Axiomatic systems in fuzzy algebra

By J. DREWNIAK*

1. Introduction

One of the most interesting problems in fuzzy set theory is that of the axiomatization of fuzzy algebra. At the beginning, it is necessary to note that there is not any agreement between authors of papers what a "fuzzy algebra" really is (cf. [1], [8], [12], [15]). So we have different fuzzy algebras and they are useful in different applications of fuzzy set theory (cf. [9], [14]).

We are going to consider different systems of axioms on the set of fuzzy sets and on the one hand — to find all common properties of different fuzzy algebras, and on the other hand — to distinguish the characteristic properties of considered algebras. We start with the recollection of definition of fuzzy sets in the following form:

Definition 1.1. A fuzzy set f in a nonempty universe X is an arbitrary function (cf. [3], [17])

$$f: X \to [0, 1].$$

Similarly (cf. [7]), an L-fuzzy set in X is a function

$$f: X \to L,$$

where L or $(L, \leqq)$ is a poset (partially ordered set), e.g. lattice or the interval of real axis.

The collection of all fuzzy sets (L-sets) in X is denoted by $F(X)$ $(F_L(X))$ or shortly by F.

In applications of fuzzy sets (cf. [13], [18]), another definition of fuzzy object is needed, not in the meaning of fuzzy subset.

Definition 1.2 ([12]). Let X and L be as in definition 1.1. Elements of the nonempty set Z are called fuzzy objects if there exists a mapping

$$M: Z \to F_L(X). \qquad (1)$$

Function $f_A = M(A)$ for $A \in Z$ is then named the membership function of fuzzy object A and $f_A(x)$ for $x \in X$ is called the membership grade of point x.

* On leave from Silesian Technical University, Gliwice, Poland; Technical University of Budapest, Department of Communication Electronics.

We shall say that two fuzzy objects $A, B \in Z$ are equal if

$$M(A) = M(B) \quad (f_A = f_B), \tag{2}$$

i.e.

$$f_A(x) = f_B(x) \quad \text{for} \quad x \in X. \tag{3}$$

The last sentence in definition 1.2 is equivalent to the assumption that mapping (1) is one to one (injection) and we can consider the inverse mapping

$$M^{-1}: M(Z) \to Z. \tag{4}$$

Remark 1.3. The particular case of membership function is that of characteristic function for a subset in X. The set of all characteristic functions

$$Ch = Ch(X) = F_{\{0,1\}}(X)$$

is contained in F whenever $\{0, 1\} \subset L$, where

$$0 = \inf L, \quad 1 = \sup L.$$

Then we can obtain different relations between Ch and $M(Z)$. For example

$$Ch \cap M(Z) = \emptyset, \quad Ch \subset M(Z) \quad \text{or} \quad M(Z) \subset Ch.$$

In this last case we see that definition 1.2 admits not entirely fuzzy objects.

Usually in theoretic papers it is assumed that $Z = F$ and then M is omitted as identity function. But if we want to write for example about fuzzy statements (cf. [1], [14], [18]), we must consider fuzzy objects different than fuzzy subsets of the universe, and the universe can be settled different in particular cases as suitable for applications (e.g. consider statements about age, height or weight of people).

In general we have three base sets: L, X and Z, and assumptions about one of these sets would have consequences in two other sets. So for $L = [0, 1]$, where there are different algebraic structures, we have greater possibilities in construction of fuzzy algebra than in the case of abstract poset L. In every case we can make use of its order by considering induced orders between fuzzy sets and between fuzzy objects.

Definition 1.4. We say that the fuzzy set $f \in F$ is contained in the fuzzy set $g \in F$ if

$$f(x) \leqq g(x) \quad \text{for} \quad x \in X \tag{5}$$

and we write

$$f \leqq g. \tag{6}$$

Similarly we say that the fuzzy object $A \in Z$ is dominated by the fuzzy object $B \in Z$ if

$$M(A) \leqq M(B) \quad (f_A \leqq f_B) \tag{7}$$

and we write

$$A \leqq B. \tag{8}$$

(The sign "$\leqq$" in (5), (6) and (8) is used as symbol for three different relations but its meaning will be understood because of the context).

