}QY^{\#}B\xrightarrow{\beta Y^{\#}B}Y^{\#}B, \tag{2.6}$$

where $i_0^{\#}$ is the run map of M defined by the homomorphic extension.



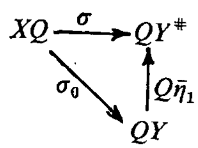
(2.7)

DEFINITION 2.5. Let $M=(Q, i, \sigma, \beta)$: $(A, X)\to(B, Y)$ be a machine in $\mathscr{K}$. We say that the initial state-output morphism i is *simple* if it can be factored thorough $Q\bar{\eta}B$: $QB\to QY^{\#}B$, i.e. there is a morphism i_0: $A\to QB$ such that



(2.8)

Similarly, the transition σ is called *simple* if there exists a natural transformation σ_0: $XQ\dot{\to} QY$ such that



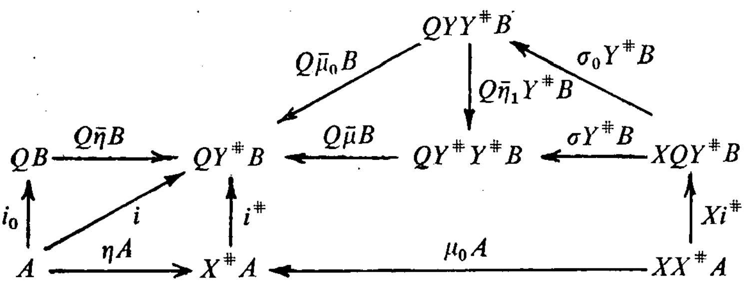
(2.9)

is commutative, where $\bar{\eta}_1$ is the *embedding* of Y into $Y^{\#}$, i.e. $\bar{\eta}_1$: $Y\xrightarrow{Y\bar{\eta}} YY^{\#}\xrightarrow{\bar{\mu}_0} Y^{\#}$.

LEMMA 2.6. Let $M=(Q, i, \sigma, \beta)$: $(A, X)\to(B, Y)$ be a machine in $\mathscr{K}$, and let i and σ be simple. Then the simple machine $M'=(Q, i_0, \sigma_0, \beta)$: $(A, X)\to(B, Y)$. where i_0 and σ_0 are as in (2.8) and (2.9), respectively, has the same response as M,

Proof. Since the final state transformation of M and that of M' is β, it is enough to prove that the corresponding run maps $i^{\#}$ and $i_0^{\#}$ coincide.

Consider the following diagram.



(2.10)

By the defining diagram (1.5) of an extended free operation, the equalities $\bar{\mu}\cdot\bar{\mu}_0Y^{\#}=\bar{\mu}_0\cdot Y\bar{\mu}$ and $\bar{\mu}\cdot\bar{\eta}Y^{\#}=1_{Y^{\#}}$ hold, thus we have

$$\bar{\mu}\cdot\bar{\eta}_1Y^{\#} = \bar{\mu}\cdot(\bar{\mu}_0\cdot Y\bar{\eta})Y^{\#} = \bar{\mu}\cdot\bar{\mu}_0Y^{\#}\cdot Y\bar{\eta}Y^{\#} = \bar{\mu}_0\cdot Y\bar{\mu}\cdot Y\bar{\eta}Y \;=$$
$$= \bar{\mu}_0\cdot Y(\bar{\mu}\cdot\bar{\eta}Y^{\#}) = \bar{\mu}\cdot Y1_{Y^{\#}} = \bar{\mu}_0.$$

Hence $Q\bar{\mu}\cdot Q\bar{\eta}_1Y^{\#}=Q\bar{\mu}_0$. Now, from the factorizations (2.8), (2.9) and the definition (2.2) of the run map $i^{\#}$, we obtain that the diagram (2.10) is completely commutative. This means that $i^{\#}$ satisfies the commutativity of diagram (2.7) which defines $i_0^{\#}$ uniquely. Thus $i^{\#}=i_0^{\#}$. □

The diagram (2.3) defines for every natural transformation $\sigma\colon XQ \dot{\rightarrow} QY^{\#}$, i.e. without σ being a transition of any machine, the extension $\sigma^{\#}\colon X^{\#}Q \dot{\rightarrow} QY^{\#}$. Alagić studied this extension in his paper [2] and proved the following theorem replaced the monad $(Y^{\#}, \bar{\eta}, \bar{\mu})$ by an arbitrary one.

THEOREM 2.7 (Alagić [2], Theorem 2.30, p. 287). Let $X, Y\colon \mathcal{K}\to\mathcal{K}$ be varietors, and $Q\colon\mathcal{K}\to\mathcal{K}$ be a functor. Then for every natural transformation $\sigma\colon XQ\dot{\rightarrow} QY^{\#}$ the extension $\sigma^{\#}\colon X^{\#}Q\dot{\rightarrow}QY^{\#}$ defined by (2.3) satisfies the commutativity of the following diagram:

$$\begin{array}{cccccc} & QY^{\#} & \xleftarrow{\;Q\bar{\mu}\;} & QY^{\#}Y^{\#} & \xleftarrow{\;\sigma^{\#}Y^{\#}\;} & X^{\#}QY^{\#} \\ {}^{Q\bar{\eta}}\nearrow & \big\uparrow{\scriptstyle \sigma^{\#}} & & & & \big\uparrow{\scriptstyle X^{\#}\sigma^{\#}} \\ Q \xrightarrow{\;\eta Q\;} & X^{\#}Q & & \xleftarrow{\;\mu Q\;} & & X^{\#}X^{\#}Q \end{array} \tag{2.11}$$

THEOREM 2.8. Let $f_1\colon X^{\#}A\to Y^{\#}B$, $f_2\colon Y^{\#}B\to Z^{\#}C$ be responses of machines $M_1\colon (A, X)\to(B, Y)$ and $M_2\colon (B, Y)\to(C, Z)$, respectively. Then the composite morphism $f_2\cdot f_1\colon X^{\#}A\to Z^{\#}C$ is again the response of a machine $M\colon (A, X)\to(C, Z)$.

Proof. Assume that machines M_1 and M_2 are specified by $M_1=(Q_1, i_1, \sigma_1, \beta_1)$, $M_2=(Q_2, i_2, \sigma_2, \beta_2)$. Consider the machine $M=(Q, i, \sigma, \beta)\colon (A, X)\to(C, Z)$, where

$$\begin{gathered} Q = Q_1Q_2, \quad \sigma = Q_1\sigma_2^{\#}\cdot\sigma_1Q_2, \\ i = A\xrightarrow{i_1} Q_1Y^{\#}B \xrightarrow{Q_1i_2^{\#}} Q_1Q_2Z^{\#}C, \quad \beta = Q_1Q_2\xrightarrow{\beta_1Q_2} Q_2\xrightarrow{\beta_2} I. \end{gathered} \tag{2.12}$$

Let us denote by $\bar{\bar{\eta}}$ and $\bar{\bar{\mu}}$ the insertion of generators and the extended free operation of Z, respectively. By the definition of the responses of M_1 and M_2, $f_2\cdot f_1=\beta_2Z^{\#}C\cdot i_2^{\#}\cdot\beta_1Y^{\#}B\cdot i_1^{\#}$. Using the naturality of β_1 we have

$$f_2\cdot f_1 = \beta_2Z^{\#}C\cdot\beta_1Q_2Z^{\#}C\cdot Q_1i_2^{\#}\cdot i_1^{\#} = (\beta_2\cdot\beta_1Q_2)Z^{\#}C\cdot Q_1i_2^{\#}\cdot i_1^{\#} = \beta Z^{\#}C\cdot Q_1i_2^{\#}\cdot i_1^{\#}.$$

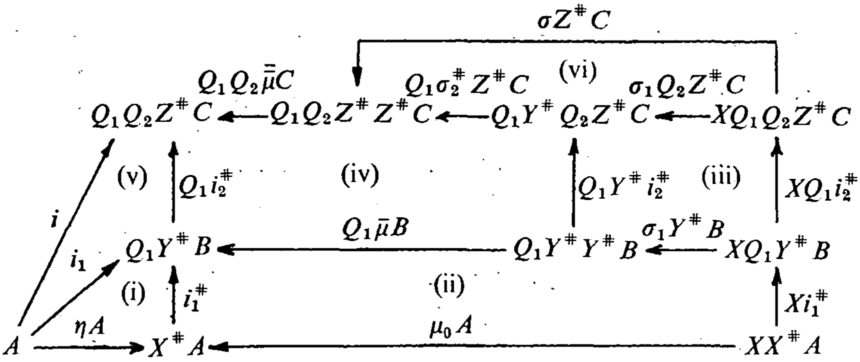
The response of M is $f_M=\beta Z^{\#}C\cdot i^{\#}$, where $i^{\#}$ is the run map of M. Thus, in order to prove that the machine M computes the composite $f_2\cdot f_1$ we need only to show that (2.13) holds

$$Q_1i_2^{\#}\cdot i_1^{\#} = i^{\#}. \tag{2.13}$$

Taking into account that the run map $i^{\#}$ is the unique morphism satisfying (2.14), it is enough to prove that the left side of (2.13) also satisfies (2.14).

$$\begin{array}{cccccc} & Q_1Q_2Z^{\#}C & \xleftarrow{\;Q_1Q_2\bar{\bar{\mu}}C\;} & Q_1Q_2Z^{\#}Z^{\#}C & \xleftarrow{\;\sigma Z^{\#}C\;} & XQ_1Q_2Z^{\#}C \\ {}^{i}\nearrow & \big\uparrow{\scriptstyle i^{\#}} & & & & \big\uparrow{\scriptstyle Xi^{\#}} \\ A \xrightarrow{\;\eta A\;} & X^{\#}A & & \xleftarrow{\;\mu_0 A\;} & & XX^{\#}A \end{array} \tag{2.14}$$

Consider the diagram (2.15) below.

