# Priority schedules of a steady job-flow pair*

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The priority schedules are discussed for a steady job-flow pair defined in [5] as a non-finite deterministic model of servicing invariably renewing demand series. Though these schedules are not dominating with respect to the utilization of the servicing processor, they are very important in practice. A method is defined for reducing the problem of evaluation of the schedules to the evaluation of simpler ones. The method is based on the reduction of the configuration constituted by the demands of job-flows. The reduction is a generalization of the Euclidean algorithm of the regular continued fraction expansion. For some configurations the reduction procedure does not prove to be finite or the evaluation procedure of the schedule of the reduced configuration is not known to be finite. For some of these configurations direct evaluation methods are given.

## 1. Introduction

In an earlier work [5] the problem of scheduling steady job-flow pairs was defined as scheduling the processor triple $\mathscr{P}=\{P_A, P_{B1}, P_{B2}\}$ to service two series $Q^{(i)}=\{C_{ij}, j=1, 2, \ldots\}$, $i=1, 2$, of task pairs $C_{ij}=(A_{ij}, B_{ij})$ demanding service of time $\eta_i \geqq 0$ and $\vartheta_i \geqq 0$ from the processor $P_A$ and $P_{Bi}$, respectively. The series $Q^{(i)}$ is a *steady job-flow* with *parameters* $\eta_i$, $\vartheta_i$ as renewing demands for processors $P_A$ and $P_{Bi}$. The steady job-flow pair is characterized by the values of the four parameters $Q=(\eta_1;\ \vartheta_1;\ \eta_2;\ \vartheta_2)$ called *configuration*. The space $\mathscr{Q}$ of configurations is the non-negative sixteenth of the four-dimensional Cartesian space.

We use below the following notations:

$$\tau_i = \eta_i + \vartheta_i, \quad i = 1, 2, \quad \eta = \eta_1 + \eta_2, \ \vartheta = \vartheta_1 + \vartheta_2, \ \gamma^{(i)} = \frac{\eta_i}{\tau_i}, \quad i = 1, 2.$$

A *schedule* is a unique determination for $t \geqq 0$ of which tasks are serviced at the moment $t$ by which processors. The demands for the processor $P_A$ can be conflicting. The schedule can be considered a decision process by which the conflicting situations are resolved and the normal continuation of service can be broken.

An important class of schedules is the set of *non-preemptive* schedules in which

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the service of any task cannot be preempted after starting until it finishes automatically. These schedules were discussed in the article [5]. A relatively simple algorithm was given to determine the optimal schedule.

The efficiency measure of schedules is the utilization of the processor $P_A$. Formally, the efficiency of a schedule $R$ is defined by the limit

$$\gamma(R) = \lim_{t\to\infty} \frac{\lambda(t)}{t} \tag{1}$$

where $\lambda(t)=\lambda(0, t)$ is the $P_A$-usage in the interval $(0, t)$. The algorithm for choosing an optimal non-preemptive schedule is based on the method of reducing the configuration which is a generalization of the well-known Euclidean algorithm of the regular continued fraction expansion. The determination of the optimal schedule takes place by the full evaluation of the elements of the dominant set of the consistent natural schedules with maximum number six. Only one reduction has to be executed. The amount of the necessary computation is well bounded and estimated.

For the *preemptive scheduling* in which preempt-resume is permitted, another set, the consistent economical schedules, is a dominant set but it is not so nicely bounded as the set of consistent natural schedules [6]. The criteria of finiteness and bounds for the cardinal of the set are not known. Neither optimal strategy nor a smaller dominant set of schedules is known. It is shown [6] that the priority schedules are not optimal either. Since the only general method for determining an optimal schedule is the full evaluation of this dominant set the optimization procedure is uncontrolled.

Though the priority schedules are neither dominant, nor actually of better efficiency than the non-preemptive schedules in general, they are of great practical importance because of their simple scheduling rule. In a *priority schedule* one of the job-flows has priority versus other(s) which means that it is serviced in the moment it needs the processor. If the processor is busy by servicing another job-flow, the service will be preempted during the service of the priority job-flow-task and resumed after that. For job-flow pairs there are only two priority schedules according to job-flows $Q^{(1)}$ and $Q^{(2)}$ as priority ones. In [6] the priority schedules were denoted by $R_{1,2}$ and $R_{2,1}$, accordingly. In the schedule $R_{i,3-i}$ $(i=1, 2)$ the job-flow $Q^{(i)}$ is scheduled without preemption and delay as when the job-flow $Q^{(3-i)}$ were not present at all. The service of $Q^{(3-i)}$ on $P_A$ takes place only in the intervals the $P_A$ is free from servicing $Q^{(i)}$. The priority schedules $R_{1,2}$ and $R_{2,1}$ of the configuration $Q=(1;\ 3;\ 5;\ 7.5)$ are illustrated by Gantt-charts in Fig. 1.

The priority scheduling of the stochastic version of job-flow pairs was studied by Arató [1] with diffusion approximation and by Tomkó [7].

For the schedules $R_{1,2}$ and $R_{2,1}$ are symmetric in the role of the job-flows $Q^{(1)}$ and $Q^{(2)}$, every fact concerning $R_{1,2}(Q)$ becomes a fact concerning $R_{2,1}(\bar{Q})$ if $\bar{Q}$ is the *conjugate configuration* of $Q$ defined as

$$\bar{Q} = (\bar{\eta}_1;\ \bar{\vartheta}_1;\ \bar{\eta}_2;\ \bar{\vartheta}_2) = (\eta_2;\ \vartheta_2;\ \eta_1;\ \vartheta_1).$$

This is why we need not word definitions and theorems depending on the order of the job-flows for both orders, only for the order $Q^{(1)}, Q^{(2)}$.

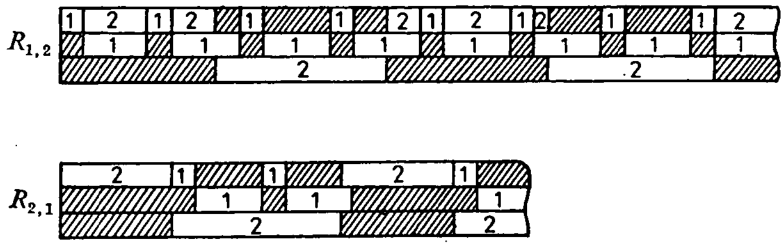



*Fig. 1*

The Gantt-charts of the priority schedules

In section 2 below we define first a method for reducing configurations $Q \in \mathcal{Q}$ into simpler, reduced configurations $Q^* \in \mathcal{Q}$. The reduction takes place by the iteration of an operator $\Delta$ to the configurations $Q_n = \Delta^n Q$ until a fixpoint $Q^* = \Delta^\nu Q$ called reduction of $Q$ is reached. We show the relationships between the parameters of $Q_n$ and $Q_m$, $n, m = 0, 1, 2, \ldots, n \neq m$. These remind one of the relationships known in the theory of continued fractions [4].

In paragraph 3 we show the connections between the characteristics of the schedules $R_{1,2}(Q_n)$ and $R_{1,2}(Q_m)$, $n \neq m$. This provides means to determine the characteristics of $R_{1,2}(Q)$ from the characteristics of $R_{1,2}(Q^*)$.

Section 4 surveys the configuration space $\mathcal{Q}$, the reduced configurations included, and give answer to the *Question* whether $R_{1,2}(Q)$ is periodic and what are its characteristics in different domains of $\mathcal{Q}$. The domain $0 < \tau_1^* < \tau_2^*$ remains unanswered in this section.

Section 5 is dealing with the above domain. The periodicity of $R_{1,2}(Q^*)$ is not cleared for the whole domain only for some parts of it. An algorithm is given for evaluating $R_{1,2}(Q^*)$ if it is periodic.

In section 6 we shall briefly deal with the connection between the $\Delta_i$-reductions defined in section 2 and $\mathcal{D}_i$-reductions given in the article [5]. Also some reference is made to the analogy between the $\Delta$-reduction and the continued fraction expansion algorithm.

Section 7 reviews the configuration space $\mathcal{Q}$ from the point of view whether the "Question" of periodicity and evaluation is answered or not, and by which theorem, if it is.

## 2. The method of $\Delta$-reduction

The transformation of configurations defined below as $\Delta$-reduction enables us to reduce the investigation of priority scheduling of some configurations to one of other configurations. This method is analogous to the reduction method applied for non-preemptive schedules by means of an operator $\mathcal{D}$ [5].

The operator $\Delta$ defined below is the $\Delta_1$ from the two operators $\Delta_i$, $i = 1, 2$, in the application of which the roles of $Q^{(1)}$ and $Q^{(2)}$ are symmetrical. We shall see later that the operator $\Delta_i$ is connected to the priority schedule $R_{i,3-i}$, $i = 1, 2$. The index 1 of $\Delta_1$ is omitted in the notation $\Delta$.