Remark 1.5. Defined order is a generalization of inclusion relation for subsets in X because in the case

$$Ch \subset F \quad \text{and} \quad A, B \subset X$$

inequality (6) can be written as

$$e_A \leqq e_B$$

which is equivalent to $A \subset B$, where

$$e_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{9}$$

Proposition 1.6. Relation (6) introduces a partial order in F and relation (8) introduces a partial order in Z, i.e. for every $A, B, C \in Z$ we have

$$A \leqq A \quad \text{(reflexivity)}, \tag{10}$$

$$A \leqq B \text{ and } B \leqq A \text{ imply } A = B \quad \text{(antisymmetry)}, \tag{11}$$

$$A \leqq B \text{ and } B \leqq C \text{ imply } A \leqq C \quad \text{(transitivity)}. \tag{12}$$

We omit the simple proof of this proposition and we consider only the case of antisymmetry (11) of relation (8). If $A, B, C \in Z$ and

$$A \leqq B \quad \text{and} \quad B \leqq A,$$

then by definition 1.4 from (7) we get

$$f_A \leqq f_B \quad \text{and} \quad f_B \leqq f_A,$$

i.e.

$$f_A(x) \leqq f_B(x) \quad \text{and} \quad f_B(x) \leqq f_A(x) \quad \text{for} \quad x \in X. \tag{13}$$

For every x we have $f_A(x), f_B(x) \in L$ and in virtue of antisymmetry in L, (13) imply (3), i.e. (2). Now by definition 1.2 we get $A = B$ which proves (11).

This property cannot be proved if the mapping (1) is not injective which makes this part of proof more interesting.

After proposition 1.6 we can say that F and Z are posets when L is a poset. Obviously beside the case of singleton X there are incomparable functions (elements) in F even then, when L is linearly ordered. So we do not have a generalization of proposition 1.6 to the case of linear order. We can look forward to properties typical in lattices under suitable assumptions about L.

In the structure of fuzzy objects we have greater variety of possibilities, because card Z can be small in comparison with card F. So it is possible that all considered fuzzy objects are comparable and $M(Z)$ forms a chain in poset F. It seems that in applications of fuzzy sets we obtain the situation described in proposition 1.6 in more natural way than definition 1.4 (cf. [16]). First we have certain dominance relation in the set Z and then we need a function M in (1) such that (8) implies (7) for every $A, B \in Z$. But the result is the same.

Now let consider an algebraic operation in the set of fuzzy sets or in the set of fuzzy objects, i.e.

$$u\colon F^n \to F \quad \text{or} \quad v\colon Z^n \to Z \tag{14}$$

for fixed $n \geqq 1$. Such operations in an ordered set can have the following properties:

Definition 1.7 ([5], Chapter 1): We shall say that an operation u is isotone (antitone) if the inequalities

$$f_i \leqq g_i \quad \text{for} \quad i = 1, 2, \ldots, n \tag{15}$$

imply

$$u(f_1, \ldots, f_n) \leqq u(g_1, \ldots, g_n) \quad \big(u(g_1, \ldots, g_n) \leqq u(f_1, \ldots, f_n)\big) \tag{16}$$

for every $(f_1, \ldots, f_n), (g_1, \ldots, g_n) \in F^n$.

Operation u is monotonic if it is isotone or antitone.

We are interested in transferring of operations from one base set to the other.

Definition 1.8. Let one of the operations (14) be given. We say that the operation $v\colon Z^n \to Z$ is induced by the operation $u\colon F^n \to F$ to the domain of M if $u\colon M(Z)^n \to M(Z)$ and v is defined by (see (4))

$$v(A_1, \ldots, A_n) = M^{-1}\big(u(M(A_1), \ldots, M(A_n))\big) \tag{17}$$

for $A_1, \ldots, A_n \in Z$.

We say that the operation $u\colon M(Z)^n \to M(Z)$ is induced by $v\colon Z^n \to Z$ to the codomain of M if u is defined by

$$u(f_1, \ldots, f_n) = M\big(v(M^{-1}(f_1), \ldots, M^{-1}(f_n))\big) \tag{18}$$

for $f_1, \ldots, f_n \in M(Z)$.

The algebraic fact described in definition 1.8 can be repeated as (cf. [4]).

Corollary 1.9. If the operations $u\colon M(Z)^n \to M(Z)$, $v\colon Z^n \to Z$ satisfy (17) then M is an isomorphism between the algebraic structures (Z, v) and $\big(M(Z), u\big)$.

Now from the known property of isomorphism we get (cf. [4]).

Proposition 1.10. The operation induced in the domain or in the codomain of an injection has such algebraic properties as the initial one.

We prove also

Proposition 1.11. The operation induced in the ordered domain or codomain of a monotonic injection by monotonic operation is also monotonic.

Proof. We prove only the first part of the proposition because the codomain of M is the domain of M^{-1} (see (1)) and we can omit the case of the codomain.