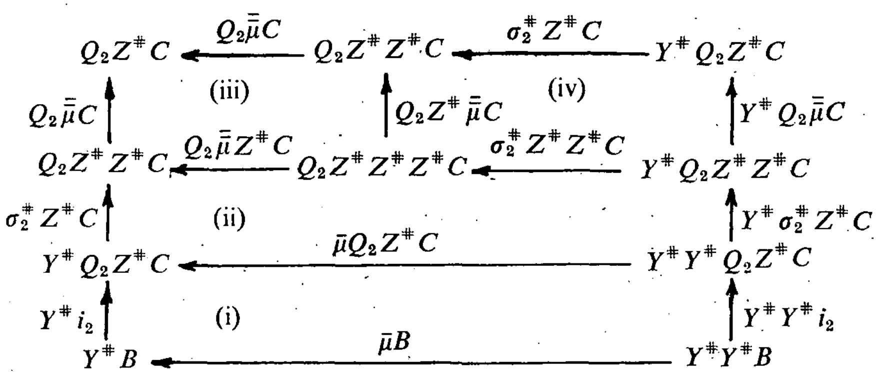


(2.15)

The subdiagrams (i) and (ii) commute by the definition of the run map $i_1^{\#}$. (iii) is a naturality square for the natural transformation σ_1. (v) and (vi) are commutative by (2.12). Thus the commutativity of (iv) is remained to prove. By Proposition 2.3 the run map $i_2^{\#}$ can be expressed by the extended transition $\sigma_2^{\#}$ of M_2 as follows

$$i_2^{\#} = Q_2\bar{\bar{\mu}}C \cdot \sigma_2^{\#}Z^{\#}C \cdot Y^{\#}i_2. \tag{2.16}$$

The diagrams (i) and (iv) in (2.17) commute, being naturality squares for $\bar{\mu}$ and $\sigma_2^{\#}$, respectively. (ii) is commutative by Theorem 2.7, finally, the commutativity of (iii) in (2.17) follows from the associativity axiom of the monad $(Z^{\#}, \bar{\bar{\eta}}, \bar{\bar{\mu}})$. Hence, (2.17) is completely commutative.



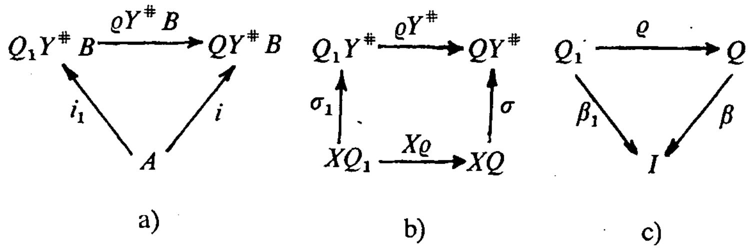
(2.17)

Putting together (2.16) and (2.17) we have

$$\begin{aligned} Q_1i_2^{\#} \cdot Q_1\bar{\mu}B &= Q_1(i_2^{\#} \cdot \bar{\mu}B) = Q_1(Q_2\bar{\bar{\mu}}C \cdot \sigma_2^{\#}Z^{\#}C \cdot Y^{\#}i_2 \cdot \bar{\mu}B) = \\ &= Q_1(Q_2\bar{\bar{\mu}}C \cdot \sigma_2^{\#}Z^{\#}C \cdot Y^{\#}Q_2\bar{\bar{\mu}}C \cdot Y^{\#}\sigma_2^{\#}Z^{\#}C \cdot Y^{\#}Y^{\#}i_2) = \\ &= Q_1Q_2\bar{\bar{\mu}}C \cdot Q_1\sigma_2^{\#}Z^{\#}C \cdot Q_1Y^{\#}(Q_2\bar{\bar{\mu}}C \cdot \sigma_2^{\#}Z^{\#}C \cdot Y^{\#}i_2) = \\ &= Q_1Q_2\bar{\bar{\mu}}C \cdot Q_1\sigma_2^{\#}Z^{\#}C \cdot Q_1Y^{\#}i_2^{\#}. \end{aligned}$$

Hence the diagram (iii) in (2.15) is commutative which completes the proof of the theorem. □

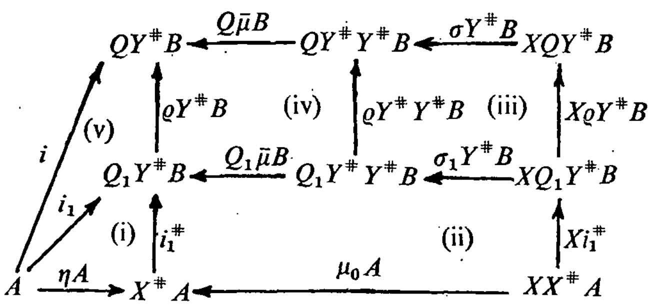
Definition 2.9. Let $M=(Q, i, \sigma, \beta)\colon (A, X)\to(B, Y)$ and $M_1=(Q_1, i_1, \sigma_1, \beta_1)\colon (A, X)\to(B, Y)$ be machines in $\mathscr{K}$. A *simulation* $\varrho\colon M_1\to M$ is a natural transformation $\varrho\colon Q_1 \dot{\to} Q$ rendering the diagrams (2.18) commutative.



(2.18)

Theorem 2.10. Let $M\colon (A, X)\to(B, Y)$ and $M_1\colon (A, X)\to(B, Y)$ be machines in $\mathscr{K}$. Whenever a simulation $\varrho\colon M_1\to M$ exists then $f_M=f_{M_1}$.

Proof. Assume that the machines M and M_1 are given by $M=(Q, i, \sigma, \beta)$, $M_1=(Q_1, i_1, \sigma_1, \beta_1)$. Then the response of M is $f_M=\beta Y^{\#}B\cdot i^{\#}$ and the response of M_1 is $f_{M_1}=\beta_1 Y^{\#}B\cdot i_1^{\#}$. Consider the diagram (2.19).



(2.19)

The diagrams (i) and (ii) in (2.19) are commutative just they define the run map $i_1^{\#}$ of M_1. Since $\varrho\colon Q_1 \dot{\to} Q$ is a simulation (iii) and (v) commute by (2.18b) and (2.18a), respectively. (iv) is a naturality square for ϱ thus (2.19) is completely commutative. Hence, we have that the morpisms $i^{\#}$ and $\varrho Y^{\#}B\cdot i_1^{\#}$ both are defined by homomorphic extensions on the same specification. The uniquenes of the homomorphic extension implies $i^{\#}=\varrho Y^{\#}B\cdot i_1^{\#}$. Finally, we have

$$f_M = \beta Y^{\#}B\cdot i^{\#} = \beta Y^{\#}B\cdot \varrho Y^{\#}B\cdot i_1^{\#} = (\beta\cdot\varrho)Y^{\#}B\cdot i_1^{\#} = \beta_1 Y^{\#}B\cdot i_1^{\#} = f_{M_1}. \quad \square$$

3. Inverse-state machines

In this section we shall develop a categorial model of Thatcher's generalized[2] sequential machine maps (see [8]), and Engelfriet's top-down tree transformations (see [5]). The term "inverse-state machine" is used here because these machines are very closely related to the inverse state transformations of Alagić [2]. We shall show that every top-down, i.e. inverse-state computation can be carried out by a machine with sutable state functor.

First, we need a theorem whose analogous one was proved in [2] and what we state as a consequence of our theorem.

THEOREM 3.1. Let (T, η', μ') be a monad and let (B, d) be a T-monad algebra in $\mathscr{K}$. Furthermore, let $X\colon \mathscr{K}\to\mathscr{K}$ be varietor and $Q\colon \mathscr{K}\to\mathscr{K}$ be a functor with right adjoint. Then for every morphism $j\colon QA\to B$ and natural transformation $\tau\colon QX\to TQ$ there exists a unique morphism $j_{\#}\colon QX^{\#}A\to B$ satisfying (3.1).

$$\begin{array}{ccccccc} & & B & \xleftarrow{d} & TB & \xleftarrow{Tj_{\#}} & TQX^{\#}A \\ & \nearrow^{j} & \uparrow{\scriptstyle j_{\#}} & & & & \uparrow{\scriptstyle \tau X^{\#}A} \\ QA & \xrightarrow{Q\eta A} & QX^{\#}A & & \xleftarrow{Q\mu_0 A} & & QXX^{\#}A \end{array} \tag{3.1}$$

Moreover, there is a bijective correspondence between triples $(j, \tau, j_{\#})$ satisfying (3.1) and triples $(i\colon A\to\bar{Q}B,\ \sigma\colon X\bar{Q}\dot{\to}\bar{Q}T,\ i^{\#}\colon X^{\#}A\to\bar{Q}B)$ satisfying (3.2), where $(Q, \bar{Q}, \nu, \varepsilon)$ is an adjunction due to Q.

$$\begin{array}{ccccccc} & & \bar{Q}B & \xleftarrow{\bar{Q}d} & \bar{Q}TB & \xleftarrow{\sigma B} & X\bar{Q}B \\ & \nearrow^{i} & \uparrow{\scriptstyle i^{\#}} & & & & \uparrow{\scriptstyle Xi^{\#}} \\ A & \xrightarrow{\eta A} & X^{\#}A & & \xleftarrow{\mu_0 A} & & XX^{\#}A \end{array} \tag{3.2}$$

Mutually inverse passages are given by (3.3) and (3.4) below.

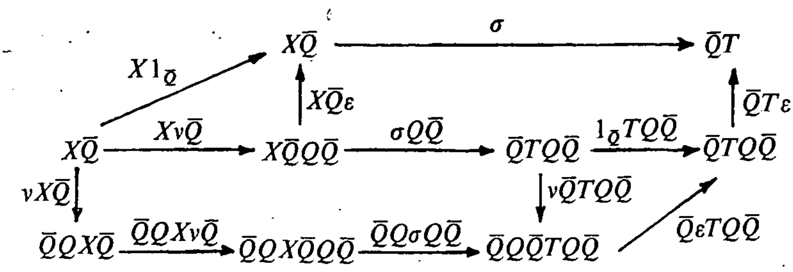
$$\begin{array}{lcl} i\colon A\to\bar{Q}B & & j\colon QA\xrightarrow{Qi}Q\bar{Q}B\xrightarrow{\varepsilon B}B \\ \sigma\colon X\bar{Q}\to\bar{Q}T & \overset{\Phi}{\mapsto} & \tau\colon QX\xrightarrow{QX\nu}QX\bar{Q}Q\xrightarrow{Q\sigma Q}Q\bar{Q}TQ\xrightarrow{\varepsilon TQ}TQ \\ i^{\#}\colon X^{\#}A\to\bar{Q}B & & j_{\#}\colon QX^{\#}A\xrightarrow{Qi^{\#}}Q\bar{Q}B\xrightarrow{\varepsilon B}B \end{array} \tag{3.3}$$

$$\begin{array}{lcl} j\colon QA\to B & & i\colon A\xrightarrow{\nu A}\bar{Q}QA\xrightarrow{\bar{Q}j}\bar{Q}B \\ \tau\colon QX\dot{\to}TQ & \overset{\Psi}{\mapsto} & \sigma\colon X\bar{Q}\xrightarrow{\nu X\bar{Q}}\bar{Q}QX\bar{Q}\xrightarrow{\bar{Q}\tau\bar{Q}}\bar{Q}TQ\bar{Q}\xrightarrow{\bar{Q}T\varepsilon}\bar{Q}T \\ j_{\#}\colon QX^{\#}A\to B & & i^{\#}\colon X^{\#}A\xrightarrow{\nu X^{\#}A}\bar{Q}QX^{\#}A\xrightarrow{\bar{Q}j_{\#}}\bar{Q}B \end{array} \tag{3.4}$$

Proof. First we show that Φ and Ψ are inverses of each other. It is a well know property of the adjunction $(Q, \bar{Q}, \nu, \varepsilon)$ that $\Psi\cdot\Phi(i)=i$, $\Phi\cdot\Psi(j)=j$. By the same argument we get $\Psi\cdot\Phi(i^{\#})=i^{\#}$, $\Phi\cdot\Psi(j_{\#})=j_{\#}$. We prove that $\Psi\cdot\Phi(\sigma)=\sigma$ and $\Phi\cdot\Psi(\tau)=\tau$.