Let the *operator* $\Delta$ be defined for any configuration $Q \in \mathcal{Q}$ by the relationships between its parameters and the parameters of the configuration $\tilde{Q} = \Delta Q = (\tilde{\eta}_1;\ \tilde{\vartheta}_1;\ \tilde{\eta}_2;\ \tilde{\vartheta}_2) \in \mathcal{Q}$. The parameters of $\tilde{Q}$ are defined by the relations

$$
\begin{aligned}
&\text{(a)}\ \tilde{\eta}_1 = \eta_1 \\
&\text{(b)}\ \vartheta_1 = l_1\tau_2 + \tilde{\vartheta}_1 \text{ where} \\
&\qquad l_1 \geqq 0 \text{ is an integer and } 0 \leqq \tilde{\vartheta}_1 < \tau_2 \text{ if } \tau_2 > 0, \\
&\qquad l_1 = 0,\ \tilde{\vartheta}_1 = \vartheta_1 \text{ if } \tau_2 = 0, \\
&\text{(c)}\ \eta_2 = k_2\tilde{\vartheta}_1 + \tilde{\eta}_2 \text{ where} \\
&\qquad k_2 \geqq 0 \text{ is an integer and } 0 < \tilde{\eta}_2 \leqq \tilde{\vartheta}_1 \text{ if } \eta_2\tilde{\vartheta}_1 > 0, \\
&\qquad k_2 = 0,\ \tilde{\eta}_2 = \eta_2 \text{ if } \eta_2\tilde{\vartheta}_1 = 0, \\
&\text{(d)}\ \vartheta_2 = l_2\tilde{\tau}_1 + \tilde{\vartheta}_2 \text{ where} \\
&\qquad l_2 \geqq 0 \text{ is an integer and } 0 \leqq \tilde{\vartheta}_2 < \tilde{\tau}_1 \text{ if } \tilde{\tau}_1 > 0, \\
&\qquad l_2 = 0,\ \tilde{\vartheta}_2 = \vartheta_2 \text{ if } \tilde{\tau}_1 = 0.
\end{aligned}
\tag{2}
$$

This definition shows that the operation $\Delta Q$ determines also an integer triple $(l_1, k_2, l_2)$ out of the configuration $\tilde{Q}$. This triple is characteristic of the configuration $Q$ from the point of view of the effect of the operator $\Delta$ on $Q$.

If $l_1+k_2+l_2=0$ then the operator $\Delta$ is *ineffective* for $Q$ and $\Delta Q = Q$. We say $Q$ that is *reduced* in this case. If $l_1+k_2+l_2>0$ then $\Delta$ is *effective* for $Q$, $\Delta Q \neq Q$ and at least one of the parameters of $\tilde{Q}$ is less than that of $Q$. Therefore the operator $\Delta$ is called a *reduction operator*. The triple $(l_1, k_2, l_2)$ is the *quotient generated by $\Delta$* applied to $Q$. $\Delta$ is defined for all points $Q$ of $\mathcal{Q}$, and $\tilde{Q} \in \mathcal{Q}$. Therefore $\Delta$ is applicable repeatedly to the transformed configurations and the series of configurations

$$Q_0 = Q, \quad Q_n = \Delta Q_{n-1}, \quad n = 1, 2, \dots,$$

can be defined for any point $Q$ of $\mathcal{Q}$. Using the powers $\Delta^n$, $n=0, 1, 2, \dots$, of the operator $\Delta$, we can write

$$Q_n = \Delta^n Q, \quad n = 0, 1, 2, \dots . \tag{3}$$

Let the series of triples generated by the series $\Delta, \Delta^2, \dots, \Delta^n, \dots$ be

$$(L)\colon\ (l_{1,0}, k_{2,0}, l_{2,0}),\ (l_{1,1}, k_{2,1}, l_{2,1}),\ \dots,\ (l_{1,n-1}, k_{2,n-1}, l_{2,n-1}),\ \dots$$

and let

$$(\Lambda)\colon\ (l_{1,0}, k_{2,0}+l_{2,0}),\ (l_{1,1}, k_{2,1}+l_{2,1}),\ \dots,\ (l_{1,n-1}, k_{2,n-1}+l_{2,n-1}),\ \dots .$$

These are the series of quotients. Let us define the length of $(L)$ and $(\Lambda)$ the index $\nu$ of the first triple for which

$$l_{1,\nu}+k_{2,\nu}+l_{2,\nu} = 0$$

if such an index exists and $\nu=\infty$ otherwise. Let us use the notation $|(L)| = |(\Lambda)| = \nu$. If $\nu<\infty$, the $Q_\nu$ is the first member in the sequence $Q_0, Q_1, \dots$ which is reduced. $\nu$ is called the *degree of compositeness* (dc) of $Q$. If $\nu<\infty$ then $Q$ is *reducible*, otherwise, it is *non-reducible*. If the dc of $Q$ is $0<\nu<\infty$ then

$$
\begin{aligned}
&\text{(a)}\ l_{1,i}+k_{2,i}+l_{2,i} > 0, \quad i = 0, 1, \dots, \nu-1, \\
&\text{(b)}\ l_{1,\nu}+k_{2,\nu}+l_{2,\nu} = 0
\end{aligned}
\tag{4}
$$

and the series ($L$) and ($\Lambda$) contain exactly $v$ non-zero members. The configuration $Q^*=Q_v$ is a reduced configuration and it is the *reduction of Q*.

From the definition (2) of $\Delta$ we can deduce the conditions of $Q^*$ to be reduced. By (2), (4b) will hold if

$$\begin{aligned} &\text{(a)}\quad 0 \leqq \vartheta_1^* < \tau_2^* \quad \text{or} \quad \tau_2^* = 0 \quad \text{and} \\ &\text{(b)}\quad 0 < \eta_2^* \leqq \vartheta_1^* \quad \text{or} \quad \eta_2^* \vartheta_1^* = 0 \quad \text{and} \\ &\text{(c)}\quad 0 \leqq \vartheta_2^* < \tau_1^* \quad \text{or} \quad \tau_1^* = 0. \end{aligned} \tag{5}$$

Conditions (5a)—(5c) are not independent of but include each other. The set $\mathcal{Q}^* \subset \mathcal{Q}$ of the reduced configurations is illustrated by planes $(\eta_1^*, \eta_2^*)$ fixed in Fig. 2a—d.

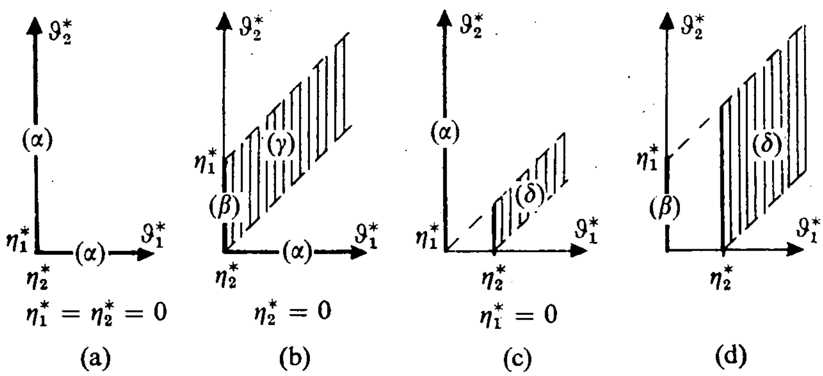



*Fig. 2*

Illustration of the set $\mathcal{Q}^*$ of reduced configurations

On the graphs we show the disjunct domains of configurations by the following lemma.

**Lemma 1.** *The operator $\Delta$ defined by* (2) *is ineffective for $Q^*$ i.e. $Q^*$ is reduced, iff one of the following conditions holds*

$$\begin{aligned} &(\alpha)\quad \tau_1^* \tau_2^* = 0 \\ &(\beta)\quad \tau_1^* \tau_2^* > 0,\ \vartheta_1^* = 0,\ 0 \leqq \vartheta_2^* < \eta_1^* \\ &(\gamma)\quad \vartheta_1^* \tau_2^* > 0,\ \eta_2^* = 0,\ 0 < \vartheta_1^* < \vartheta_2^* < \tau_1^* \\ &(\delta)\quad \vartheta_1^* \eta_2^* > 0,\ \eta_2^* \leqq \vartheta_1^* < \tau_2^*,\ 0 \leqq \vartheta_2^* < \tau_1^*. \end{aligned} \tag{6}$$

*Proof.* In either domain of (6α)–(6δ) every of the conditions (5a)—(5c) holds. Conditions (6α)—(6δ) are, therefore, sufficient for $Q^*$ to be reduced. To see the necessity it is easy to verify that one of (6α)—(6δ) holds if (5a)—(5c) are true [4]. □

Let the number series ($\lambda$) defined as $\lambda_{2i}=l_{1,i}$, $\lambda_{2i+1}=k_{2,i}+l_{2,i}$, $i=0, 1, \dots$. The following lemma shows that no zero value in the series ($\lambda$) between $l_{1,0}$ and

$k_{2,\nu-1}+l_{2,\nu-1}$ exists. This means that the parameters of both job-flows are reduced in the transformation $Q_i \to Q_{i+1}$, $i=1, 2, \ldots, \nu-2$. They are the transformations $Q_0 \to Q_1$ and $Q_{\nu-1} \to Q_\nu$ only in which it is possible that only one of the job-flows be reduced: $Q^{(2)}$ in $Q_0 \to Q_1$ and $Q^{(1)}$ in $Q_{\nu-1} \to Q_\nu$. This fact is expressed by the relations concerning $(\lambda)$

$$l_{1,0} \geqq 0, \; k_{2,i}+l_{2,i} > 0, \; 0 \leqq i < \nu-1, \; l_{1,i} > 0, \; 1 \leqq i \leqq \nu-1, \; k_{2,\nu-1}+l_{2,\nu-1} \geqq 0. \quad (7)$$