Let u be isotone, i.e. (15) imply (16). Assume that

$$A_i \leqq B_i \quad \text{for} \quad A_i, B_i \in Z, \quad i = 1, \ldots, n, \tag{19}$$

and put

$$f_i = M(A_i), \quad g_i = M(B_i), \quad i = 1, \ldots, n. \tag{20}$$

Now if M is also isotone as in definition 1.4, then from (8) we get (7) and from (19) and (20) we get (15). Therefore from (16) and (20) it follows

$$u\big(M(A_1), \ldots, M(A_n)\big) \leqq u\big(M(B_1), \ldots, M(B_n)\big)$$

and both parts of this inequality belong to $M(Z)$ under the conditions of definition

1.8. But the inverse M^{-1} of the isotone mapping M is also isotone and we obtain

$$M^{-1}(u(M(A_1), \ldots, M(A_n))) \leqq M^{-1}(u(M(B_1), \ldots, M(B_n))),$$

i.e.

$$v(A_1, \ldots, A_n) \leqq v(B_1, \ldots, B_n)$$

in virtue of (17). Thus the operation v is also isotone and monotonic.

If u or M is antitone then very similar argumentation finishes the proof.

Now we can see that the algebraic structure can be transformed only between Z and $M(Z)$ if $M(Z) \neq F$. We cannot use definition 1.8 if the operation u does not introduce any substructure into $M(Z)$ (if the set $M(Z)$ is not closed under operation u). Also if v is given we obtain a new structure only in $M(Z)$ but not in F. Thus the general assumption $M(Z)=F$ and even $Z=F$ can be very useful (and it is often used).

Another situation is between F and L. Every algebraic operation in L induces a similar operation in F (cf. [7]) but inverse transferring is impossible. None of the operations defined in F can be transformed to the set L independently of $x \in X$ (obviously if we omit all operations just induced from L to F).

So if we do not assume any algebraic operation in L we cannot induce a unique algebraic structure there similar to the considered one in F (different possibilities can be considered if we restrict all $f \in F$ to a fixed point $x_0 \in X$).

At that stage we can give the most general statement about the meaning of the phrase "fuzzy algebra".

Definition 1.12. By a fuzzy algebra (algebra of fuzzy sets, algebra of fuzzy objects) we mean every algebraic structure in F or in Z such that

($*$) every its operation is monotonic (definition 1.7) in the ordered structure induced from L (definition 1.4).

A fuzzy algebra is named "ordinary" one if the following assumptions are fulfilled (cf. remark 1.3):

($*\,*$) $0=\inf L \in L$, $1=\sup L \in L$, $Ch \subset M(Z)$,

($*\,*\,*$) every its algebraic operation restricted to Ch is identical to one of the set-theoretical operations as union, intersection, difference, complementation or symmetric difference.

In the contrary we speak about "special" fuzzy algebra.

Condition ($*$) can be written in a weak form under the assumption that the operations are monotonic in each variable separatively, but if we consider only unary operations or binary associative operations then it is equivalent to ($*$) (cf. [5], Chapter 1). Assumption about L in ($*\,*$) is equivalent to boundedness of poset L. At last assumption ($*\,*\,*$) guarantes that the considered algebra is a generalization of certain part of the set algebra.

Now we can overlook different papers regarding the fuzzy set theory and consider different further assumptions accepted in the fuzzy algebra. We select only a few papers which are principally concerning about operations and axioms of fuzzy algebras.

2. The first definition of Zadeh

I think it is forgotten now that Zadeh [17] has given a very simple argumentation for introducting his "max" and "min" operations. He writes that intuitively

Z1 the union of two fuzzy sets is the smallest fuzzy set containing both these sets;

Z2 the intersection of two fuzzy sets is the largest fuzzy set which is contained in both these sets.

It is a definition as natural as possible, because in the order structure it is equivalent to the definition of union and intersection in the set theory. For the case $L=[0, 1]$ Zadeh [17] proved that Z1 and Z2 are equivalent to "max" and "min" operations in F. It is usually proved in the lattice theory (cf. [2]) that operations of supremum and infimum for subsets containing only two elements are equivalent to the lattice operations $\vee$ and $\wedge$. So Zadeh's definition and proof can be used in every lattice and we have

Theorem 2.1. If $L=(L, \vee, \wedge)$ is a lattice, then Z1 and Z2 are equivalent to

$$f\vee g = \sup\{f, g\} \quad \text{and} \quad f\wedge g = \inf\{f, g\} \quad \text{for} \quad f, g\in F, \tag{21}$$

where

$$\begin{aligned} (f\vee g)(x) &= \sup\{f(x), g(x)\} = f(x)\vee g(x), \\ (f\wedge g)(x) &= \inf\{f(x), g(x)\} = f(x)\wedge g(x) \end{aligned} \tag{22}$$

for $x\in X$.

The following result is from Brown [3].

Theorem 2.2. If L is a lattice, then F with operations (21) is a lattice, too.

As we remarked above, the operations (21) can be reduced to the set-theoretical operations whenever $0, 1\in L$ (see $(**)$), they are also monotonic and we have

Corollary 2.3. If L is a lattice with 0 and 1 then the operations (21) introduce in F an ordinary fuzzy algebra which is a lattice algebra.