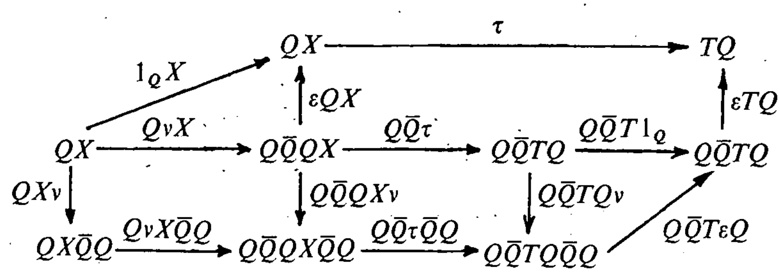
$$\Psi\cdot\Phi(\sigma) = \Psi(\varepsilon TQ\cdot Q\sigma Q\cdot QX\nu) = \bar{Q}T\varepsilon\cdot\bar{Q}(\varepsilon TQ\cdot Q\sigma Q\cdot QX\nu)\bar{Q}\cdot\nu X\bar{Q} =$$
$$= \bar{Q}T\varepsilon\cdot\bar{Q}\varepsilon TQ\bar{Q}\cdot\bar{Q}Q\sigma Q\bar{Q}\cdot\bar{Q}QX\nu\bar{Q}\cdot\nu X\bar{Q}.$$

Consider the diagram (3.5) whose triangular parts are commutative according to the triangular identities of the adjunction $(Q, \bar{Q}, \nu, \varepsilon)$. The other two parts of (3.5) commute since they are naturality squares for ν and σ, respectively. Thus we have $\Psi \cdot \Phi(\sigma) = \sigma$.



(3.5)

The following diagram also commutes by the adjunction identity $\varepsilon Q \cdot Q\nu = 1_Q$, and the naturality of ν, τ and ε.



(3.6)

Hence,

$$\Phi \cdot \Psi(\tau) = \Phi(\bar{Q}T\varepsilon \cdot \bar{Q}\tau\bar{Q} \cdot \nu X\bar{Q}) = \varepsilon TQ \cdot Q(\bar{Q}T\varepsilon \cdot \bar{Q}\tau\bar{Q} \cdot \nu X\bar{Q})Q \cdot QX\nu =$$
$$= \varepsilon TQ \cdot Q\bar{Q}T\varepsilon Q \cdot Q\bar{Q}\tau\bar{Q}Q \cdot Q\nu X\bar{Q}Q \cdot QX\nu = \tau \cdot 1_Q X = \tau \cdot 1_{QX} = \tau.$$

Let us prove that the passages Φ and Ψ preserve satisfyability of the appropriate diagrams. Assume that a triple $(i, \sigma, i^{\#})$ satisfies (3.2), Then,

$$\Phi(i^{\#}) \cdot Q\eta A = \varepsilon B \cdot Qi^{\#} \cdot Q\eta A = \varepsilon B \cdot Q(i^{\#} \cdot \eta A) = \varepsilon B \cdot Qi = \Phi(i).$$

Thus the triangular part of (3.1) holds.

$$\Phi(i^{\#}) \cdot Q\mu_0 A = \varepsilon B \cdot Qi^{\#} \cdot Q\mu_0 A = \varepsilon B \cdot Q(i^{\#} \cdot \mu_0 A) = \varepsilon B \cdot Q(\bar{Q}d \cdot \sigma B \cdot Xi^{\#}) =$$
$$= \varepsilon B \cdot Q\bar{Q}d \cdot Q\sigma B \cdot QXi^{\#}.$$

One of the adjunction identities says $1_{\bar{Q}} = \bar{Q}\varepsilon \cdot \nu\bar{Q}$ and hence $1_{QX\bar{Q}B} = QX1_{\bar{Q}}B = QX(\bar{Q}\varepsilon \cdot \nu\bar{Q})B = QX\bar{Q}\varepsilon B \cdot QX\nu\bar{Q}B$, which yields $\Phi(i^{\#}) \cdot Q\mu_0 A = \varepsilon B \cdot Q\bar{Q}d \cdot Q\sigma B \cdot (QX\bar{Q}\varepsilon B \cdot QX\nu\bar{Q}B) \cdot QXi^{\#}$. Application of commutations for the natural transformations ε, $\varepsilon T \cdot Q\sigma$, $\Phi(\sigma)$ and $\Phi(\sigma)=\varepsilon TQ \cdot Q\sigma Q \cdot QXv$ produces

$$\Phi(i^{\#}) \cdot Q\mu_0 A = d \cdot \varepsilon TB \cdot Q\sigma B \cdot QX\bar{Q}\varepsilon B \cdot QXv\bar{Q}B \cdot QXi^{\#} =$$
$$= d \cdot T\varepsilon B \cdot \varepsilon TQ\bar{Q}B \cdot Q\sigma Q\bar{Q}B \cdot QXv\bar{Q}B \cdot QXi^{\#} = d \cdot T\varepsilon B \cdot (\varepsilon TQ \cdot Q\sigma Q \cdot QXv)\bar{Q}B \cdot QXi^{\#} =$$
$$= d \cdot T\varepsilon B \cdot \Phi(\sigma)\bar{Q}B \cdot QXi^{\#} = d \cdot T\varepsilon B \cdot TQi^{\#} \cdot \Phi(\sigma)X^{\#}A =$$
$$= d \cdot T(\varepsilon B \cdot Qi^{\#}) \cdot \Phi(\sigma)X^{\#}A = d \cdot T\Phi(i^{\#}) \cdot \Phi(\sigma)X^{\#}A.$$

Thus, the triple $(j, \tau, j_{\#})=(\Phi(i), \Phi(\sigma), \Phi(i^{\#}))$ satisfies (3.1).

Conversely, let us suppose that the left side $(j, \tau, j_{\#})$ of (3.4) makes (3.1) commutative. Then, for the right side of (3.4), we have

$$\Psi(j_{\#}) \cdot \eta A = \bar{Q}j_{\#} \cdot vX^{\#}A \cdot \eta A = \bar{Q}j_{\#} \cdot \bar{Q}Q\eta A \cdot vA =$$
$$= \bar{Q}(j_{\#} \cdot Q\eta A) \cdot vA = \bar{Q}j \cdot vA = \Psi(j).$$

This means that the triangular part of (3.2) is satisfied. Let us see the other part of (3.2). By the definition (3.4) of Ψ and the naturality of v we have

$$\Psi(j_{\#}) \cdot \mu_0 A = \bar{Q}j_{\#} \cdot vX^{\#}A \cdot \mu_0 A = \bar{Q}j_{\#} \cdot \bar{Q}Q\mu_0 A \cdot vXX^{\#}A =$$
$$= \bar{Q}(j_{\#} \cdot Q\mu_0 A) \cdot vXX^{\#}A = \bar{Q}(d \cdot Tj_{\#} \cdot \tau X^{\#}A) \cdot vXX^{\#}A =$$
$$= \bar{Q}d \cdot \bar{Q}Tj_{\#} \cdot \bar{Q}\tau X^{\#}A \cdot vXX^{\#}A.$$

From the adjunction identity $1_Q=\varepsilon Q \cdot Qv$ follows $1_{\bar{Q}TQX^{\#}A}=\bar{Q}T1_Q X^{\#}A= =\bar{Q}T(\varepsilon Q \cdot Qv)X^{\#}A=\bar{Q}T\varepsilon QX^{\#}A \cdot \bar{Q}TQvX^{\#}A$, thus we get

$$\Psi(j_{\#}) \cdot \mu_0 A = \bar{Q}d \cdot \bar{Q}Tj_{\#} \cdot \bar{Q}T\varepsilon QX^{\#}A \cdot \bar{Q}TQvX^{\#}A \cdot \bar{Q}\tau X^{\#}A \cdot vXX^{\#}A.$$

Using the naturality of $\bar{Q}T\varepsilon$ and $\bar{Q}\tau \cdot vX$ we conclude

$$\Psi(j) \cdot \mu_0 A = \bar{Q}d \cdot \bar{Q}T\varepsilon B \cdot \bar{Q}TQ\bar{Q}j_{\#} \cdot \bar{Q}TQvX^{\#}A \cdot \bar{Q}\tau X^{\#}A \cdot vXX^{\#}A =$$
$$= \bar{Q}d \cdot \bar{Q}T\varepsilon B \cdot \bar{Q}TQ(\bar{Q}j_{\#} \cdot vX^{\#}A) \cdot (\bar{Q}\tau \cdot vX)X^{\#}A =$$
$$= \bar{Q}d \cdot \bar{Q}T\varepsilon B \cdot \bar{Q}TQ\Psi(j_{\#}) \cdot (\bar{Q}\tau \cdot vX)X^{\#}A = \bar{Q}d \cdot \bar{Q}T\varepsilon B \cdot (\bar{Q}\tau \cdot vX)\bar{Q}B \cdot X\Psi(j_{\#}) =$$
$$= \bar{Q}d \cdot (\bar{Q}T\varepsilon \cdot \bar{Q}\tau\bar{Q} \cdot vX\bar{Q})B \cdot X\Psi(j_{\#}) = \bar{Q}d \cdot \Psi(\tau)B \cdot X\Psi(j_{\#}).$$

Thus the triple $(i, \sigma, i^{\#})=(\Psi(j), \Psi(\tau), \Psi(j_{\#}))$ satisfies (3.2). The existential statement of the Theorem can be obtained as follows. For given morphism $j\colon QA \to B$ and natural transformation $\tau\colon QX \dot{\to} TQ$ consider $i:=\Phi(j)$, $\sigma:=\Phi(\tau)$ and take the unique $i^{\#}$ satisfying (3.2). This $i^{\#}$ exists because $(X^{\#}A, \mu_0 A)$ is a free X-algebra. Then, as we have shown, $(\Psi(i), \Psi(\sigma), \Psi(i^{\#}))$ satisfies (3.1). But $\Psi(i)=j$ and $\Psi(\sigma)=\tau$, hence $(j, \tau, \Psi(i^{\#}))$ satisfies (3.1). The uniqueness of $j_{\#}$ in (3.1) follows from the facts that Ψ is bijective and $i^{\#}$ is unique in (3.2). This completes the proof of Theorem 3.1. □

The following statement was proved in another way in Alagić [2] (see Theorem 3.10 pp. 297) replaced $(Y^{\#}, \bar{\eta}, \bar{\mu})$ by an arbitrary monad.