In any circumstances, the following relations hold for $i=0, 1, \ldots$:

$$\begin{aligned} &\text{(a)} \quad \vartheta_{1,i}-\vartheta_{1,i+1} = l_{1,i}\tau_{2,i}, \quad \tau_{1,i}-\tau_{1,i+1} = l_{1,i}\tau_{2,i} \\ &\text{(b)} \quad \eta_{2,i}-\eta_{2,i+1} = k_{2,i}\vartheta_{1,i+1}, \quad \tau_{2,i}-\tau_{2,i+1} = (k_{2,i}+l_{2,i})\vartheta_{1,i+1}+l_{2,i}\eta_1 \qquad (8) \\ &\text{(c)} \quad \vartheta_{2,i}-\vartheta_{2,i+1} = l_{2,i}\tau_{1,i+1}. \end{aligned}$$

**Lemma 2.** *Let*

$$k_{2,I}+l_{2,I} = 0, \quad I \geqq 0, \quad \text{or} \quad l_{1,I} = 0, \quad I \geqq 1,$$

*be the first zero value after $l_{1,0}$ in the series $(\lambda)$ if such one exists. Then all members in $(\lambda)$ following it are zeros and the degree of compositeness of $Q$ is as follows:*

$$\begin{aligned} &\textit{in case } k_{2,0}+l_{2,0} = 0: \quad && \nu = 0 \quad \textit{if} \quad l_{1,0} = 0 \\ & && \nu = 1 \quad \textit{if} \quad l_{1,0} > 0, \\ &\textit{in cases } I > 0: && \nu = I \quad \textit{if} \quad l_{1,I} = 0 \\ & && \nu = I+1 \quad \textit{if} \quad k_{2,I}+l_{2,I} = 0, \; l_{1,I} > 0. \end{aligned}$$

*Proof.* If $l_{1,0}=k_{2,0}+l_{2,0}=0$ $(I=0)$ then $Q_0$ is reduced by definition and $\nu=0$. If $l_{1,I}>0$ but $k_{2,I}+l_{2,I}=0$, $I\geqq 0$, then $\nu>I$ and $\vartheta_{1,I+1}<\tau_{2,I}$, $\tau_{2,I+1}=\tau_{2,I}$ from (2), and, therefore, $\vartheta_{1,I+1}<\tau_{2,I+1}$ and so $l_{1,I+1}=0$ and $\tau_{1,I+2}=\tau_{1,I+1}$. If, however, $l_{1,I+1}=0$, $I\geqq 1$, then $\tau_{1,I+2}=\tau_{1,I+1}$. But in this case $\eta_{2,I+2}=\eta_{2,I+1}$ and $\vartheta_{2,I+2}=\vartheta_{2,I+1}$ from (2) and so $Q_{I+2}=Q_{I+1}$. This means $\nu \leqq I+1$. □

The following lemma shows the part of $\mathcal{Q}$ in which non-reducibility is possible.

**Lemma 3.** *To any $Q \in \mathcal{Q}$ there exists a finite integer $\nu' \geqq 0$ for which the configuration*

$$Q_{\nu'} = \Delta^{\nu'} Q$$

*is either reduced or defective with*

$$\eta_1 \vartheta_{2,\nu'} = 0.$$

*Proof.* If $\eta_1=0$, there is nothing to prove. Let $\eta_1>0$. If $l_{2,i}>0$ then from (2d) we get

$$\vartheta_{2,i}-\vartheta_{2,i+1} = l_{2,i}\tau_{1,i+1} \geqq \tau_{1,i+1} \geqq \eta_1 > 0$$

and, therefore, the value of $\vartheta_{2,i}$ decreases at least by $\eta_1$. This means that only a finite number of positive $l_{2,i}$ members in the series $l_{2,0}, l_{2,1}, \ldots$ can exist and there exists an $i_0 \geqq 0$ so that

$$l_{2,i} = 0, \quad \vartheta_{2,i} = \vartheta_{2,i_0} \quad \text{if} \quad i \geqq i_0.$$

If $\vartheta_{2,i_0}=0$ then $\nu'=i_0$. Let $\vartheta_{2,i_0}>0$. If $l_{1,i}>0$ then from (2b) we get

$$\vartheta_{1,i}-\vartheta_{1,i+1} = l_{1,i}\tau_{2,i} \geqq \tau_{2,i} \geqq \vartheta_{2,i_0} > 0$$

and, therefore, the value of $\vartheta_{1,i}$ decreases at least by $\vartheta_{2,i_0}$. This means that only a finite number of positive $l_{1,i}$ member can exist in $(\lambda)$. If $l_{1,i'}$ is the last positive $l_{1,i}$ member then $v'=i'+1$ and $Q_{v'}$ is reduced. □

By Lemma 3 only the cases

$$\eta_1\vartheta_2 = 0 \tag{9}$$

remain questionable in regard to reducibility. The following lemma concerns these cases.

**Lemma 4.** *Any $Q\in\mathcal{Q}$ with (9) is either reducible or*

$$Q_n \to (\eta_1; 0; 0; 0) \quad as \quad n \to \infty.$$

*In the latter case*

$$\vartheta_{1,n}\tau_{2,n} > 0 \tag{10}$$

*after any finite step $n$. This case comes true if*

$$\tau_1\tau_2 > 0,\ \eta_2\vartheta_2 = 0,\ \vartheta_1 \text{ and } \vartheta_2 \text{ are rationally independent.} \tag{11}$$

*Proof.* $Q$ is reduced if $\tau_2=0$. Let now $\tau_2>0$.

If $\vartheta_2=0, \eta_2>0$, the reduction procedure will be equivalent to the regular continued fraction expansion of the number

$$\xi = \frac{\vartheta_1}{\tau_2} \tag{12}$$

with the restriction that the number $n+1$ of the partial quotients $[b_0, b_1, \ldots, b_n]$ must be chosen odd in finite cases because $\eta_2^*$ cannot be zero by definition (2). This choice is always possible [3]. The number of the partial quotients and the steps of reduction will be finite exactly when $\xi$ is a rational number [3]. The reduction results in $Q^*=(\eta_1;\ 0;\ \eta_2^*;\ 0)$. If (11) holds, neither $\vartheta_{1,i}$ nor $\eta_{2,i}$ becomes zero in finite steps and (10) is true.

Let now $\vartheta_2>0$. Then $\eta_1=0$ from (9). If $\vartheta_1=0$ then $Q$ is reduced. Let, therefore, $\vartheta_1>0$ as well.

If $\eta_2=0$, the reduction procedure becomes equivalent to the continued fraction expansion of $\xi$ and it is finite exactly when $\xi$ is a rational number. The reduction results in $Q^*=(0;\ \vartheta_1^*;\ 0;\ 0)$ or $Q^*=(0;\ 0;\ 0;\ \vartheta_2^*)$. If $\vartheta_1$ and $\tau_2$ are rationally independent, the expansion procedure is infinite and neither of $\vartheta_{1,i}$ and $\vartheta_{2,i}$ will be zero for finite $i$ and (10) holds.

Let $\eta_2>0$ as well. Suppose $Q$ is not-reducible, i.e., the degree of compositeness $v=\infty$. By Lemma 2 all members of $(\lambda)$ are positive after $l_{1,0}$. From (8) we can write for any $i>0$:

$$\vartheta_{1,i}-\vartheta_{1,i+1} = l_{1,i}\tau_{2,i} = l_{1,i}[(k_{2,i}+l_{2,i})\vartheta_{1,i+1}+\tau_{2,i+1}] \geqq$$
$$\geqq \max(\vartheta_{1,i+1}, \eta_{2,i+1}, \vartheta_{2,i+1}).$$

If either of the parameters $\vartheta_1, \eta_2, \vartheta_2$ remained bounded from below by a positive number $\alpha>0$, then $\vartheta_1$ would be decreased by at least $\alpha$ in every step of reduction. After $\vartheta_1/\alpha$ steps $\vartheta_{1,i}$ would surely become negative which is a contradiction. Thus none of $\vartheta_{1,i}, \eta_{2,i}, \vartheta_{2,i}$ could be bounded by an $\alpha>0$, and $Q_i\to$ $\to(0;\ 0;\ 0;\ 0)$ if $i\to\infty$. This proves (10).

In cases (11) we have shown that $\nu=\infty$ and (10) holds. But from (2) we get

$$\vartheta_{1,i}-\vartheta_{1,i+1} = l_{1,i}[(k_{2,i}+l_{2,i})\vartheta_{1,i+1}+\tau_{2,i+1}+l_{2,i}\eta_1] \geqq$$
$$\geqq \max(\vartheta_{1,i+1}, \eta_{2,i+1}, \vartheta_{2,i+1})$$

also in these cases and the parameters cannot remain bounded from below and so $Q_i\to(\eta_1;\ 0;\ 0;\ 0)$ as $i\to\infty$. $\square$

From Lemma 3 and Lemma 4 we can assert that $\nu=\infty$ can hold only for defective configurations for which $\eta_1=0$ and for configurations for which $\vartheta_{2,\nu'}=0$ for some $\nu'\geqq 0$. We cannot exactly show the domains or points of $\mathcal{Q}$ in which $Q$ is non-reducible. We know such subsets of $\mathcal{Q}$ but not all such points.

The relationships below are true independently of the finiteness of $\nu$ and the relation of $\nu$ and $n$. These relationships concern the parameters of $Q$ and $Q_n$ and $Q_n$ and $Q_{n+1}$.