If the lattice L is nonbounded (which is possible only for infinite lattices — cf. [2]) then the operations (21) introduce in F a special fuzzy algebra which is a lattice algebra, too.

This corollary stressed the importance of assumptions about the poset L in definition 1.12. Under additional assumptions it is possible to consider further lattice properties (distributivity, completeness) or even continuity of operations (21) in the interval topology (cf. [7]), but we have not any further problems why the union and the intersection of fuzzy sets has form (21). (I think that none in the world has examined why the set-theoretical sum is the "sum" but it is not a "composition" of sets, because it was so named and that is all.) Obviously we can introduce many other operations which will have other names and will compose other fuzzy algebras. For example Zadeh [17] proposed other operations as the complement $1-f$, the arithmetic product fg, the arithmetic sum $f+g-fg$, and the absolute difference $|f-g|$, which can be considered for arbitrary $f, g\in F$ in the case $L=[0, 1]$. All these operations will be reduced in $L=\{0, 1\}$ to the ordinary set-theoretical operations and thus form in F different ordinary fuzzy algebras. There were also defined the sum $f+g$ and the convex combination $hf+(1-h)g$, which cannot be reduced to ordinary set-theoretical operations and so they form special fuzzy algebras. We do not consider more precisely all these algebras because of the great literature on the case $L=[0, 1]$ (e.g. almoust the entire book of Kaufmann [10] treates the case $L=[0, 1]$).

Now remains the problem, what we can say about an ordinary fuzzy algebra if L is not a lattice. In this case we cannot use the natural definitions Z1 and Z2, because it is possible that the needed elements do not exist in F.

If we want to preserve as much as possible from the definition (22) in a bounded poset L, we can use the following extension of the lattice operations:

$$(f\vee g)(x)=\begin{cases}\sup\{f(x),g(x)\} & \text{if supremum exists,}\\ 1 & \text{otherwise;}\end{cases} \tag{23}$$

$$(f\wedge g)(x)=\begin{cases}\inf\{f(x),g(x)\} & \text{if infimum exists,}\\ 0 & \text{otherwise.}\end{cases} \tag{24}$$

These operations are idempotent and commutative and also can be reduced to the set-theoretical operations in the case $L=\{0, 1\}$. Unfortunately operations (23) and (24) are not associative what is illustrated by

Example 2.4. Let

$$L=\{(0,0),\ (0,1/3),\ (1/3,0),\ (1/3,2/3),\ (2/3,1/3),\ (2/3,1),\ (1,2/3),\ (1,1)\}$$

be the poset with partial order induced in Cartesian product. It is bounded and $0=(0, 0)$, $1=(1, 1)$ but it is not a lattice, because e.g. $\sup\{a, b\}$ and $\inf\{a, b\}$ do not exist for

$$a=(1/3,2/3),\quad b=(2/3,1/3),\quad c=(1,2/3),\quad d=(0,1/3).$$

By (23) we compute

$$a\vee b=1 \quad\text{and}\quad b\vee c=c$$

so

$$(a\vee b)\vee c=1 \quad\text{and}\quad a\vee(b\vee c)=a\vee c=c\neq 1.$$

Similarly by (24) we get

$$(a\wedge b)\wedge d=0 \quad\text{and}\quad a\wedge(b\wedge d)=d\neq 0,$$

thus none of these operations is associative and in consequence they are not very interesting as algebraic operations. Moreover operations (23) and (24) are not monotonic in the poset L because we have

$$b<c \quad\text{and}\quad d<a$$

and simultaneously

$$a\vee b=1>a\vee c=c \quad\text{and}\quad b\vee d=b<b\vee a=1,$$

$$a\wedge b=0<a\wedge c=a \quad\text{and}\quad b\wedge d=d>b\wedge a=0.$$

Therefore operations (23) and (24) do not form any fuzzy algebra in F and it is not a simple way to introduce a fuzzy algebra in F if L is not a lattice.

Another problem related paper [17] brings the definition of the complement of the fuzzy set. Namely, the natural meaning of the word "complement" in the set theory is "the smallest set in the universe which in the union with the given set makes the universe", or it means "the greatest set in universe disjoint with the given one". So independently of Zadeh's definition

Z3 the (arithmetic) complement of a fuzzy set is the arithmetic complementation of its values to 1 in $L=[0, 1]$.

We can consider two other definitions

Z3′ the (union) complement of a fuzzy set is the smallest fuzzy set which in union with the given set makes e_X (see (9));

Z3″ the (intersection) complement of a fuzzy set is the greatest fuzzy set disjoint with the given set.

We propose to name these three complements by arithmetic, union and intersection complement, respectively. It is evident that definitions Z3′ and Z3″ can be used in the case of complete lattice L while the definition Z3 can be extended to the case of complemented lattice L. However, the use of definitions Z3′ and Z3″ is a little confounding because as complements we always obtain the elements of Ch (see remark 1.3).