STATEMENT 3.2. Let X, Y be varietors in $\mathcal{K}$ and let $Q: \mathcal{K} \to \mathcal{K}$ be a functor having right adjoint. Then for every natural transformation $\tau: QX \dot{\to} Y^{\#}Q$ there is a unique $\tau_{\#}: QX^{\#} \dot{\to} Y^{\#}Q$ defined by

$$\begin{array}{ccccccc} & & Y^{\#}Q & \xleftarrow{\bar{\mu}Q} & Y^{\#}Y^{\#}Q & \xleftarrow{Y^{\#}\tau_{\#}} & Y^{\#}QX^{\#} \\ & \overset{\bar{\eta}Q}{\nearrow} & \uparrow{\scriptstyle \tau_{\#}} & & & & \uparrow{\scriptstyle \tau X^{\#}} \\ Q & \xrightarrow{Q\eta} & QX^{\#} & & \xleftarrow{\qquad Q\mu_0 \qquad} & & QXX^{\#} \end{array} \tag{3.7}$$

Proof. Let A be an object of $\mathcal{K}$. As $(Y^{\#}, \bar{\eta}, \bar{\mu})$ is a monad it is evident that $(Y^{\#}QA, \bar{\mu}QA)$ is an $Y^{\#}$-monad algebra. Take $j := \bar{\eta}QA: QA \to Y^{\#}QA$ and apply Theorem 3.1 for this j and τ above. We have that there exists a unique $j_{\#}$: $QX^{\#}A \to Y^{\#}QA$ denoted by $\tau_{\#}A$ which renders (3.8) commutative.

$$\begin{array}{ccccccc} & & Y^{\#}QA & \xleftarrow{\bar{\mu}QA} & Y^{\#}Y^{\#}QA & \xleftarrow{Y^{\#}\tau_{\#}A} & Y^{\#}QX^{\#}A \\ & \overset{\bar{\eta}QA}{\nearrow} & \uparrow{\scriptstyle \tau_{\#}A} & & & & \uparrow{\scriptstyle \tau X^{\#}A} \\ QA & \xrightarrow{Q\eta A} & QX^{\#}A & & \xleftarrow{\qquad Q\mu_0 A \qquad} & & QXX^{\#}A \end{array} \tag{3.8}$$

Thus we need only to show that $\tau_{\#}A$ in (3.8) is natural in A. The proof is straightforward. □

DEFINITION 3.3. Let A, B be objects of $\mathcal{K}$ and let X, Y be varietors in $\mathcal{K}$. An *inverse-state machine*

$$M = (Q, \alpha, \tau, j): (A, X) \to (B, Y)$$

in $\mathcal{K}$ consists of the following data:

$Q: \mathcal{K} \to \mathcal{K}$ a functor, the *state functor,* having right adjoint,
$\alpha: I \dot{\to} Q$ a natural transformation, the *initial state* transformation,
$\tau: QX \dot{\to} Y^{\#}Q$ a natural transformation, the *transition,*
$j: QA \to Y^{\#}B$ a morphism, the *final state-output* morphism.

DEFINITION 3.4. Let $M = (Q, \alpha, \tau, j): (A, X) \to (B, Y)$ be an inverse-state machine in $\mathcal{K}$. The morphism f_M *computed* by M or the *response* of M is defined by

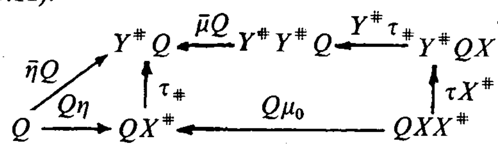
$$f_M: X^{\#}A \xrightarrow{\alpha X^{\#}A} QX^{\#}A \xrightarrow{j_{\#}} Y^{\#}B, \tag{3.9}$$

where $j_{\#}$ is the *(inverse-state) run map* defined to be the unique morphism

$$\begin{array}{ccccccc} & & Y^{\#}B & \xleftarrow{\bar{\mu}B} & Y^{\#}Y^{\#}B & \xleftarrow{Y^{\#}j_{\#}} & Y^{\#}QX^{\#}A \\ & \overset{j}{\nearrow} & \uparrow{\scriptstyle j_{\#}} & & & & \uparrow{\scriptstyle \tau X^{\#}A} \\ QA & \xrightarrow{Q\eta A} & QX^{\#}A & & \xleftarrow{\qquad Q\mu_0 A \qquad} & & QXX^{\#}A \end{array} \tag{3.10}$$

according to Theorem 3.1.

By Statement 3.2 we define the *extended transition* of the inverse-state machine M by the diagram (3.11).



(3.11)

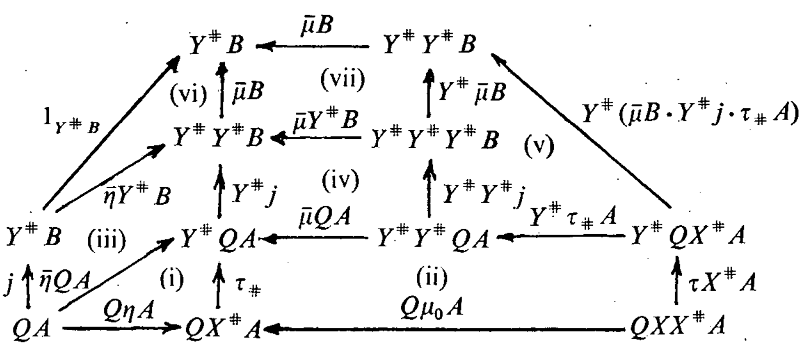
We shall show that the response of an inverse-state machine can be expressed in terms of the extended transition.

LEMMA 3.5. Let $M=(Q, \alpha, \tau, j)$: $(A, X)\to(B, Y)$ be an inverse-state machine in $\mathscr{K}$. The response of M is

$$f_M = \bar{\mu}B \cdot Y^{\#} j \cdot \tau_{\#} A \cdot \alpha X^{\#} A, \tag{3.12}$$

where $\tau_{\#}$ is the extended transition of M.

Proof. Because of the fact that the run map $j_{\#}$ of M is unique in (3.10) it is sufficient to prove that substituting the morphism $\bar{\mu}B \cdot Y^{\#} j \cdot \tau_{\#} A$ for $j_{\#}$, (3.10) remaines commutative. Consider the diagram



(3.13)

(i) and (ii) are commutative by the diagram (3.11) of the extended transition $\tau_{\#}$. (iii) and (iv) are naturality squares for $\bar{\eta}$ and $\bar{\mu}$, respectively, hence they commute. The commutativity of (vi) and (vii) follows directly from the monad identities of $(Y^{\#}, \bar{\eta}, \bar{\mu})$. (v) just expresses the value of the functor $Y^{\#}$ on a composite morphism. Thus the whole diagram is commutative which ends the proof of the Lemma. □

THEOREM 3.6. Given inverse-state machine $M=(Q, \alpha, \tau, j)$: $(A, X)\to(B, Y)$ there is a machine $\overline{M}$: $(A, X)\to(B, Y)$ computing the response of M.

Proof. Let $\bar{Q}$ be a right adjoint of Q, and denote the corresponding adjunction by $(Q, \bar{Q}, \nu, \varepsilon)$. Define a machine $M=(Q, i, \sigma, \beta)$ by

$$\begin{aligned} i&: A \xrightarrow{\nu A} \bar{Q}QA \xrightarrow{\bar{Q}j} \bar{Q}Y^{\#}B, \\ \sigma&: X\bar{Q} \xrightarrow{\nu X\bar{Q}} \bar{Q}QX\bar{Q} \xrightarrow{\bar{Q}\tau\bar{Q}} \bar{Q}Y^{\#}Q\bar{Q} \xrightarrow{\bar{Q}Y^{\#}\varepsilon} \bar{Q}Y^{\#}, \\ \beta&: \bar{Q} \xrightarrow{\alpha\bar{Q}} Q\bar{Q} \xrightarrow{\varepsilon} I. \end{aligned} \tag{3.14}$$

We are going to prove that $f_M = f_{\bar{M}}$. By the notations above

$$f_M = j_{\#} \cdot \alpha X^{\#} A, \quad f_{\bar{M}} = \beta Y^{\#} B \cdot i^{\#}, \tag{3.15}$$

where $j_{\#}$ and $i^{\#}$ are the run maps of M and $\bar{M}$, respectively. Thus the triple $(j, \tau, j_{\#})$ satisfies (3.10) and hence, by Theorem 3.1 the triple $(i, \sigma, \bar{Q} j_{\#} \cdot \nu X^{\#} A)$ satisfies the commutativity of the diagram which defines the run map $i^{\#}$ of $\bar{M}$. The uniqueness of the homomorphic extension implies

$$i^{\#} = \bar{Q} j_{\#} \cdot \nu X^{\#} A. \tag{3.16}$$

Thus we have

$$f_{\bar{M}} = (\varepsilon \cdot \alpha \bar{Q}) Y^{\#} B \cdot \bar{Q} j_{\#} \cdot \nu X^{\#} A = \varepsilon Y^{\#} B \cdot \alpha \bar{Q} Y^{\#} B \cdot \bar{Q} j_{\#} \cdot \nu X^{\#} A. \tag{3.17}$$