As the definition (2) of $Q_{i+1}=\Delta Q_i$, we get

$$\begin{matrix} \eta_{1,i} = \eta_{1,i+1}, & \eta_{2,i} = k_{2,i}\vartheta_{1,i+1}+\eta_{2,i+1} \\ \vartheta_{1,i} = l_{1,i}\tau_{2,i}+\vartheta_{1,i+1}, & \vartheta_{2,i}= l_{2,i}\tau_{1,i+1}+\vartheta_{2,i+1}. \end{matrix} \quad i = 0, 1, \dots \tag{13}$$

From the same definition we can obtain the relationship between the parameters of $Q_n$ and $Q_{n+1}$ in the following form:

$$\begin{aligned} \eta_{1,n} &= \eta_{1,n+1} \\ \vartheta_{1,n} &= l_{1,n}l_{2,n}\eta_{1,n+1}+[l_{1,n}(k_{2,n}+l_{2,n})+1]\vartheta_{1,n+1}+l_{1,n}\eta_{2,n+1}+l_{1,n}\vartheta_{2,n+1} \\ \eta_{2,n} &= k_{2,n}\vartheta_{1,n+1}+\eta_{2,n+1} \\ \vartheta_{2,n} &= l_{2,n}\eta_{1,n+1}+l_{2,n}\vartheta_{1,n+1}+\vartheta_{2,n+1} \end{aligned} \tag{14}$$

$$\begin{aligned} \tau_{1,n} &= [l_{1,n}(k_{2,n}+l_{2,n})+1]\tau_{1,n+1}+l_{1,n}\tau_{2,n+1}-l_{1,n}k_{2,n}\eta_1 \\ \tau_{2,n} &= (k_{2,n}+l_{2,n})\tau_{1,n+1}+\tau_{2,n+1}-k_{2,n}\eta_1 \end{aligned} \tag{15}$$

$$\begin{aligned} \eta_{1,n+1} &= \eta_{1,n} \\ \vartheta_{1,n+1} &= \vartheta_{1,n}-l_{1,n}\eta_{2,n}-l_{1,n}\vartheta_{2,n} \\ \eta_{2,n+1} &= -k_{2,n}\vartheta_{1,n}+(l_{1,n}k_{2,n}+1)\eta_{2,n}+l_{1,n}k_{2,n}\vartheta_{2,n} \\ \vartheta_{2,n+1} &= -l_{2,n}\eta_{1,n}-l_{2,n}\vartheta_{1,n}+l_{1,n}l_{2,n}\eta_{2,n}+(l_{1,n}l_{2,n}+1)\vartheta_{2,n} \end{aligned} \tag{14'}$$

$$\begin{aligned} \tau_{1,n+1} &= \tau_{1,n}-l_{1,n}\tau_{2,n} \\ \tau_{2,n+1} &= -(k_{2,n}+l_{2,n})\tau_{1,n}+[l_{1,n}(k_{2,n}+l_{2,n})+1]\tau_{2,n}+k_{2,n}\eta_1. \end{aligned} \tag{15'}$$

As the parameter $\eta_1$ is not concerned during reduction, $\eta_{1,n}=\eta_1$, $n=0, 1, \dots$, and it can be separated from the other parameters.

From the relationships (14) the connection between the parameters of any two $Q_n$ and $Q_{n'}$, $n\neq n'$, especially between the parameters of $Q=Q_0$ and $Q_n$ can be obtained. To make the further relationships more compact we have to introduce some series of integers, vectors and matrices as follow.

Let $(X)$ be the formal notation of the infinite sequence:

$(X)$: $X_0, X_1, X_2, \dots, X_n, \dots$

and let $|(X)|$ be the index of the first member of $(X)$ from which all members are the same, if such a member exists. This is called the length of $(X)$.

We have already defined the two series $(L)$ and $(\Lambda)$. The members of $(Q)$ are the configurations $Q_n=(\eta_1;\ \vartheta_{1,n};\ \eta_{2,n};\ \vartheta_{2,n})$. The lengths of $(L)$, $(\Lambda)$, $(Q)$ are the same $v$, the dc of the configuration $Q_0=Q$. Let $(0)$ be the series of the identically zero members with the length 0. We have referred to the series $(\lambda)$ the members of which are

$$(\lambda):\quad \lambda_{2i}=l_{1,i},\quad \lambda_{2i+1}=k_{2,i}+l_{2,i};\quad i=0,1,\ldots .$$

Define also the series

$$(k):\quad k_n=k_{2,n},\quad n=0,1,\ldots$$

and

$$(l):\quad l_n=l_{2,n},\quad n=0,1,\ldots .$$

We define now a set of new series necessary to writing down the relationships among the parameters of $(Q)$. The definitions are recursive for $i, n=0, 1, \ldots$.

$$(A):\quad A_n=\lambda_n A_{n-1}+A_{n-2}\quad \text{with}\quad A_{-2}=0,\quad A_{-1}=1$$

$$(B):\quad B_n=\lambda_n B_{n-1}+B_{n-2}\quad \text{with}\quad B_{-2}=1,\quad B_{-1}=0$$

$$(C):\quad C_{2i}=A_{2i},\quad C_{2i+1}=k_{2,i}C_{2i}+C_{2i-1}\quad \text{with}\quad C_{-1}=0$$

$$(D):\quad D_{2i}=B_{2i},\quad D_{2i+1}=k_{2,i}D_{2i}+D_{2i-1}\quad \text{with}\quad D_{-1}=0$$

$$(B'):\quad B'_{2i}=\lambda_{2i}B'_{2i-1}+B'_{2i-2},\quad B'_{2i+1}=\lambda_{2i+1}B'_{2i}+B'_{2i-1}+k_{2,i}\quad \text{with}\quad B'_{-2}=0,\ B'_{-1}=0$$

$$(B''):\quad B''_{2i}=\lambda_{2i}B''_{2i-1}+B''_{2i-2},\quad B''_{2i+1}=\lambda_{2i+1}B''_{2i}+B''_{2i-1}-k_{2,i}\quad \text{with}\quad B''_{-2}=1,\ B''_{-1}=0$$

$$(D'):\quad D'_{2i}=B'_{2i},\ D'_{2i+1}=k_{2,i}D'_{2i}+D'_{2i-1}+k_{2,i}\quad \text{with}\quad D'_{-1}=0$$

$$(D''):\quad D''_{2i}=B''_{2i},\quad D''_{2i+1}=k_{2,i}D''_{2i}+D''_{2i-1}-k_{2,i}\quad \text{with}\quad D''_{-1}=0$$

Define the following sequences of vectors and matrices for $n=0, 1, \ldots$ as well.

$$(\underline{\tilde{Q}}):\quad \underline{\tilde{Q}}_n=\begin{pmatrix}\tilde{\vartheta}_{1,n}\\ \tilde{\eta}_{2,n}\\ \tilde{\vartheta}_{2,n}\end{pmatrix}$$

$$(\underline{\tilde{\tau}}):\quad \underline{\tilde{\tau}}_n=\begin{pmatrix}\tilde{\tau}_{1,n}\\ \tilde{\tau}_{2,n}\end{pmatrix}$$

with

$$\begin{aligned}
\tilde{\vartheta}_{1,n}&=\vartheta_{1,n}+(B'_{2n-2}+1)\eta_1\\
\tilde{\eta}_{2,n}&=-\eta_{2,n}+D'_{2n-1}\eta_1\\
\tilde{\vartheta}_{2,n}&=-\vartheta_{2,n}+(B'_{2n-1}-D'_{2n-1})\eta_1\\
\tilde{\tau}_{1,n}&=\tau_{1,n}+B'_{2n-2}\eta_1\\
\tilde{\tau}_{2,n}&=-\tau_{2,n}+B'_{2n-1}\eta_1
\end{aligned}\tag{16}$$

$$(\underline{\underline{D}}_+):\quad \underline{\underline{D}}_{n,n+1} = \begin{pmatrix} 1 & \lambda_{2n} \\ \lambda_{2n+1} & \lambda_{2n}\lambda_{2n+1}+1 \end{pmatrix} = \begin{pmatrix} 1 & l_{1,n} \\ k_{2,n}+l_{2,n} & l_{1,n}(k_{2,n}+l_{2,n})+1 \end{pmatrix}$$

$$(\underline{\underline{\Delta}}_+):\quad \underline{\underline{\Delta}}_{n,n+1} = \begin{pmatrix} 1 & l_{1,n} & l_{1,n} \\ k_{2,n} & l_{1,n}k_{2,n}+1 & l_{1,n}k_{2,n} \\ l_{2,n} & l_{1,n}l_{2,n} & l_{1,n}l_{2,n}+1 \end{pmatrix}$$

$$(\underline{\underline{D}}):\quad \underline{\underline{D}}_n = \begin{pmatrix} B_{2n-2} & A_{2n-2} \\ B_{2n-1} & A_{2n-1} \end{pmatrix}$$

$$(\underline{\underline{\Delta}}):\quad \underline{\underline{\Delta}}_n = \begin{pmatrix} B_{2n-2} & A_{2n-2} & A_{2n-2} \\ D_{2n-1} & C_{2n-1}+1 & C_{2n-1} \\ B_{2n-1}-D_{2n-1} & A_{2n-1}-C_{2n-1}-1 & A_{2n-1}-C_{2n-1} \end{pmatrix}.$$

We remark at once that

$$\underline{\tilde{Q}}_0 = \begin{pmatrix} \tau_{1,0} \\ -\eta_{2,0} \\ -\vartheta_{2,0} \end{pmatrix} = \underline{\tilde{Q}},\quad \underline{\tilde{\tau}}_0 = \begin{pmatrix} \tau_{1,0} \\ -\tau_{2,0} \end{pmatrix} = \underline{\tilde{\tau}} \tag{17}$$

and that the $D$-matrices can be obtained from the corresponding $\Delta$-matrices by summing up the two last rows and omitting one of the last two equal columns.