3. The axiom system of Bellman and Giertz

Many authors find the paper [1] very useful (cf. [6], [8], [16]), so we too are going to use it. The paper treates the naturality of Zadeh's "max" and "min" operations. We have already remarked above that it is a hard work to add something interesting to Zadeh's own argumentation in Z1 and Z2. We give here a short review of this new argumentation from paper [1].

Let Z denote the set of fuzzy objects named "fuzzy statements". Then the existence of two binary operations "and" and "or" is required, but we have not exact information about mapping (1). Thus it is impossible to consider the induced operations (18) in the set of membership functions. Authors in [1] could not use a definition like definition 1.8 and introduced operations in F by system of axioms. They assumed that $P, S\colon F^2 \to F$ are such that (we use different notation)

$$f_{A \text{ and } B} = P(f_A, f_B), \quad f_{A \text{ or } B} = S(f_A, f_B) \tag{25}$$

for every $A, B \in Z$ and its dependence on the membership functions can be described by

$$P(f, g)(x) = p(f(x), g(x)), \quad S(f, g)(x) = s(f(x), g(x)), \tag{26}$$

where functions

$$p, s\colon [0, 1]^2 \to [0, 1]$$

fulfil the following system of axioms:

BG1 p and s are nondecreasing and continuous in both variables;
BG2 p and s are symmetric $(p(x, y)=p(y, x), s(x, y)=s(y, x))$;
BG3 $p(x, x)$ and $s(x, x)$ are strictly increasing in x;
BG4 $p(x, y) \leqq \min(x, y), s(x, y) \geqq \max(x, y)$;
BG5 $p(1, 1)=1, s(0, 0)=0$;
BG6 logically equivalent statements have equal membership functions (grades).

Further they deduced from this axioms the system of functional equations for functions p and s, and they proved that this system of functional equations and inequalities (see BG4) has a unique solution

$$p(x, y) = \min(x, y), \quad s(x, y) = \max(x, y) \quad \text{for} \quad x, y \in [0, 1]. \tag{27}$$

The mentioned system of equations and inequalities was discussed in details in Hamacher's paper [8] and in Kóczy's dissertation [11] and we do not want to say any more about it. However, we devote a little time to the consideration of the above BG1—BG6 axioms.

I think that for the consequences of the prescribed axiom system almost all depends on the meaning of BG6. We show that it is difficult to find a correct meaning of BG6.

First, let us suppose that operations "and" and "or" fulfil in Z the propositional calculus of conjunction and disjunction. Then we have e.g.

"A and B" is equivalent to "B and A",
"A or B" is equivalent to "B or A",
"A and A" is equivalent to "A",
"A or A" is equivalent to "A"

for arbitrary $A, B \in Z$, and we can omit axioms BG2 and BG5 as implied from BG6. Moreover we can write

$$p(x, x) = x, \quad s(x, x) = x \quad \text{for} \quad x \in [0, 1] \tag{28}$$

and it is more interesting because of

Theorem 3.1. If the functions $p, s\colon [0, 1]^2 \to [0, 1]$ fulfil BG4, (28) and

$$p \text{ and } s \text{ are nondecreasing in both variables,} \tag{29}$$

then we obtain (27).

Proof. Let $x, y \in [0, 1]$, $x \leqq y$. Thus from (29) and (28) we get

$$x = p(x, x) \leqq p(x, y) \leqq p(y, y) = y,$$
$$x = s(x, x) \leqq s(x, y) \leqq s(y, y) = y$$

and therefore

$$p(x, y) \geqq \min(x, y), \quad s(x, y) \leqq \max(x, y).$$

This together with BG4 proves (27).

This short theorem contains more informations about "max" and "min" operations than all information contained in paper [1] because we use exactly only axiom BG4 and our assumption (29) is weaker than BG1, and assumption (28) is a very special case of BG6. It seems, we must be very satisfied because of this great reduction of the axiom system BG1—BG6. However, we are not satisfactory because of the unnatural assumption BG4. Namely, assumption (29) is equivalent to condition (*) from the definition of fuzzy algebra (see definition 1.12) and if we omit (29) we can obtain an algebraic structure different from the fuzzy algebra (cf. example 2.4). Assumption (28) can be admitted as a natural extension of this law from the algebra of sets and we cannot say anything similar about BG4.

It was only the first part of our consideration of axiom BG6. If we admit a part of propositional calculus in Z we can ask why not admit the whole propositional calculus in Z with all operations used in logic. Thus axiom BG6 can be understood as the assumption that Z is a Boolean algebra of fuzzy objects and then it can be supposed that paper [1] is devoted to transferring of this algebra on the set of fuzzy sets.