Consider the diagram below.

$$\begin{array}{ccccc}
\bar{Q} Y^{\#} B & \xrightarrow{\alpha \bar{Q} Y^{\#} B} & Q \bar{Q} Y^{\#} B & \xrightarrow{\varepsilon Y^{\#} B} & Y^{\#} B \\
\uparrow {\scriptstyle \bar{Q} j_{\#}} & & \uparrow {\scriptstyle Q \bar{Q} j_{\#}} & & \uparrow {\scriptstyle j_{\#}} \\
\bar{Q} Q X^{\#} A & & Q \bar{Q} Q X^{\#} A & \xrightarrow{\varepsilon Q X^{\#} A} & Q X^{\#} A \\
\uparrow {\scriptstyle \nu X^{\#} A} & & \uparrow {\scriptstyle Q \nu X^{\#} A} & \nearrow {\scriptstyle 1_Q X^{\#} A} & \\
X^{\#} A & \xrightarrow{\alpha X^{\#} A} & Q X^{\#} A & &
\end{array} \tag{3.18}$$

The triangular part of (3.18) is commutative by reason of the adjunction identity $\varepsilon Q \cdot Q\nu = 1_Q$, and the other two parts of (3.18) commute being naturality squares for α and ε, respectively. Putting together (3.17) and (3.18) we have

$$f_{\bar{M}} = j_{\#} \cdot 1_Q X^{\#} A \cdot \alpha X^{\#} A = j_{\#} \cdot \alpha X^{\#} A = f_M. \quad \square$$

Now we state the dual of Theorem 3.6.

THEOREM 3.7. Let $M = (\bar{Q}, i, \sigma, \beta)\colon (A, X) \to (B, Y)$ be a machine in $\mathcal{K}$ such that its state functor $\bar{Q}$ has a left adjoint. Then the response of M can be computed by an inverse-state machine.

Proof. Let $(Q, \bar{Q}, \nu, \varepsilon)$ be an adjunction. Define an inverse-state machine $M = (Q, \alpha, \tau, j)\colon (A, X) \to (B, Y)$ by

$$\begin{aligned}
\alpha &\colon I \xrightarrow{\nu} \bar{Q} Q \xrightarrow{\beta \bar{Q}} Q, \\
\tau &\colon QX \xrightarrow{QX\nu} QX\bar{Q}Q \xrightarrow{Q\sigma Q} Q\bar{Q}Y^{\#}Q \xrightarrow{\varepsilon Y^{\#} Q} Y^{\#} Q, \\
j &\colon QA \xrightarrow{Qi} Q\bar{Q}Y^{\#} B \xrightarrow{\varepsilon Y^{\#} B} Y^{\#} B.
\end{aligned} \tag{3.19}$$

In consequence of Theorem 3.6 it is sufficient to prove that applying the construction (3.14) for the data in (3.19) we get back the specification of the machine M, i.e.

$$i = \bar{Q} j \cdot \nu A, \quad \sigma = \varepsilon Y^{\#} \bar{Q} \cdot \bar{Q} \tau \bar{Q} \cdot \nu X \bar{Q}, \quad \beta = \varepsilon \cdot \alpha \bar{Q}. \tag{3.20}$$

The first two equalities of (3.19) have already been proved in Theorem 3.1 in context that Φ and Ψ are inverses of each other. The remaining $\beta=\varepsilon\cdot\alpha\bar{Q}$ is obvious from the adjunction identity

$$1_{\bar{Q}}=\bar{Q}\varepsilon\cdot\nu\bar{Q};\ \varepsilon\cdot\alpha\bar{Q}=\varepsilon\cdot(\beta Q\cdot\nu)\bar{Q}=\varepsilon\cdot\beta Q\bar{Q}\cdot\nu\bar{Q}=\beta\cdot\bar{Q}\varepsilon\cdot\nu\bar{Q}=\beta\cdot 1_{\bar{Q}}=\beta. \quad \square$$

THEOREM 3.8. Let M_1: $(A, X)\to(B, Y)$ and M_2: $(B, Y)\to(C, Z)$ be inverse-state machines in $\mathcal{K}$. Then the composite morphism $f_{M_2}\cdot f_{M_1}$: $X^{\#}A\to Z^{\#}C$ can be again computed by an inverse state machine.

Proof. Assume that M_1 has a state functor Q_1 and M_2 has a state functor Q_2. Denote a right adjoint of Q_1 and Q_2 by $\bar{Q}_1$ and $\bar{Q}_2$, respectively. By Theorem 3.6 the responses f_{M_1} and f_{M_2} can be computed by machines whose state functors are $\bar{Q}_1$ and $\bar{Q}_2$, respectively. Now apply Theorem 2.8 which says that the composite morphism $f_{M_2}\cdot f_{M_1}$ is the response of a machine with state functor $\bar{Q}_1\bar{Q}_2$. According to Theorem 3.7 if the composite functor $\bar{Q}_1\bar{Q}_2$ has left adjoint then the morphism $f_{M_1}\cdot f_{M_2}$ can be computed by an inverse-state machine. But, it is a well known result in category theory that the composite functors yield an adjunction, i.e. Q_2Q_1 is left adjoint to $\bar{Q}_1\bar{Q}_2$ (see [7], Theorem 8.1, pp. 101). $\square$

4. Generalized sequential machines in categories

The concept of generalized sequential machines in categories having binary products is developed in this section. A generalized sequential machine is a machine whose state functor is a product-functor and its final state transformation is a projection.

We also investigate sequential machines, i.e. machines working sequentially, moreover, elementary input produces an elementary output. Morphisms computed by generalized sequential as well as sequential machines in a category are characterized.

Throughout this section we assume that a category $\mathcal{K}$ with binary products is given.

DEFINITION 4.1. Fix a choice of a product diagram $A\xleftarrow{p}A\times B\xrightarrow{q}B$ for every given pair (A, B) of objects of $\mathcal{K}$, and given morphisms f: $A'\to A$, g: $B'\to B$ define the morphism $f\times g$: $A'\times B'\to A\times B$ by

$$\begin{array}{ccccc} A & \xleftarrow{p} & A\times B & \xrightarrow{q} & B \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f\times g} & & \uparrow{\scriptstyle g} \\ A' & \xleftarrow{p'} & A'\times B' & \xrightarrow{q'} & B' \end{array} \tag{4.1}$$

It is well known that in this case each object S of $\mathcal{K}$ induces a functor $S\times -: \mathcal{K}\to\mathcal{K}$ by

$$(S\times -)A:=S\times A,\quad (S\times -)f:=1_S\times f. \tag{4.2}$$

These functors are called *product functors*. It is obvious from (4.1) that the family of projections πA: $S\times A\to A$ constitute a natural transformation $\pi: (S\times -)\dot{\to}I$, called projection transformation. For orbitrary morphisms $h_1\colon C\to A$, $h_2\colon C\to B$ we use the notation (h_1, h_2) for the unique morphism satisfying (4.3) below.

$$\begin{array}{ccccc} A & \xleftarrow{\;p\;} & A\times B & \xrightarrow{\;q\;} & B \\ & {}^{h_1}\nwarrow & \uparrow{\scriptstyle (h_1,\, h_2)} & \nearrow^{h_2} & \\ & & C & & \end{array} \tag{4.3}$$

According to (4.1) and (4.3) we have the following identities:

$$(f\times g)\cdot(h_1, h_2) = (f\cdot h_1, g\cdot h_2) \tag{4.4}$$

$$(f\times g)\cdot(f_1\times g_1) = (f\cdot f_1)\times(g\cdot g_1) \tag{4.5}$$

$$(h_1, h_2)\cdot h = (h_1\cdot h, h_2\cdot h) \tag{4.6}$$

DEFINITION 4.2. A *generalized sequential machine* in $\mathcal{K}$ is a machine $M=(Q, i, \sigma, \beta)\colon (A, X)\to(B, Y)$ whose state functor Q is a product-functor induced by an object S of $\mathcal{K}$, and the final state transformation is the projection $S\times - \dot{\to} I$. Thus, a generalized sequential machine can be specified by

$M=(S, i, \sigma)\colon (A, X)\to(B, Y)$, where S is an object of $\mathcal{K}$, the *state object*,
$i\colon A\to S\times Y^{\#}B$ is a $\mathcal{K}$-morphism, the *initial state-output* morphism,
$\sigma\colon X(S\times -)\dot{\to}(S\times -)Y^{\#}$ is a natural transformation, the *transition*.

The *response* of a generalized sequential machine $M=(S, i, \sigma)\colon (A, X)\to(B, Y)$ is defined to be the response of the machine $M'=(S\times -, i, \sigma, \pi)\colon (A, X)\to(B, Y)$, where π is the projection $S\times - \dot{\to} I$.

Now we give a definition of sequential machines in a category. A sequential machine is a simple machine whose state functor is a product functor and whose final state transformation is the projection.