The foregoing entities simplify the relationships between the parameters of the members of $(Q)$. The proof of the relationships will be automatic by means of the relationships of the following lemma. The relationships are interesting on their own right as well. To simplify writing we use the following determinant notation:

$$H_n(x, y) = \begin{vmatrix} x_n & y_n \\ x_{n-1} & y_{n-1} \end{vmatrix} = x_n y_{n-1} - x_{n-1} y_n,\quad n = 1, 2, \ldots, \tag{18}$$

for any two series $(x)$ and $(y)$. From this definition the relation

$$H_n(y, x) = -H_n(x, y) \tag{19}$$

is trivial. (18)—(19) can be interpreted for $n=-1,0$ as well if the values $x_{-2}, y_{-2}, x_{-1}, y_{-1}$ are also given.

**Lemma 5.** *Among the entities defined beforehand, the following relationships hold. For* $i, n=-1, 0, 1, \ldots$

$$H_n(A, B) = (-1)^{n-1} \tag{20}$$

$$(A_n, B_n),\ (A_n, A_{n-1}),\ (B_n, B_{n-1}),\ (A_{n-1}, B_{n-1}) \tag{21}$$

*are relatively prime integer pairs**

$$A_{2i+1} = \sum_{j=0}^{i-1} (k_{2,j}+l_{2,j})A_{2j}+1,\quad B_{2i+1} = \sum_{j=0}^{i-1} (k_{2,j}+l_{2,j})B_{2j}$$
$$C_{2i+1} = \sum_{j=0}^{i-1} k_{2,j}A_{2j},\quad D_{2i+1} = \sum_{j=0}^{i-1} k_{2,j}B_{2j} \tag{22}$$

* 0 and 1 are considered relatively prime integers.

*(with the definition* $\sum_{j=0}^{-1} x_j = 0$*)*

$$B'_n + B''_n = B_n, \quad D'_n + D''_n = D_n. \tag{23}$$

*For* $i, n = 0, 1, \ldots$

$$\begin{aligned} H_{2i}(B, A) &= H_{2i-1}(A, B) &&= 1 \\ H_{2i}(B', A) &= H_{2i-1}(A, B') &&= 1 - C_{2i-1} \\ H_{2i}(B'', A) &= H_{2i-1}(A, B'') &&= 1 + C_{2i-1} \\ H_{2i}(B', B) &= H_{2i-1}(B, B') &&= -D_{2i-1} \\ H_{2i}(B'', B) &= H_{2i-1}(B, B'') &&= D_{2i-1} \\ H_{2i}(B'', B') &= H_{2i-1}(B', B'') &&= D_{2i-1}; \end{aligned} \tag{24}$$

$$\begin{aligned} &A_{2i} D_{2i} - B_{2i} C_{2i} = 0, \quad A_{2i-1} D_{2i-1} - B_{2i-1} C_{2i-1} = B'_{2i-1} \\ &A_{2i+1} D_{2i} - B_{2i+1} C_{2i} = A_{2i-1} D_{2i} - B_{2i-1} C_{2i} = 1 \\ &A_{2i} D_{2i+1} - B_{2i} C_{2i+1} = A_{2i} D_{2i-1} - B_{2i} C_{2i-1} = B'_{2i}; \end{aligned} \tag{25}$$

*if* $(k) = (0)$ *then*

$$\begin{gathered} C_{2i+1} = D_{2i+1} = 0, \quad (B') = (0), \quad (B'') = (B) \\ (D') = (0), \; D''_{2i} = B_{2i}, \; D''_{2i+1} = 0; \end{gathered} \tag{26}$$

*if* $(l) = (0)$ *then*

$$\begin{gathered} B'_{2i} = B_{2i} - 1, \quad B'_{2i+1} = B_{2i+1}, \quad B''_{2i} = 1, \quad B''_{2i+1} = 0 \\ D'_{2i} = B_{2i-1}, \quad D'_{2i+1} = B_{2i+1}, \quad D''_{2i} = 1, \quad D''_{2i+1} = 0; \end{gathered} \tag{26'}$$

$$\underline{\underline{D}}_{n+1} = \underline{\underline{D}}_{n,n+1} \underline{\underline{D}}_n, \quad \underline{\underline{\Delta}}_{n+1} = \underline{\underline{\Delta}}_{n,n+1} \underline{\underline{\Delta}}_n \quad \text{with} \quad \underline{\underline{D}}_0 = \underline{\underline{I}}, \quad \underline{\underline{\Delta}}_0 = \underline{\underline{I}}, \tag{27}$$

$$\begin{gathered} \underline{\underline{D}}^{-1}_{n,n+1} = \begin{pmatrix} \lambda_{2n}\lambda_{2n+1} + 1 & -\lambda_{2n} \\ -\lambda_{2n+1} & 1 \end{pmatrix} = \begin{pmatrix} l_{1,n}(k_{2,n} + l_{2,n}) + 1 & -l_{1,n} \\ -(k_{2,n} + l_{2,n}) & 1 \end{pmatrix} \\ \underline{\underline{\Delta}}^{-1}_{n,n+1} = \begin{pmatrix} l_{1,n}(k_{2,n} + l_{2,n}) + 1 & -l_{1,n} & -l_{1,n} \\ -k_{2,n} & 1 & 0 \\ -l_{2,n} & 0 & 1 \end{pmatrix}. \end{gathered} \tag{28}$$

$$\underline{\underline{D}}_n^{-1} = \begin{pmatrix} A_{2n-1} & -A_{2n-2} \\ -B_{2n-1} & B_{2n-2} \end{pmatrix}$$

$$\underline{\underline{\Delta}}_n^{-1} = \begin{pmatrix} A_{2n-1} & -A_{2n-2} & -A_{2n-2} \\ -B_{2n-1} & B_{2n-2} + 1 & B_{2n-2} \\ -B_{2n-1} & B_{2n-2} - 1 & B_{2n-2} \end{pmatrix}$$

$$\underline{\underline{D}}_{n,n+1} = \begin{pmatrix} 1 & 0 \\ k_{2,n} + l_{2,n} & 1 \end{pmatrix} \begin{pmatrix} 1 & l_{1,n} \\ 0 & 1 \end{pmatrix}$$

$$\underline{\underline{D}}^{-1}_{n,n+1} = \begin{pmatrix} 1 & -l_{1,n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(k_{2,n} + l_{2,n}) & 1 \end{pmatrix} \tag{29}$$

$$\underline{\underline{\Delta}}_{n,n+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{2,n} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ k_{2,n} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l_{1,n} & l_{1,n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{\Delta}}_{n,n+1}^{-1} = \begin{pmatrix} 1 & -l_{1,n} & -l_{1,n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -k_{2,n} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{2,n} & 0 & 1 \end{pmatrix}$$

*The determinant* $\det(\underline{\underline{X}})$ *for every matrix encounters above is*

$$\det(\underline{\underline{X}}) = 1. \tag{30}$$

*Proof.* Taking into account definition (18), we easily see (20) and (24) for $n=-1$ and $i, n=0$. The other relations (21)—(26′) can be checked for the least index by the definitions of the entities. Using the recursive definitions of the series, we can verify (20), (22)—(26′) by induction. (21) follows from (20) because every common divisor of the pairs must divide $(-1)^{n-1}$ and is, therefore, $\pm 1$. (27) can be verified by executing the multiplications. The inverse matrices (28) can be verified most simply by multiplying them with the corresponding original matrices and using (20)—(25). The factorizations (29) can simply be checked by executing the assigned multiplications. (30) is trivial for every matrix encountering. □

After Lemma 5 we can now easily prove

**Theorem 1.** *For any configuration* $Q \in \mathcal{Q}$ *the following relationships between the parameters of* $(Q)$ *hold:*

$$\begin{aligned} &\underline{\tilde{Q}}_{n+1} = \underline{\underline{\Delta}}_{n,n+1}\underline{\tilde{Q}}_n, \quad \underline{\tilde{Q}}_n = \underline{\underline{\Delta}}_{n,n+1}^{-1}\underline{\tilde{Q}}_{n+1}, \quad \underline{\tilde{Q}}_n = \underline{\underline{\Delta}}_n\underline{\tilde{Q}}, \quad \underline{\tilde{Q}} = \underline{\underline{\Delta}}_n^{-1}\underline{\tilde{Q}}_n \\ &\underline{\tilde{\tau}}_{n+1} = \underline{\underline{D}}_{n,n+1}\underline{\tilde{\tau}}_n, \quad \underline{\tilde{\tau}}_n = \underline{\underline{D}}_{n,n+1}^{-1}\underline{\tilde{\tau}}_{n+1}, \quad \underline{\tilde{\tau}}_n = \underline{\underline{D}}_n\underline{\tilde{\tau}}, \quad \underline{\tilde{\tau}} = \underline{\underline{D}}_n^{-1}\underline{\tilde{\tau}}_n. \end{aligned} \tag{31}$$