We have remarked after proposition 1.11 that the structure induced in $M(Z)$ can be different from that in F (obviously in the case $M(Z)\neq F$). However, there is assumed here the transferring of the Boolean algebra on the whole F, what is impossible in the case $L=[0, 1]$ (it is possible if L is a Boolean algebra, cf. [3]).

The last remark about axiom BG6 has moral meaning. It is not right to suppose that "fuzzy statements" are "logically equivalent" in the same manner as logical sentences are in the propositional calculus. If there are "fuzzy statements" they can be totally unlogical and it is the main reason of the different "fuzzy" investigations.

4. Hamacher's axiom system

Paper [8] contains a very interesting method of the generalization of the set-theoretical operations but two things make reading difficult:

a) many proofs are omitted without a hint, how or where they were obtained;

b) lack of the list of references (in my copy).

The author creates the following system of axioms for two operations p, s: $[0, 1]^2\to[0, 1]$ (we change notations):

H1 p and s are associative,

H2 p and s are continuous,

H3 p in $(0, 1]$ and s in $[0, 1)$ are injections in both variables,

H4 $p(x, x)=x\Leftrightarrow x=1$ for $x\in(0, 1]$ and $s(x, x)=x\Leftrightarrow x=0$ for $x\in[0, 1)$.

These axioms are considered independently for p and s and both operations form certain semigroups in the intervals from H3, respectively. Axiom H3 with continuity H2 gives strict monotonicity of p and s in both variables and these together with H1 imply that (cf. [5]) p and s are strictly increasing in $(0, 1]$ and $[0, 1)$, respectively. It is a stronger property than $(*)$ in definition 1.12 and stronger than in natural models of those operations for $L=\{0, 1\}$. Thus the author must exclude certain boundary points in H3 and H4. It is noted in [8] that H3 admits only one idempotent case

$$p(x,x) = x \quad \text{in} \quad (0,1] \quad \text{and} \quad s(x,x) = x \quad \text{in} \quad [0,1).$$

In this situation axiom H4 is equivalent to the assumption that for functions

$$p_a(x) = p(a,x) \quad \text{in} \quad (0,1] \tag{30}$$

and

$$s_b(x) = s(b,x) \quad \text{in} \quad [0,1) \tag{31}$$

there exist such $a=1$ and $b=0$ that suitable functions p_1 and s_0 are surjections. Indeed we have

Lemma 4.1. Under assumptions H1—H3 if there exists $u<1$ such that

$$p(u, u) = u, \tag{32}$$

then none of the operations (30) is a surjection.

Similarly if there exists $v>0$ such that

$$s(v, v) = v, \tag{33}$$

then none of the operations (31) is a surjection.

Proof. Because of the unicity of the idempotents for both operations we have

$$p(1, 1) \neq 1 \quad \text{and} \quad s(0, 0) \neq 0$$

and therefore

$$p(1, 1) < 1 \quad \text{and} \quad s(0, 0) > 0.$$

Thus by monotonity

$$p(x, y) \leqq p(1, 1) < 1 \quad \text{for} \quad x, y \in (0, 1]$$

and

$$s(x, y) \geqq s(0, 0) > 0 \quad \text{for} \quad x, y \in [0, 1).$$

Therefore none of the functions (30) or (31) obtain the value $p(x, y)=1$ or $s(x, y)=0$, respectively, and none of them is a surjection.

It is a strange situation, because in paper [8] one theorem tells that every idempotent for operations p or s is an identity element and this implies the mentioned unicity of idempotents. But every identity element forms the identity bijection and we get

$$p_u(x) = x \quad \text{for} \quad x \in (0, 1]$$

from (30) and (32), and also

$$s_v(x) = x \quad \text{for} \quad x \in [0, 1)$$

from (31) and (33). This contradicts the thesis of lemma. Thus the assumptions $u<1$ and $v>0$ are not fulfilled for any $u \in (0, 1]$ and $v \in [0, 1)$. Therefore we have proved

Lemma 4.2. Under assumptions H1—H3 if u fulfils (32) then $u=1$, and if v fulfils (33) then $v=0$.

This result is not else than the first implication in axiom H4. Thus we can assume only the second implication from H4, i.e.

$$p(1, 1) = 1 \quad \text{and} \quad s(0, 0) = 0$$

and it is exactly axiom BG5 from paper [1]. Now we have

Theorem 4.3. The system of axioms H1—H4 is equivalent to the system of axioms H1—H3 and BG5.

Our consideration about lemma 4.1 brings one more result, because of the mentioned equivalence between idempotents and identity elements and thus axiom BG5 (under assumption H1—H3) is equivalent to

H4′ $p(1, x) = p(x, 1) = x$ and $s(x, 0) = s(0, x) = x$ for $x \in [0, 1]$.

We have

Theorem 4.4. The system of axioms H1—H4 is equivalent to the system of axioms H1—H3 and H4'.