DEFINITION 4.3. Let A, B be objects of $\mathcal{K}$ and let X, Y be varietors in $\mathcal{K}$. A *sequential machine*

$$M = (S, i_0, \sigma_0)\colon (A, X) \to (B, Y)$$

in $\mathcal{K}$ consists of the following data:
an object S of $\mathcal{K}$, the state object,
a $\mathcal{K}$-morphism $i_0\colon A\to S\times B$, the initial state-output,
a natural transformation $\sigma_0\colon X(S\times -)\dot{\to}(S\times -)Y$, the transition.
The response of a sequential machine $M=(S, i_0, \sigma_0)$ is the composite morphism $f_M=\pi Y^{\#}B\cdot i_0^{\#}$, where $\pi\colon S\times - \dot{\to} I$ is the projection and $i_0^{\#}$ is the run map of M defined by

$$\begin{array}{ccccccc} S\times B & \xrightarrow{1_S\times\bar\eta B} & S\times Y^{\#}B & \xleftarrow{1_S\times\bar\mu_0 B} & S\times Y{\cdot}Y^{\#}B & \xleftarrow{\sigma_0 Y^{\#}B} & X(S\times Y^{\#}B) \\ \uparrow{\scriptstyle i_0} & & \uparrow{\scriptstyle i_0^{\#}} & & & & \uparrow{\scriptstyle Xi_0^{\#}} \\ A & \xrightarrow{\eta A} & X^{\#}A & & \xleftarrow{\mu_0 A} & & XX^{\#}A \end{array} \tag{4.7}$$

DEFINITION 4.4. Let A, B be objects of $\mathscr{K}$ and let X, Y be varietors in $\mathscr{K}$. A morphism $f\colon X^\# A \to Y^\# B$ is called *initial-segment preserving* if there is a natural transformation

$$\lambda\colon X(X^\# A \times -) \dot{\to} Y^\#, \tag{4.8}$$

such that

$$\begin{array}{ccccc} X^\# A & & \xrightarrow{\quad f \quad} & & Y^\# B \\ \uparrow{\scriptstyle \mu_0 A} & & & & \uparrow{\scriptstyle \bar{\mu} B} \\ XX^\# A & \xrightarrow{X(1_{X^\# A},\, f)} & X(X^\# A \times Y^\# B) & \xrightarrow{\lambda Y^\# B} & Y^\# Y^\# B \end{array} \tag{4.9}$$

THEOREM 4.5. A morphism $f\colon X^\# A \to Y^\# B$ can be computed by a generalized sequential machine in $\mathscr{K}$ if and only if f is initial-segment preserving.

Proof. Assume that a morphism $f\colon X^\# A \to Y^\# B$ is computed by a generalized sequential machine $M=(S, i, \sigma)\colon (A, X) \to (B, Y)$. Thus, $f=f_M=\pi Y^\# B \cdot i^\#$, where π is the projection transformation $S \times - \dot{\to} I$ and $i^\#$ is the run map of M defined by (4.10) below.

$$\begin{array}{ccccc} & S\times Y^\# B & \xleftarrow{1_S \times \bar{\mu} B} S\times Y^\# Y^\# B \xleftarrow{\sigma Y^\# B} & X(S\times Y^\# B) \\ {\scriptstyle i} \nearrow & \uparrow{\scriptstyle i^\#} & & \uparrow{\scriptstyle Xi^\#} \\ A \xrightarrow{\eta A} & X^\# A & \xleftarrow{\qquad \mu_0 A \qquad} & XX^\# A \end{array} \tag{4.10}$$

Denote by p the projection $S \leftarrow S\times Y^\# B$, and let

$$r\colon X^\# A \xrightarrow{i^\#} S\times Y^\# B \xrightarrow{p} S. \tag{4.11}$$

It can be seen by the identity (4.5) that the morphism $r\colon X^\# A \to S$ induces a natural transformation $(r\times -)\colon X^\# A \times - \dot{\to} S\times -$ by

$$(r\times -)C\colon r\times 1_C\colon X^\# A\times C \to S\times C \tag{4.12}$$

for each object C of $\mathscr{K}$. Consider the natural transformation

$$\lambda\colon X(X^\# A\times -) \xrightarrow{X(r\times -)} X(S\times -) \xrightarrow{\sigma} (S\times -)Y^\# \xrightarrow{\pi Y^\#} Y^\#. \tag{4.13}$$

We shall prove that this λ satisfies (4.9) with the response morphism f. First, we show that $i^\# = (r, f)$. Because $S \xleftarrow{p} S\times Y^\# B \xrightarrow{\pi Y^\# B} Y^\# B$ is a product diagram $(p, \pi Y^\# B) = 1_{S\times Y^\# B}$. Thus we have

$$i^\# = 1_{S\times Y^\# B}\cdot i^\# = (p, \pi Y^\# B)\cdot i^\# = (p\cdot i^\#, \pi Y^\# B\cdot i^\#) = (r, f). \tag{4.14}$$

By (4.4) we obtain from (4.14)

$$i^\# = (r\cdot 1_{X^\# A},\ 1_{Y^\# B}\cdot f) = (r\times 1_{Y^\# B})\cdot(1_{X^\# A}, f). \tag{4.15}$$

Taking into account (4.10) and (4.15) we have

$$\begin{aligned} f\cdot\mu_0 A &= \pi Y^{\#}B\cdot i^{\#}\cdot\mu_0 A = \pi Y^{\#}B\cdot(1_S\times\bar{\mu}B)\cdot\sigma Y^{\#}B\cdot Xi^{\#} = \\ &= \bar{\mu}B\cdot\pi Y^{\#}Y^{\#}B\cdot\sigma Y^{\#}B\cdot Xi^{\#} = \bar{\mu}B\cdot(\pi Y^{\#}\cdot\sigma)Y^{\#}B\cdot Xi^{\#} = \\ &= \bar{\mu}B\cdot(\pi Y^{\#}\cdot\sigma)Y^{\#}B\cdot X((r\times 1_{Y^{\#}B})\cdot(1_{Y^{\#}A}, f)) = \\ &= \bar{\mu}B\cdot(\pi Y^{\#}\cdot\sigma)Y^{\#}B\cdot X(r\times -)Y^{\#}B\cdot X(1_{X^{\#}A}, f) = \\ &= \bar{\mu}B\cdot(\pi Y^{\#}\cdot\sigma\cdot X(r\times -))Y^{\#}B\cdot X(1_{X^{\#}A}, f). \end{aligned}$$

Applying the definition (4.13) of the natural transformation λ we conclude that

$$f\cdot\mu_0 A = \bar{\mu}B\cdot\lambda Y^{\#}B\cdot X(1_{X^{\#}A}, f),$$

which proves the commutativity of (4.9).

Conversely, assume that a morphism $f\colon X^{\#}A\to Y^{\#}B$ is initial-segment preserving, i.e. there is a natural transformation $\lambda\colon X(X^{\#}A\times -)\overset{\cdot}{\to} Y^{\#}$ rendering the diagram (4.9) commutative. For each object C of $\mathscr{K}$ let us denote by ϱC the projection $X^{\#}A\leftarrow X^{\#}A\times C$. We show that the composite morphism

$$\begin{aligned} \sigma C\colon X(X^{\#}A\times -)\,C = X(X^{\#}A\times C)&\xrightarrow{(\mu_0 A\cdot X\varrho C,\,\lambda C)} X^{\#}A\times Y^{\#}C = \\ &= (X^{\#}A\times -)Y^{\#}C \end{aligned} \tag{4.16}$$

is natural in C, thus we get a natural transformation

$$\sigma\colon X(X^{\#}A\times -)\overset{\cdot}{\to}(X^{\#}A\times -)Y^{\#}. \tag{4.17}$$

Let $h\colon C\to D$ be an arbitrary morphism. We have to prove that

$$\begin{array}{ccc} X(X^{\#}A\times C) & \xrightarrow{\sigma C} & X^{\#}A\times Y^{\#}C \\ {\scriptstyle X(X^{\#}A\times -)h}\downarrow & & \downarrow{\scriptstyle (X^{\#}A\times -)Y^{\#}h} \\ X(X^{\#}A\times D) & \xrightarrow{\sigma D} & X^{\#}A\times Y^{\#}D. \end{array} \tag{4.18}$$

By (4.4) and the definition of the product-functor $X^{\#}A\times -$ we have

$$\begin{aligned} \sigma D\cdot X(X^{\#}A\times -)h &= (\mu_0 A\cdot X\varrho D, \lambda D)\cdot X(1_{X^{\#}A}\times h) = \\ &= (\mu_0 A\cdot X(\varrho D\cdot(1_{X^{\#}A}\times h)), \lambda D\cdot X(1_{X^{\#}A}\times h)). \end{aligned}$$

From (4.1) follows $\varrho D\cdot(1_{X^{\#}A}\times h) = 1_{X^{\#}A}\cdot\varrho C = \varrho C$, hence using the naturality of λ we obtain

$$\begin{aligned} \sigma D\cdot X(X^{\#}A\times -)h &= (\mu_0 A\cdot X\varrho C, Y^{\#}h\cdot\lambda C) = \\ &= (1_{X^{\#}A}\times Y^{\#}h)\cdot(\mu_0 A\cdot X\varrho C, \lambda C) = (X^{\#}A\times -)Y^{\#}h\cdot\sigma C. \end{aligned}$$

Thus the diagram (4.18) is commutative.

Let us define the generalized sequential machine

$$M = (X^{\#}A, i, \sigma)\colon (A, X)\to(B, Y)$$

by σ in (4.16) and put

$$i:\ A \xrightarrow{\eta A} X^{\#}A \xrightarrow{(1_{X^{\#}A},\, f)} X^{\#}A \times Y^{\#}B. \tag{4.19}$$

We show that f is the response of M, i.e.

$$f = \pi Y^{\#}B \cdot i^{\#}, \tag{4.20}$$

where π is the projection transformation $X^{\#}A \times - \overset{\cdot}{\to} I$ and $i^{\#}$ is the run map of M:

$$\begin{array}{ccccc}
 & X^{\#}A\times Y^{\#}B & \xleftarrow{1_{X^{\#}A}\times \bar{\mu}B} X^{\#}A\times Y^{\#}Y^{\#}B & \xleftarrow{\sigma Y^{\#}B} & X(X^{\#}A\times Y^{\#}B) \\
 {}^{i}\nearrow & \uparrow{\scriptstyle i^{\#}} & & & \uparrow{\scriptstyle Xi^{\#}} \\
A \xrightarrow{\eta A} & X^{\#}A & \xleftarrow{\qquad\qquad \mu_0 A \qquad\qquad} & & XX^{\#}A
\end{array} \tag{4.21}$$

In order to prove (4.20) it is enough to verify that $i^{\#}=(1_{X^{\#}A}, f)$. We do this by observing from the following that $(1_{X^{\#}A}, f)$ is an X-homomorphic extension by the same specification as $i^{\#}$, which means (4.21).

a) $(1_{X^{\#}A}, f)\cdot\eta A = i$, by definition (4.19) of i.

b) $(1_{X^{\#}A}, f)\cdot\mu_0 A = (1_{X^{\#}A}, \bar{\mu}B)\cdot\sigma Y^{\#}B\cdot X(1_{X^{\#}A}, f)$.