*Proof.* The relationships in the second and fourth columns follow from those of the first and third columns. The relationships in the third column follow from the ones of the first column because of (17) and the recursions (27). The relationships of the first column are to be verified. This can be done by (14′)—(15′) and definitions (16) and $(\underline{\underline{D}}_+)$, $(\underline{\underline{\Delta}}_+)$. By (16)

$$\begin{aligned} \tilde{\vartheta}_{1,n+1} &= \vartheta_{1,n+1} + (B'_{2n}+1)\eta_1 \\ \tilde{\eta}_{2,n+1} &= -\eta_{2,n+1} + D'_{2n+1}\eta_1 \\ \tilde{\vartheta}_{2,n+1} &= -\vartheta_{2,n+1} + (B'_{2n+1} - D'_{2n+1})\eta_1 \\ \tilde{\tau}_{1,n+1} &= \tau_{1,n+1} + B'_{2n}\eta_1 \\ \tilde{\tau}_{2,n+1} &= -\tau_{2,n+1} + B'_{2n+1}\eta_1. \end{aligned}$$

From (14′)—(15′) and $(B')$, $(D')$, (16)

$$\begin{aligned} \tilde{\vartheta}_{1,n+1} &= \vartheta_{1,n} - l_{1,n}\eta_{2,n} - l_{1,n}\vartheta_{2,n} + [l_{1,n}B'_{2n-1} + B'_{2n-2} + 1]\eta_1 = \\ &= \vartheta_{1,n} + (B'_{2n-2}+1)\eta_1 + l_{1,n}[-\eta_{2,n} + D'_{2n-1}\eta_1] + l_{1,n}[-\vartheta_{2,n} + (B'_{2n-1} - D'_{2n-1})\eta_1] = \\ &= \tilde{\vartheta}_{1,n} + l_{1,n}\tilde{\eta}_{2,n} + l_{1,n}\tilde{\vartheta}_{2,n}, \end{aligned}$$

$$\begin{aligned}\tilde{\eta}_{2,n+1} &= k_{2,n}\vartheta_{1,n}-(l_{1,n}k_{2,n}+1)\eta_{2,n}-l_{1,n}k_{2,n}\vartheta_{2,n}+[k_{2,n}B'_{2n}+D'_{2n-1}+k_{2,n}]\eta_1 = \\ &= k_{2,n}[\vartheta_{1,n}-l_{1,n}(\eta_{2,n}+\vartheta_{2,n})+(l_{1,n}B'_{2n-1}+B'_{2n-2}+1)\eta_1]-\eta_{2,n}+D'_{2n-1}\eta_1 = \\ &= k_{2,n}[\vartheta_{1,n}+(B'_{2n-2}+1)\eta_1]+(l_{1,n}k_{2,n}+1)[-\eta_{2,n}+D'_{2n-1}\eta_1]+ \\ &\qquad +l_{1,n}k_{2,n}[-\vartheta_{2,n}+(B'_{2n-1}-D'_{2n-1})\eta_1] = \\ &= k_{2,n}\tilde{\vartheta}_{1,n}+(l_{1,n}k_{2,n}+1)\tilde{\eta}_{2,n}+l_{1,n}k_{2,n}\tilde{\vartheta}_{2,n},\end{aligned}$$

$$\begin{aligned}\tilde{\vartheta}_{2,n+1} &= l_{2,n}\eta_1+l_{2,n}\vartheta_{1,n}-l_{1,n}l_{2,n}\eta_{2,n}-(l_{1,n}l_{2,n}+1)\vartheta_{2,n}+[l_{2,n}B'_{2n}+B'_{2n-1}-D'_{2n-1}]\eta_1 = \\ &= l_{2,n}[\eta_1+\vartheta_{1,n}-l_{1,n}(\eta_{2,n}+\vartheta_{2,n})+(l_{1,n}B'_{2n-1}+B'_{2n-2})\eta_1]- \\ &\qquad -\vartheta_{2,n}+(B'_{2n-1}-D'_{2n-1})\eta_1 = \\ &= l_{2,n}[\vartheta_{1,n}+(B'_{2n-2}+1)\eta_1]+l_{1,n}l_{2,n}[-\eta_{2,n}+D'_{2n-1}\eta_1]+ \\ &\qquad +(l_{1,n}l_{2,n}+1)[-\vartheta_{2,n}+(B'_{2n-1}-D'_{2n-1})\eta_1] = \\ &= l_{2,n}\tilde{\vartheta}_{1,n}+l_{1,n}l_{2,n}\tilde{\eta}_{2,n}+(l_{1,n}l_{2,n}+1)\tilde{\vartheta}_{2,n}.\end{aligned}$$

These are exactly the relationship $\underline{\tilde{Q}}_{n+1}=\underline{\underline{A}}_{n,n+1}\underline{\tilde{Q}}_n$. Taken into account that $\tilde{\tau}_{1,n}=\tilde{\vartheta}_{1,n}$ and $\tilde{\tau}_{2,n}=\tilde{\eta}_{2,n}+\tilde{\vartheta}_{2,n}$ and summing up the last two equations, we get the relationship $\underline{\tilde{\tau}}_{n+1}=\underline{\underline{D}}_{n,n+1}\underline{\tilde{\tau}}_n$. □

This theorem is applicable to relate the parameters of a configuration $Q$ and its reduction $Q^*$ if the latter does exist.

## 3. The priority schedule and the reduction

In our previous article [6] we discussed the so-called consistent economical schedules (CESs) which represent a dominant set. There also the priority schedules were defined and shown as specific CESs. This means that the priority schedules $R_{1,2}$ and $R_{2,1}$ possess all the characteristics every CES possesses. There we illustrated the CESs by graphs which showed the basic characteristics of the CESs such as periodicity, the succession of the so-called typical and critical situations etc. The specific characteristics of $R_{i,3-i}$ $(i=1, 2)$ is that no task type $A_i$ can be preempted and, therefore, the job-flow $Q^{(3-i)}$ is always delayed whenever a cycle $C_{3-i,j}$ of it finishes in such a moment when a task type $A_i$ is under service or is ready for service. These are the critical situations type $\sigma_{3-i,1}$ and $\sigma_0$, respectively, defined in [6]. The delay can be $0\leqq d_{3-i}\leqq\eta_i$ and after finishing the service of $A_i$ the situation will be the same as the situation after finishing the first task $A_{i1}$. Since $R_{i,3-i}$ is consistent, the continuation of the servicing process after the two task-finishing points passes off similarly. This means that $R_{i,3-i}$ is *periodic* with a period represented by the schedule section between the two task-finishing points. If $\vartheta_i>0$ then the task $A_{3-i,1}$ begins immediately after the finishing point $t'_i=\eta_i$ of the task $A_{i1}$ in $R_{i,3-i}$. This situation is called *$\beta_i$-situation* [5, 6]. This situation returns next to the first delay of $Q^{(3-i)}$ after $t'_i$. The $\beta_i$-situation returns, however, whenever a cycle $C_{3-i,j}$ finishes during the service of a task type $A_i$ if $\vartheta_i>0$. If $\vartheta_i=0$ then the initial situation $\sigma_0$ returns at the point $t'_i$ immediately and, because of the consistency, the scheduling of the job-flow $Q^{(i)}$ is repeated. The period consists then of a cycle $C_i$ of $Q^{(i)}$ and the job-flow $Q^{(3-i)}$ fails to be scheduled. The efficiency of $R_{i,3-i}$ will be $\gamma=1$, the possible maximum, if $\eta_i>0$. But this schedule is by no means acceptable in practice. $R_{3-i,i}$ has efficiency $\gamma=1$ as well if $\eta_i>0, \vartheta_i=0$ unless $\tau_{3-i}=0$. If $\tau_i=0$ and $\vartheta_{3-i}>0$, the schedules $R_{1,2}$ and $R_{2,1}$ are degenerated with a finite length and some modification of the scheduling strategy is needed to produce practically acceptable schedules. This problem and generally the scheduling specialities of *degenerate* job-flow pairs (for which $\tau_1\tau_2=0$) were discussed in [4]. In spite of this fact we cannot keep degenerate and *defective* configurations (with zero value parameters) away from further discussion because the reduction of a nondefective configuration $Q$ can lead to defective reduced configuration $Q^*$.

Confining ourselves to the priority schedules $R_{1,2}(Q)$, $Q\in\mathcal{Q}$, which always start with the service of the task $A_{11}$, we know that $R_{1,2}(Q)$ is periodic if $\vartheta_1=0$ or the $\beta_1$-situation returns. A period is the section of the schedule between the point $t_1'=\eta_1$ and the first recurrence point $T_1^*>t_1'$ of $\beta_1$ if $\vartheta_1>0$. $R_{1,2}$ is not periodic if $\vartheta_1>0$ and the recurrence point of $\beta_1$ does not exist. In this case $Q^{(2)}$ cannot be delayed out of the starting delay of value $\eta_1$ and the preemptions. This means that the finishing times $f(i)$ of the cycles $C_{2,i}$, $i=1, 2, \ldots$, of $Q^{(2)}$ can be written as

$$f(i) = \eta_1+i\tau_2+\chi(i)\eta_1 \tag{32}$$

where $\chi(i)$ is an integer depending on $i$, the number of preemptions of the first $i$ $C_2$-cycles. (32) is valid only until the first recurrence of the $\beta_1$-situation. Suppose the $\beta_1$-situation recurs first after the $\mu_2$th cycle-finishing point. The length of period $p$ is then the distance between $t_1'$, the start-point of $C_{2,1}$, and $T_1^*$, the start-point of $C_{2,\mu_2+1}$, which consists of $\mu_2$ demand cycles of $Q^{(2)}$; $\varkappa_2=\chi(\mu_2)$ services of preempting $A_1$-tasks and the last delay $d_2$ of $Q^{(2)}$, if any, i.e.