A great part of paper [8] contains considerations about the class of functions fulfilling axioms H1—H4. We remark here only three results:

a) every function

$$p\colon [0, 1]^2 \to [0, 1] \tag{34}$$

fulfilling axioms H1—H3 has the form

$$p(x, y) = f^{-1}(f(x)+f(y)) \tag{35}$$

with the continuous, monotonic real function f defined in $[0, 1]$;

b) every rational function (34) fulfilling axioms H1—H4 has the form

$$p(x, y) = \frac{dxy}{a+(d-a)(x+y-xy)} \tag{36}$$

with suitable constants a and d.

c) if function (34), fulfilling H1—H4 is a polynomial then

$$p(x, y) = xy. \tag{37}$$

At first we use formula (35). Let $a>0$ and

$$f(x) = x^a \quad \text{for} \quad x \in [0, 1].$$

We get

$$p(x, y) = (x^a+y^a)^{1/a}$$

and it indeed fulfils axioms H1—H3 but the function

$$p\colon [0, 1]^2 \to [0, 2^{1/a}]$$

is different from (34) and it does not fulfil H4. Thus formula (35) admits operations over our interest. So we put a question:

I. Is there any assumption about function f, under which every function (35) is of the type (34)?

We put

$$p(x, y) = \frac{xy}{(2-x^a-y^a+x^a y^a)^{1/a}} \quad \text{for} \quad x, y \in [0, 1], \quad a > 0. \tag{38}$$

and now it is a good example of irrational functions fulfilling the system of axioms H1—H4. We also ask:

II. Does exist a finite-parametric formula for the class of all functions (34) fulfilling axioms H1—H4?

At last put $a=1$ in (38). We get

$$p(x, y) = \frac{xy}{2-x-y+xy} \tag{39}$$

and it is example of rational function which fulfils axiom system H1—H4. We could find it between rational solutions in (36).

At the finish of this part, we remark that using formulas (25), (26) we obtain

Corollary 4.5. Functions (34) from class (36) introduce in F an ordinary fuzzy algebra which is a commutative semigroup with identity.

It is also interesting, that under assumptions H1—H4 Hamacher proved the inequalities similar to BG4 with strict inequality.

5. The axiomatic system of Kóczy

The papers [12] and [13] contain the reachest system of axioms of fuzzy algebra. We have used these papers in many places in our introduction, and our definition 1.2 is exactly the first axiom of paper [12]. Thus all our considerations are made in terminology of paper [12]. Now we rewrite the other axioms from this paper.

K2 card $Z \geqq 2$ and $(Z, \vee, \wedge, ')$ is algebraic structure with operations $\vee: Z^2 \to Z$, $\wedge: Z^2 \to Z$ and $': Z \to Z$;

K3 there exist an element $0 \in Z$ called zero and the operations in Z fulfil

$$A \vee B = B \vee A, \tag{40}$$

$$(A \vee B) \vee C = A \vee (B \vee C), \tag{41}$$

$$A'' = A, \tag{42}$$

$$A \vee 0 = A, \quad A \wedge 0 = 0, \tag{43}$$

$$(A \vee B)' = A' \wedge B' \tag{44}$$

for every $A, B, C \in Z$;

K4 under order induced in F from L (see definition 1.4) mapping (1) fulfils (here $f_A = M(A)$):

$$f_P > f_Q \quad \text{for} \quad P = (A \wedge B) \vee (A \wedge C) \neq 0, \quad Q = A \wedge (B \vee C) \neq 0', \tag{45}$$

$$f_P < f_Q \quad \text{for} \quad P = (A \vee B) \wedge (A \vee C) \neq 0', \quad Q = A \vee (B \wedge C) \neq 0, \tag{46}$$

$$f_{A \vee B} > f_A \quad \text{for} \quad A \neq 0', \ B \neq 0, \tag{47}$$

$$f_{A \wedge B} < f_A \quad \text{for} \quad A \neq 0, \ B \neq 0', \tag{48}$$

$$f_A - f_B = f_{A'} - f_{B'} \tag{49}$$

for arbitrary $A, B, C \in Z$;

K4′ under order in F it is assumed that

$$f_{A \vee A} > f_A \quad \text{for} \quad A \neq 0, \ A \neq 0', \tag{50}$$

$$f_{A \wedge A} < f_A \quad \text{for} \quad A \neq 0, \ A \neq 0', \tag{51}$$

$$f_{A \vee A} > f_{B \vee B} \quad \text{iff} \quad f_A > f_B, \tag{52}$$

$$f_{A \wedge A} > f_{B \wedge B} \quad \text{iff} \quad f_A > f_B \tag{53}$$

for arbitrary $A, B \in Z$;

K5 there is admitted at most one solution U for every of the equations

$$A \vee U = B \quad (A, B \in Z,\ A \neq 0'), \tag{54}$$

$$A \wedge U = B \quad (A, B \in Z,\ A \neq 0); \tag{55}$$

K5′ there is assumed exactly one solution U for every of the equations (54), (55);

K6 L is a interval of real axis and (cf. notation (25), (26)),

$$f_{A\wedge B} = p(f_A, f_B),\ f_{A\vee B} = s(f_A, f_B),\ f_{A'} = c(f_A), \tag{56}$$

where functions $p, s\colon L^2 \to L$ and $c\colon L \to L$ are continuously differentiable.