Applying (4.6), (4.9) and (4.4) in this order we have

$$(1_{X^{\#}A}, f)\cdot\mu_0 A = (\mu_0 A, f\cdot\mu_0 A) = (\mu_0 A, \bar{\mu}B\cdot\lambda Y^{\#}B\cdot X(1_{X^{\#}A}, f)) =$$
$$= (1_{X^{\#}A}\times\bar{\mu}B)\cdot(\mu_0 A, \lambda Y^{\#}B\cdot X(1_{X^{\#}A}, f)).$$

By (4.3) $\varrho Y^{\#}B\cdot(1_{X^{\#}A}, f) = 1_{X^{\#}A}$ holds, thus

$$(1_{X^{\#}A}, f)\cdot\mu_0 A = (1_{X^{\#}A}\times\bar{\mu}B)\cdot(\mu_0 A\cdot\times 1_{X^{\#}A}, \lambda Y^{\#}B\cdot X(1_{X^{\#}A}, f)) =$$
$$= (1_{X^{\#}A}\times\bar{\mu}B)\cdot(\mu_0 A\cdot X(\varrho Y^{\#}B\cdot(1_{X^{\#}A}, f)), \lambda Y^{\#}B\cdot X(1_{X^{\#}A}, f)) =$$
$$= (1_{X^{\#}A}\times\bar{\mu}B)\cdot(\mu_0 A\cdot X\varrho Y^{\#}B, \lambda Y^{\#}B)\cdot X(1_{X^{\#}A}, f).$$

Taking the definition (4.16) of the natural transformation σ we conclude that

$$(1_{X^{\#}A}, f)\cdot\mu_0 A = (1_{X^{\#}A}\times\bar{\mu}B)\cdot\sigma Y^{\#}B\cdot X(1_{X^{\#}A}, f)$$

which completes the proof of the theorem.

Corollary 4.6. Let A be an object of $\mathcal{K}$ and let X be a varietor in $\mathcal{K}$. The object $X^{\#}A$ is universal in the sense that for every generalized sequential machine M: $(A, X)\to(B, Y)$ there is a generalized sequential machine M': $(A, X)\to(B, Y)$ whose state object is $X^{\#}A$, and M' computes the response of M.

Now we give a characterization of morphisms computed by sequential machines in $\mathcal{K}$.

THEOREM 4.7. Let X, Y be varietors in $\mathscr{K}$ and let A, B be objects of $\mathscr{K}$. A morphism $f\colon X^{\#}A\to Y^{\#}B$ can be computed by a sequential machine in $\mathscr{K}$ iff the following two conditions are satisfied:

i) there is a morphism $f_0\colon A\to B$ such that

$$\begin{array}{ccc} X^{\#}A & \xrightarrow{f} & Y^{\#}B \\ \eta A\uparrow & & \uparrow\bar{\eta}B \\ A & \xrightarrow{f_0} & B \end{array} \tag{4.22}$$

ii) there is a natural transformation $\lambda_0\colon X(X^{\#}A\times -)\dot{\to} X$ making (4.23) commutative.

$$\begin{array}{ccccc} X^{\#}A & & \xrightarrow{\qquad\qquad f \qquad\qquad} & & Y^{\#}B \\ \mu_0 A\uparrow & & & & \uparrow\bar{\mu}_0 B \\ XX^{\#}A & \xrightarrow{X(1_{X^{\#}A},f)} & X(X^{\#}A\times Y^{\#}B) & \xrightarrow{\lambda_0 Y^{\#}B} & YY^{\#}B \end{array} \tag{4.23}$$

Proof. Assume that a sequential machine $M=(S, i_0, \sigma_0)\colon (A, X)\to(B, Y)$ computes $f\colon X^{\#}A\to Y^{\#}B$. Let us take the generalized sequential machine $M'= =(S, i, \sigma)\colon (A, X)\to(B, Y)$, where

$$\begin{aligned} i &:= A\xrightarrow{i_0} S\times B\xrightarrow{1_S\times\bar{\eta}B} S\times Y^{\#}B, \\ \sigma &:= X(S\times -)\xrightarrow{\sigma_0}(S\times -)Y\xrightarrow{(S\times -)\bar{\eta}_1}(S\times -)Y^{\#}. \end{aligned} \tag{4.24}$$

Remember that $\bar{\eta}_1=\bar{\mu}_0\cdot Y\bar{\eta}$. Then, by Lemma 2.6, the machine M' computes the response of M, i.e. the morphism f. Therefore $f=\pi Y^{\#}B\cdot i^{\#}$, where $\pi\colon S\times -\dot{\to} I$ is the projection and $i^{\#}$ is the run map of M'. Thus we have from (2.2)

$$f\cdot\eta A = \pi Y^{\#}B\cdot i^{\#}\cdot\eta A = \pi Y^{\#}B\cdot i = \pi Y^{\#}B\cdot(1_S\times\bar{\eta}B)\cdot i_0 = \bar{\eta}B\cdot\pi B\cdot i_0.$$

Hence, taking f_0 to be $\pi B\cdot i_0$ the condition i) of Theorem 4.7 will be satisfied. According to Theorem 4.5 there is a natural transformation $\lambda\colon X(X^{\#}A\times -)\to Y^{\#}$ such that for this λ and f the diagram (4.9) is commutative. Moreover, by (4.13), λ has the form

$$\lambda = X(X^{\#}A\times -)\xrightarrow{X(r\times -)} X(S\times -)\xrightarrow{\sigma}(S\times -)Y^{\#}\xrightarrow{\pi Y^{\#}} Y^{\#}. \tag{4.25}$$

Now let us define the natural transformation λ_0 by

$$\lambda_0 = X(X^{\#}A\times -)\xrightarrow{X(r\times -)} X(S\times -)\xrightarrow{\sigma_0}(S\times -)Y\xrightarrow{\pi Y} Y. \tag{4.26}$$

Since (4.9) holds for λ in (4.25) it is enough to prove

$$\bar{\mu}\cdot\lambda Y^{\#} = \bar{\mu}_0\cdot\lambda_0 Y^{\#}.$$

By (4.24), (4.25), (4.26) and the naturality of π we have

$$\begin{aligned} \bar{\mu}\cdot\lambda Y^{\#} &= \bar{\mu}(\pi Y^{\#}\cdot\sigma\cdot X(r\times -))Y^{\#} = \bar{\mu}\cdot(\pi Y^{\#}\cdot(S\times -)\bar{\eta}_1\cdot\sigma_0\cdot X(r\times -))Y^{\#} = \\ &= \bar{\mu}\cdot(\bar{\eta}_1\cdot\pi Y\cdot\sigma_0\cdot X(r\times -))Y^{\#} = \bar{\mu}\cdot(\bar{\eta}_1\cdot\lambda_0)Y^{\#} = \bar{\mu}\cdot\bar{\eta}_1 Y^{\#}\cdot\lambda_0 Y^{\#}. \end{aligned}$$

But we have already proved in Lemma 2.6 that $\bar{\mu}\cdot\bar{\eta}_1 Y^{\#}=\bar{\mu}_0$, thus we obtain $\bar{\mu}\cdot\lambda Y^{\#}=\bar{\mu}_0\cdot\lambda_0 Y^{\#}$.

Conversely, assume that the conditions i) and ii) are satisfied for a morphism $f\colon X^{\#}A\to Y^{\#}B$. If we take $\lambda=\bar{\eta}_1\lambda_0$ we have $\bar{\mu}\cdot\lambda Y^{\#}=\bar{\mu}\cdot(\bar{\eta}_1\cdot\lambda_0)Y^{\#}=$ $=\bar{\mu}\cdot\bar{\eta}_1 Y^{\#}\cdot\lambda_0 Y^{\#}=\bar{\mu}_0\cdot\lambda_0 Y^{\#}$. Thus (4.23) implies that the λ above and f satisfies (4.9), and hence by Theorem 4.5 there is generalized sequential machine $M=(X^{\#}A, i, \sigma)$ computing the morphism f. In the sense of Lemma 2.6 it is sufficient to prove that the initial state-output morphism i and the transition σ of M are simple. Since the initial state-output i of M is defined in Theorem 4.5 by

$$i\colon A\xrightarrow{\eta A} X^{\#}A\xrightarrow{(1_{X^{\#}A},\, f)} X^{\#}A\times Y^{\#}B,$$

thus, if we take i_0 to be $(\eta A, f_0)$ for the f_0 in condition i), then

$$(X^{\#}A\times -)\bar{\eta}B\cdot i_0=(1_{X^{\#}A}\times\bar{\eta}B)\cdot(1_{X^{\#}A}, f_0)=(\eta A, \bar{\eta}B\cdot f_0)=$$
$$=(\eta A, f\cdot\eta A)=(1_{X^{\#}A}, f)\cdot\eta A=i.$$

This means that i is simple in the sense of Definition 2.5. The transition σ of M has the form (α, λ) for some α by Theorem 4.5. From $\lambda=\bar{\eta}_1\cdot\lambda_0$ we conclude that σ is simple. This completes the proof of the theorem. □

THEOREM 4.8. The family of the generalized sequential machine morphisms in $\mathscr{K}$ is closed under composition.

Proof. Let $M_1=(S_1, i_1, \sigma_1)\colon (A, X)\to(B, Y)$ and $M_2=(S_2, i_2, \sigma_2)\colon (B, Y)\to$ $\to(C, Z)$ be generalized sequential machines in $\mathscr{K}$ computing the morphisms $f_1\colon$ $X^{\#}A\to Y^{\#}B$, $f_2\colon Y^{\#}B\to Z^{\#}C$, respectively. By Theorem 2.8 the composite morphism $f_2\cdot f_1\colon X^{\#}A\to Z^{\#}C$ can be computed by a machine

$$M=(Q, i, \sigma, \beta)\colon (A, X)\to(C, Z)$$

where $Q=(S_1\times -)(S_2\times -)$,

$$\begin{aligned} i&=A\xrightarrow{i_1} S_1\times Y^{\#}B\xrightarrow{(S_1\times -)i_2^{\#}}(S_1\times -)(S_2\times -)Z^{\#}C=S_1\times(S_2\times Z^{\#}C),\\ \beta&=(S_1\times -)(S_2\times -)\xrightarrow{(S_1\times -)\pi_2}(S_1\times -)\xrightarrow{\pi_1} I.\end{aligned}\tag{4.27}$$