$$p = T_1^*-t_1' = \mu_2\tau_2+\varkappa_2\eta_1+\varepsilon_2\eta_1 \tag{33}$$

where $\mu_2>0, \varkappa_2\geqq 0$ are integers and

$$0 \leqq \varepsilon_2 \leqq 1. \tag{34}$$

In both points $t_1'$ and $T_1^*$ a task type $A_1$ finishes and, as a result of priority, the service of the job-flow $Q^{(1)}$ goes on continually without break and delay and an integer number of $C_1$-cycles are serviced in the period between $t_1'$ and $T_1^*$. Let this number be denoted by $\mu_1$. Thus

$$p = \mu_1\tau_1, \tag{33'}$$

where $\mu_1>0$. Let us call $\mu_1$ and $\mu_2$ the *cycle numbers*, $\varkappa_2$ the *preemption number* and $\varepsilon_2$ the *relative delay*. These are the *characteristics* of $R_{1,2}$ and they are denoted by the quaternary

$$\Pi_{1,2} = (\mu_1; \mu_2; \varkappa_2; \varepsilon_2). \tag{35}$$

If $\vartheta_1=0$ then $R_{1,2}$ will be periodic with $p=\tau_1=\eta_1$ which accords with (33) and (33') if we define the characteristics as

$$\Pi_{1,2} = (1; 0; 0; 1). \tag{35'}$$

Another degenerate case must be discussed yet. This is when $\vartheta_1>0$ and $\tau_2=0$.

Scheduling this configuration with the priority of $Q^{(1)}$ the cycles $C_{2,j}$ with length 0 will be scheduled infinite times after the first, $A_{11}$, task and the further section of the schedule $R_{1,2}(Q)$ is undefined. Without modification of the strategy the obtained section of $R_{1,2}(Q)$ can be considered as periodic with length $p=0$ and the period consists of a $C_2$-cycle. In this exceptional case let the characteristics of $R_{1,2}(Q)$ be defined as

$$\Pi_{1,2} = (0; 1; 0; 0). \tag{35''}$$

From definition (1) of the efficiency $\gamma(R)$ of a schedule $R$ the efficiency of a periodic schedule can be obtained as

$$\gamma(R) = \frac{a_R}{p_R} \quad \left(\frac{0}{0} = 0!\right), \tag{1'}$$

where $p_R \geqq 0$ is the *length of the period* of $R$ and $a_R \geqq 0$ is the *$P_A$-usage* time in a period of $R$ and the quotient is defined as zero if both of $a_R$ and $p_R$ are zeros.

By the characteristics (35) of a priority schedule $R_{1,2}(Q)$ the $P_A$-usage is composed exactly from the service times of $A_1$-tasks of number $\mu_1$ and from the service times of $A_2$-tasks of number $\mu_2$ and, therefore,

$$a_{1,2} = \mu_1\eta_1 + \mu_2\eta_2. \tag{36}$$

We have proved

**Theorem 2.** *If for any configuration $Q \in \mathcal{Q}$ the priority schedule $R = R_{1,2}(Q)$ is periodic then the length of the period $p$ and the $P_A$-usage $a$ can be written in the forms*

$$p = \mu_1\tau_1 = \mu_2\tau_2 + (\varkappa_2 + \varepsilon_2)\eta_1, \tag{37}$$

$$a = \mu_1\eta_1 + \mu_2\eta_2, \tag{38}$$

*where integers $\mu_1 \geqq 0$, $\mu_2 \geqq 0$, $\varkappa_2 \geqq 0$ and real $0 \leqq \varepsilon_2 \leqq 1$ are the characteristics*

$$\Pi = (\mu_1; \mu_2; \varkappa_2; \varepsilon_2)$$

*of $R$ with the specialities*

| $Q$ | $\mu_1$ | $\mu_2$ | $\varkappa_2$ | $\varepsilon_2$ |
|---|---|---|---|---|
| $\vartheta_1 > 0,\ \tau_2 = 0$ | 0 | 1 | 0 | 0 |
| $\vartheta_1 = 0$ | 1 | 0 | 0 | 1 |
| $\vartheta_1\tau_2 > 0$ | $>0$ | $>0$ | $\geqq 0$ | $\in [0, 1]$ |

(39)

*Proof.* After the preliminary discussion there is nothing to prove. □

Let us inspect now the influence of the reduction step defined by (2) on the periodicity and the characteristics of a priority schedule $R_{1,2}(Q)$. Denote by

$$(R): \qquad R_n = R_{1,2}(Q_n), \quad n = 0, 1, 2, \ldots,$$

the sequence of priority schedules of the sequence of configurations $(Q)$.

Fig. 3 illustrates the influence of the reduction step $Q_n \to Q_{n+1}$ on the corresponding priority schedules. The transformation $R_n \to R_{n+1}$ defined implicitly is shown in three substeps $R_n \to R_n'$, $R_n' \to R_n''$, $R_n'' \to R_{n+1}$ corresponding to the substeps (2b)—(2d) as transformations $Q_n \to Q_n'$, $Q_n' \to Q_n''$, $Q_n'' \to Q_{n+1}$. This decomposition of the transformation $Q_n \to Q_{n+1}$ corresponds to the factorization (29) of the matrix $\underline{\underline{\Delta}}_{n,n+1}$ of the transformation. The series of configurations in Fig. 3 is $Q_n=(1;\ 15.5;\ 5;\ 7.5)$, $Q_n=(1;\ 3;\ 5;\ 7.5)$, $Q_n=(1;\ 3;\ 2;\ 7.5)$, $Q_{n+1}=$ $=(1;\ 3;\ 2;\ 3.5)$.

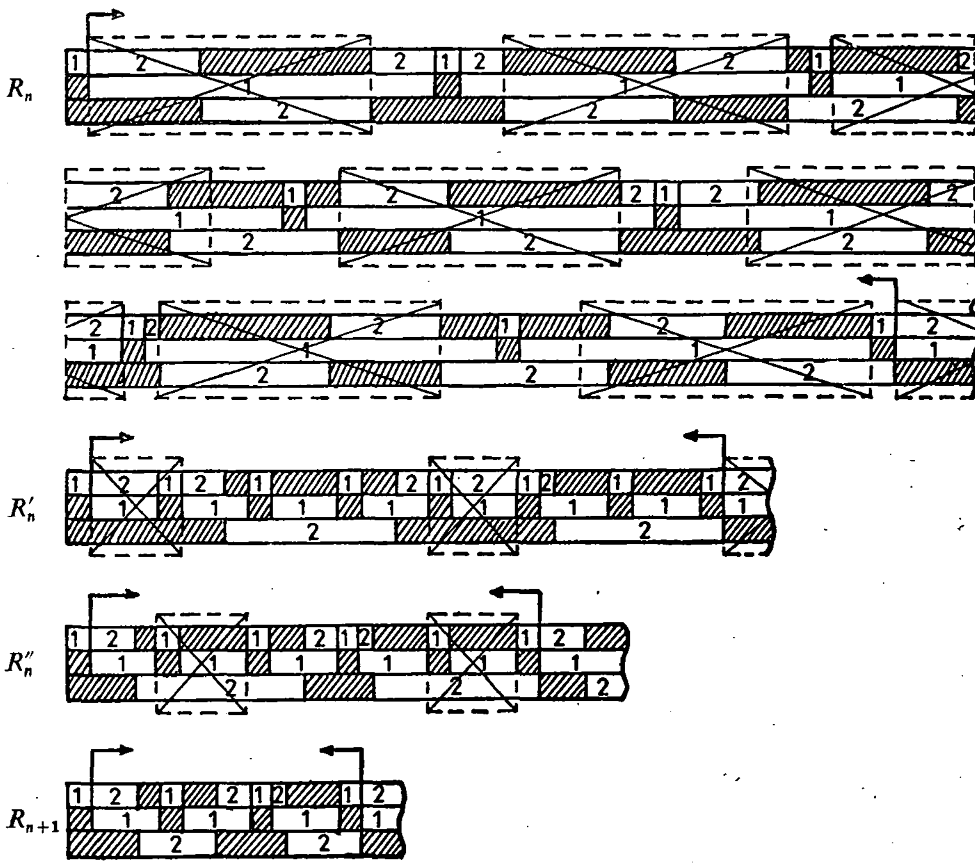



*Fig. 3*

The influence of the substeps of the reduction $Q_{n+1}=\Delta Q_n$ on the priority schedule $R_{1,2}$

The sequence of $R_n$, $R'_n$, $R''_n$, $R_{n+1}$ shows that these schedules are periodic at once and the transformation $Q_n \to Q_{n+1}$ does not influence the existence of periodicity of priority schedules. This means that the members of the sequence $(R)$ are simultaneously periodic or not periodic at all.