It is possible that this is not the final form of Kóczy's work upon axiomatization of fuzzy algebra. The form presented in papers [12] and [13] has some reticences. For example in fact it is not precised what kind of order is considered in F (we wrote in K4 our supposition only) and it is also not precised, what kind of continuous differentiation is possible in L (and we suppose that L is in the real axis).

Now we precise some consequences of the above axioms.

Proposition 5.1. Under assumptions K2 and K3 the operation $\wedge$ has the following "dual" properties:

$$A \wedge B = B \wedge A, \tag{57}$$

$$(A \wedge B) \wedge C = A \wedge (B \wedge C), \tag{58}$$

$$A \wedge I = A, \quad A \vee I = I, \tag{59}$$

$$(A \wedge B)' = A' \vee B' \tag{60}$$

for arbitrary $A, B, C \in Z$, where

$$I = 0'. \tag{61}$$

Proof. Let $A, B, C \in Z$. From (42) and (44) we get

$$A \vee B = (A \vee B)'' = (A' \wedge B')'. \tag{62}$$

First we prove the "dual" formula

$$A \wedge B = (A' \vee B')' \tag{63}$$

Indeed, it follows from (42) and (44) that

$$A \wedge B = A'' \wedge B'' = (A')' \wedge (B')' = (A' \vee B')'.$$

Now using (42) in (63) we get (60):

$$(A \wedge B)' = (A' \vee B')'' = A' \vee B'.$$

(63) and (40) gives now (57):

$$A \wedge B = (A' \vee B')' = (B' \vee A')' = B \wedge A.$$

In a similar way from (63), (60) and (41) we get

$$(A \wedge B) \wedge C = ((A \wedge B)' \vee C')' = ((A' \vee B') \vee C')' =$$
$$= (A' \vee (B' \vee C'))' = (A' \vee (B \wedge C)')' = A \wedge (B \wedge C),$$

which gives (58). Now from (42) and (61) we have

$$I' = 0. \tag{64}$$

By (61)—(64) and (43) we obtain

$$A \wedge I = (A' \vee I')' = (A' \vee 0)' = A'' = A,$$

$$A \vee I = (A' \wedge I')' = (A' \wedge 0)' = 0' = I,$$

which completes the proof.

Immediately from (43) and (59) we get

Proposition 5.2 (idempotent and absorption cases). Under assumptions K2 and K3 we have

$$0 \vee 0 = 0, \qquad 0 \wedge 0 = 0,$$

$$I \vee I = I, \qquad I \wedge I = I,$$

$$A \vee (A \wedge 0) = A, \quad A \wedge (A \vee 0) = A \wedge A,$$

$$A \wedge (A \vee I) = A, \quad A \vee (A \wedge I) = A \vee A,$$

$$0 \vee (0 \wedge A) = 0, \quad 0 \wedge (0 \vee A) = 0,$$

$$I \wedge (I \vee A) = I, \quad I \vee (I \wedge A) = I$$

for every $A \in Z$.

Proposition 5.3. Under assumption K2 and K4 or K4′

a) Z contains only two idempotents 0 and I,
b) if $\operatorname{card} L = 2$ then $\operatorname{card} Z = 2$.

Proof. Case a) is a consequence of strict inequalities from (47), (48), (50) and (51).

If $\operatorname{card} L = 2$ then L can be considered as Boolean algebra and then F is a Boolean algebra, too (cf. [3]). Then every element of F is a idempotent of both binary operations and (by homomorphism M) every element of Z is an idempotent. This together with a) ends the proof.

Our considerations of axiom system K2—K6 will be continued in further papers.

6. Conclusion

The axiomatic method of the introduction of fuzzy algebra has great meaning in the development of fuzzy set theory, obviously if the axiom system admits a broader class of operations as it was done e.g. in papers [8] and [12]. In the contrary, if the axiom system is constructed for the purpose of characterizing one given operation as in paper [1], it would have greater meaning in the theory of functional equations then in fuzzy set theory.

The interesting direction in considerations of different fuzzy algebras brings papers [9] and [16] where it is proved that different fuzzy algebras can be useful for different applications.

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SILESIAN UNIVERSITY
DEPT. OF MATHEMATIC
UL. BANKOWA 14.
40—007 KATOWICE
POLAND

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