Here $\pi_1\colon S_1\times -\stackrel{\cdot}{\to} I$, $\pi_2\colon S_2\times -\stackrel{\cdot}{\to} I$ are the projection transformations. The object map of the composite functor $(S_1\times -)(S_2\times -)$ is $(S_1\times -)(S_2\times -)D=$ $=(S_1\times -)(S_2\times D)=S_1\times(S_2\times D)$ for any object D of $\mathscr{K}$. Since the category $\mathscr{K}$ has binary products we may recall the well known result (see Mac Lane [7], pp. 73. Proposition 1) which asserts that there is an isomorphism

$$\alpha_{S_1,S_2}\colon S_1\times(S_2\times D)\cong(S_1\times S_2)\times D$$

natural in S_1, S_2 and D, moreover, $\alpha_{S_1,S_2,D}$ commutes with the projections to S_1, S_2 and D, respectively. Thus there is a natural transformation

$$\varphi\colon (S_1\times -)(S_2\times -)\stackrel{\cdot}{\to}(S_1\times S_2)\times -$$

with inverse ψ (i.e., both $\varphi\cdot\psi$ and $\psi\cdot\varphi$ are the identity natural transformations on the corresponding functors),

$$\psi\colon (S_1\times S_2)\times - \stackrel{\cdot}{\longrightarrow} (S_1\times -)(S_2\times -)$$

such that $\pi\cdot\varphi=\pi_1\cdot(S_1\times -)\pi_2$, where $\pi\colon (S_2\times S_1)\times - \stackrel{\cdot}{\to} I$ is the projection. Consider the generalized sequential machine

$$M' = \big((S_1\times S_2)\times -, i', \sigma', \pi\big)\colon (A, X)\to(C, Z)$$

where i' and σ' are defined by i and σ in (4.27) as follows

$$i' = A\xrightarrow{i}(S_1\times -)(S_2\times -)Z^{\#}C\xrightarrow{\varphi Z^{\#}C}((S_1\times S_2)-)Z^{\#}C, \tag{4.28}$$
$$\sigma' = \varphi Z^{\#}\cdot\sigma\cdot X\psi.$$

By Theorem 2.10 it is sufficient to prove that φ is a simulation $\varphi\colon M\to M'$. We have to show the equalities

$$i' = \varphi Z^{\#}C\cdot i,\quad \sigma'\cdot X\varphi = \varphi Z^{\#}\cdot\sigma,\quad \pi\cdot\varphi=\beta. \tag{4.29}$$

The first equality of (4.29) holds by (4.28). As $\beta=\pi_1\cdot(S_1\times -)\pi_2$, thus $\pi\cdot\varphi=\beta$. Using the definition (4.28) of σ' and the equality $\psi\cdot\varphi=1_{(S_1\times -)(S_2\times -)}$ we have

$$\sigma'\cdot X\varphi = \varphi Z^{\#}\cdot\sigma\cdot X\psi\cdot X\varphi = \varphi Z^{\#}\cdot\sigma\cdot X(\psi\cdot\varphi) = \varphi Z^{\#}\cdot\sigma\cdot X1_{(S_1\times -)(S_2\times -)} = \varphi Z^{\#}\cdot\sigma.$$

This proves that φ is a simulation and completes the proof of the theorem. □

Finally, we show that the computational capacity of the generalized sequential machines in a category and that of the process transformations of Arbib and Manes are equal.

Definition 4.9 (Arbib and Manes [4]). Let A, B be objects of $\mathcal{K}$ and let X, Y be varietors in $\mathcal{K}$. A *process transformation* $T\colon (A, X)\to(B, Y)$ in $\mathcal{K}$ is $T=(S, d, t, k, \beta)$, where

(S, d) is an X-algebra, the *state algebra*,
$t\colon A\to S$ is the *initial state*,
$k\colon A\to Y^{\#}B$ is the *initial throughput*,
$\beta\colon X(S\times -)\stackrel{\cdot}{\to} Y^{\#}$ is a natural transformation, the *output*.

The *response* of T is the morphism $g\colon X^{\#}A\to Y^{\#}B$ defined by

$$\begin{array}{ccccccc} & & Y^{\#}B & \xleftarrow{\bar{\mu}B} & Y^{\#}Y^{\#}B & \xleftarrow{\beta Y^{\#}B} & X(S\times Y^{\#}B) \\ & \nearrow k & \uparrow g & & & & \uparrow X(r, g) \\ A & \xrightarrow{\eta A} & X^{\#}A & & \xleftarrow{\mu_0 A} & & XX^{\#}A \end{array} \tag{4.30}$$

where $r\colon X^{\#}A \to S$ is the reachability map of (t, d), i.e. the homomorphic extension

$$\begin{array}{ccccc} & & S & \xleftarrow{\;d\;} & XS \\ & \overset{t}{\nearrow} & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle Xr} \\ A & \xrightarrow{\;\eta A\;} & X^{\#}A & \xleftarrow{\;\mu_0 A\;} & XX^{\#}A \end{array} \tag{4.31}$$

THEOREM 4.10. *A morphism $g\colon X^{\#}A \to Y^{\#}B$ is the response of a process transformation iff g can be computed by a generalized sequential machine in $\mathscr{K}$.*

Proof. Assume that a morphism $g\colon X^{\#}A \to Y^{\#}B$ is the response of a process transformation $T=(S, d, t, k, \beta)\colon (A, X) \to (B, Y)$. For each object C of $\mathscr{K}$ let

$$S \xleftarrow{\varrho C} S \times C \xrightarrow{\pi C} C \tag{4.32}$$

be the product diagram, and define the morphism $\sigma C\colon X(S\times C) \to (S\times -)Y^{\#}C$ by the composite

$$\sigma C\colon X(S\times C) \xrightarrow{(d\cdot X\varrho C,\, \beta C)} S\times Y^{\#}C. \tag{4.33}$$

One can check by an easy coputation that σC in (4.32) is natural in C, i.e. we get a natural transformation

$$\sigma\colon X(S\times -) \stackrel{.}{\to} (S\times -)Y^{\#}.$$

Consider the generalized sequential machine $M=(S, i, \sigma)\colon (A, X) \to (B, Y)$, where $i=(t, k)$ and σ is defined in (4.32). We prove that this machine computes the morphism g, i.e. $f_M = g$. The response of M is $f_M = \pi Y^{\#}B \cdot i^{\#}$, where $i^{\#}$ is the run map of M, i.e. the unique morphism satisfying both (4.34) and (4.35) below

$$i^{\#}\cdot \eta A = i, \tag{4.34}$$

$$i^{\#}\cdot \mu_0 A = (1_S \times \bar{\mu}B)\cdot \sigma Y^{\#}B \cdot Xi^{\#}. \tag{4.35}$$

Since $\pi Y^{\#}B\cdot (r, g) = g$, it is enough to prove that $i^{\#} = (r, g)$. We do this by observing that the morphism (r, g) satisfies (4.34) and (4.35) in place of $i^{\#}$, i.e. (4.36) and (4.37) hold

$$(r, g)\cdot \eta A = i, \tag{4.36}$$

$$(r, g)\cdot \mu_0 A = (1_S \times \bar{\mu}B)\cdot \sigma Y^{\#}B\cdot X(r, g). \tag{4.37}$$

By the triangular part of (4.30) and (4.31) we have

$$(r, g)\cdot \eta A = (r\cdot \eta A, g\cdot \eta A) = (t, k),$$

thus (4.36) holds. Again by (4.30) and (4.31)

$$(r, g)\cdot \mu_0 A = (r\cdot \mu_0 A, g\cdot \mu_0 A) = \big(d\cdot Xr, \bar{\mu}B\cdot \beta Y^{\#}B\cdot X(r, g)\big). \tag{4.38}$$

From the definition (4.33) of σ it follows that $\pi Y^{\#}Y^{\#}B\cdot \sigma Y^{\#}B = \beta Y^{\#}B$, and hence, using the naturality of π we obtain

$$\begin{aligned}(r, g)\cdot \mu_0 A &= \big(d\cdot Xr, \bar{\mu}B\cdot \pi Y^{\#}Y^{\#}B\cdot \sigma Y^{\#}B\cdot X(r, g)\big) = \\ &= \big(d\cdot Xr, \pi Y^{\#}B\cdot (1_S\times \bar{\mu}B)\cdot \sigma Y^{\#}B\cdot X(r, g)\big).\end{aligned} \tag{4.39}$$

Because (4.32) is a product diagram we have

$$d\cdot Xr = d\cdot X(\varrho Y^{\#}B\cdot(r,g)) = \varrho Y^{\#}B\cdot(d\cdot X\varrho Y^{\#}B\times(r,g),\ \bar{\mu}B\cdot\beta Y^{\#}B\cdot X(r,g)) =$$

$$= \varrho Y^{\#}B\cdot(d\cdot X\varrho Y^{\#}B, \bar{\mu}B\cdot\beta Y^{\#}B)\cdot X(r,g) =$$

$$= \varrho Y^{\#}B\cdot(1_S\times\bar{\mu}B)\cdot(d\cdot X\varrho Y^{\#}B,\ \beta Y^{\#}B)\cdot X(r,g).$$

And by the definition (4.33) of σ

$$d\cdot Xr = \varrho Y^{\#}B\cdot(1_S\times\bar{\mu}B)\cdot\sigma Y^{\#}B\cdot X(r,g). \tag{4.40}$$

Putting toghether (4.39), (4.40) and the equality $1_{S\times Y^{\#}B}=(\varrho Y^{\#}B, \pi Y^{\#}B)$ we conclude

$$(r,g)\cdot\mu_0 A = (\varrho Y^{\#}B, \pi Y^{\#}B)\cdot(1_S\times\bar{\mu}B)\cdot\sigma Y^{\#}B\cdot X(r,g) = (1_S\times\bar{\mu}B)\cdot\sigma Y^{\#}B\cdot X(r,g).$$

Thus (4.37) holds, which ends the proof of the "only if" part.

Conversely, assume that a morphism $f\colon X^{\#}A\to Y^{\#}B$ can be computed by a generalized sequential machine in $\mathscr{K}$. Then, by Theorem 4.5, the morphism f is initial-segment preserving, i.e. there is a natural transformation

$$\lambda\colon X(X^{\#}A\times -)\to Y^{\#},$$

such that the diagram (4.9) is commutative. Now consider the process transformation $T=(X^{\#}A, \mu_0 A, f\cdot\eta A, \eta A, \lambda)\colon (A, X)\to(B, Y)$. It is obvious that $1_{X^{\#}A}$ is the reacability map of $(\eta A, \mu_0 A)$. Hence, taking into account the defining diagram (4.30) of a process transformation we obtain that (4.9) defines the response of T, which is f.

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