Let us introduce the following symbolics. Denote the characteristics of $R_n$ by

$$(\Pi): \qquad \Pi_n = (\mu_{1,n};\ \mu_{2,n};\ \varkappa_{2,n};\ \varepsilon_{2,n}), \quad n = 0, 1, 2, \ldots$$

and let the vectors $\underline{\mu}_n$ and $\underline{\pi}_n$ be defined as

$$(\underline{\pi}):\quad \underline{\mu}_n = \begin{pmatrix} \mu_{1,n} \\ \mu_{2,n} \end{pmatrix}, \quad n = 0, 1, \ldots$$

$$(\underline{\mu}):\quad \underline{\pi}_n = \begin{pmatrix} \mu_{1,n} \\ \mu_{2,n} \\ \varkappa_{2,n} \end{pmatrix}, \quad n = 0, 1, \ldots$$

and let the matrices $\underline{\underline{M}}_n$ and $\underline{\underline{M}}_{n,n+1}$ be defined as

$$(\underline{\underline{M}}):\quad \underline{\underline{M}}_n = \begin{pmatrix} B_{2n-2} & A_{2n-2} & B'_{2n-2} \\ B_{2n-1} & A_{2n-1} & B'_{2n-1} \\ 0 & 0 & 1 \end{pmatrix}, \quad n = 0, 1, \ldots$$

$$(\underline{\underline{M}}_+):\quad \underline{\underline{M}}_{n,n+1} = \begin{pmatrix} 1 & l_{1,n} & 0 \\ k_{2,n}+l_{2,n} & l_{1,n}(k_{2,n}+l_{2,n})+1 & k_{2,n} \\ 0 & 0 & 1 \end{pmatrix}, \quad n = 0, 1, \ldots$$

**Lemma 6.** *For the matrices* $(\underline{\underline{M}})$ *and* $(\underline{\underline{M}}_+)$ *the following relationships hold for* $n=0, 1, \ldots$

$$\underline{\underline{M}}_{n+1} = \underline{\underline{M}}_{n,n+1}\underline{\underline{M}}_n, \quad \text{with} \quad \underline{\underline{M}}_0 = \underline{\underline{I}} \tag{40}$$

$$\underline{\underline{M}}_n^{-1} = \begin{pmatrix} A_{2n-1} & -A_{2n-2} & C_{2n-1} \\ -B_{2n-1} & B_{2n-2} & -D_{2n-1} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{M}}_{n,n+1}^{-1} = \begin{pmatrix} l_{1,n}(k_{2,n}+l_{2,n})+1 & -l_{1,n} & l_{1,n}k_{2,n} \\ -(k_{2,n}+l_{2,n}) & 1 & -k_{2,n} \\ 0 & 0 & 1 \end{pmatrix} \tag{41}$$

$$\underline{\underline{M}}_{n,n+1} = \begin{pmatrix} 1 & 0 & 0 \\ l_{2,n} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ k_{2,n} & 1 & k_{2,n} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & l_{1,n} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{M}}_{n,n+1}^{-1} = \begin{pmatrix} 1 & -l_{1,n} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -k_{2,n} & 1 & -k_{2,n} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -l_{2,n} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{42}$$

*The determinant* $\det(\underline{\underline{X}})$ *for every matrix encountered above is*

$$\det(\underline{\underline{X}}) = 1. \tag{43}$$

*Proof.* (40) can be verified by executing the matrix production and using the definitions of $(A)$, $(B)$, $(B')$. The verification of (41) is easy by multiplying the matrices with their inverses and using (20)—(25). The factorizations (42) are obvious by executing the multiplications. (43) is trivial. □

Now we prove our main result.

**Theorem 3.** *For any configuration $Q\in\mathcal{Q}$ the whole sequence $(R)$ of priority schedules of the sequence of configurations $(Q)$ is periodic at once and the following relationships hold among the members of the sequence $(\Pi)$ of characteristics:*

$$\varepsilon_{2,n} = \varepsilon_2 \tag{44}$$

and

$$\begin{array}{llll} \underline{\mu}_{n+1} = \underline{\underline{D}}_{n,n+1}^{-T}\underline{\mu}_n, & \underline{\mu}_n = \underline{\underline{D}}_{n,n+1}^{T}\underline{\mu}_{n+1}, & \underline{\mu}_n = \underline{\underline{D}}_n^{-T}\underline{\mu}, & \underline{\mu} = \underline{\underline{D}}_n^{T}\underline{\mu}_n \\ \underline{\pi}_{n+1} = \underline{\underline{M}}_{n,n+1}^{-T}\underline{\pi}_n, & \underline{\pi}_n = \underline{\underline{M}}_{n,n+1}^{T}\underline{\pi}_{n+1}, & \underline{\pi}_n = \underline{\underline{M}}_n^{-T}\underline{\pi}, & \underline{\pi} = \underline{\underline{M}}_n^{T}\underline{\pi}_n \end{array} \tag{45}$$

*for* $n=0, 1, 2, \ldots,$ *where* $\underline{\underline{X}}^{-T}$ *denotes the transpose of the inverse of matrix* $\underline{\underline{X}}$.

*Proof.* The second and fourth columns of (45) follow from the first and third. The first line follows from the second because the $D$-matrices are the $2\times2$ submatrices of the $M$-matrices as their definitions show. The relationships of the third column follow from the ones of the first in consequence of (27) and (40). The first relationship of the first line of (45) remains to be proved with (44). To go on with the proof we need the following triads.

Define

$$\varphi(i) = \left[\frac{f(i)}{\tau_1}\right] \quad \text{and} \quad \varrho(i) = f(i)-\varphi(i)\tau_1, \quad i = 1, 2, \ldots \tag{46}$$

as *moduli* and *residua* of the cycle-finishing times $f(i)$ of $Q^{(2)}$.

$$\varrho(i) \equiv f(i) \pmod{\tau_1} \quad \text{and} \quad 0 \leqq \varrho(i) < \tau_1. \tag{47}$$

For the cycle-finishing times the decomposition (32) is true until the first recurrence of the $\beta_1$-situation. Substituting this into $\varrho(i)$ in (46) we get

$$\varrho(i) = \eta_1 + i\tau_2 + \chi(i)\eta_1 - \varphi(i)\tau_1. \tag{48}$$

The *triads*

$$H(i) = \big(\varphi(i), i, \chi(i)\big), \quad i = 1, 2, \ldots$$

for $Q$ are determined by the priority schedule $R=R_{1,2}(Q)$. We saw earlier that the periodicity of $R$ is true if for a finite $i$ there exists a triad $H(i)$ for which

$$0 \leqq \varrho(i) \leqq \eta_1,$$

because the $\beta_1$-situation recurs exactly in this case. The length $p$ of the period is determined by the first such $i$ and $H(i)$ because the first recurrence point $T_1^*$ of the $\beta_1$-situation is the $A_1$-task-finishing point next $f(i)$ which is by time $\eta_1-\varrho(i)$ later than $f(i)$, that is

$$T_1^* = f(i)+\eta_1-\varrho(i).$$

From this

$$p = T_1^* - t_1' = f(i)-\varrho(i) = \eta_1 + i\tau_2 + \chi(i)\eta_1 - \varrho(i).$$

On the other hand

$$p = \varphi(i)\tau_1 = i\tau_2 + \big(\chi(i)+\varepsilon_2\big)\eta_1$$

from which

$$\varrho(i) = (1-\varepsilon_2)\eta_1 \quad \text{and} \quad \varepsilon_2 = 1-\varrho(i)/\eta_1.$$

We have got that $R$ is periodic if and only if there exists a triad $H(i)$ for which

$$0 \leqq \varepsilon_2\eta_1 = \varphi(i)\tau_1 - i\tau_2 - \chi(i)\eta_1 \leqq \eta_1. \tag{49}$$

Since the member of triads determined by $R$ are monotonic with each other, there exists a unique minimum $i$ satisfying (49). Let

$$\mu_1 = \varphi(i), \quad \mu_2 = i, \quad \varkappa_2 = \chi(i), \quad \varepsilon_2 = 1 - \varrho(i)/\eta_1 \tag{49'}$$

with this $i$. Then the so defined $\Pi_n$ are the characteristics of $R_n$. $\mu_{2,n}$ is the minimum value of $i$ for which (49) holds for $R_n$, i.e.

$$0 \leqq \mu_{1,n}\tau_{1,n} - \mu_{2,n}\tau_{2,n} - \varkappa_{2,n}\eta_1 = \varepsilon_{2,n}\eta_1 \leqq \eta_1. \tag{50}$$

Let us see the first substep $Q_n \to Q_n'$. Substitute from (2b) $\tau_{1,n} = l_{1,n}\tau_{2,n} + \tau_{1,n+1}$ into (50) and we get

$$0 \leqq \mu_{1,n}\tau_{1,n+1} - (\mu_{2,n} - l_{1,n}\mu_{1,n})\tau_{2,n} - \varkappa_{2,n}\eta_1 = \varepsilon_{2,n}\eta_1 \leqq \eta_1. \tag{50'}$$

This means that

$$H_n' = (\mu_{1,n}, \mu_{2,n} - l_{1,n}\mu_{1,n}, \varkappa_{2,n})$$

is a triad for $R_n' = R_{1,2}(Q_n')$ for which (49) holds. Because the correspondence between parameters of $Q_n$ and $Q_n'$ is unique, $H_n'$ must also be the minimum triad for which (49) holds. This means that the characteristics of $R_n'$ are

$$\mu_{1,n}' = \mu_{1,n}, \quad \mu_{2,n}' = \mu_{2,n} - l_{1,n}\mu_{1,n}, \quad \varkappa_{2,n}' = \varkappa_{2,n}, \quad \varepsilon_{2,n}' = \varepsilon_{2,n}.$$

The matrix of this transformation is the transpose of the first factor of $\underline{\underline{M}}_{n,n+1}^{-1}$ in (42).

Substitute now the expression $\eta_{2,n} = k_{2,n}\vartheta_{1,n+1} + \eta_{2,n+1}$ from (2c) into (50') correspondingly to the transformation $Q_n' \to Q_n''$. We obtain unambiguously the inequality

$$0 \leqq (\mu_{1,n}' - k_{2,n}\mu_{2,n}')\tau_{1,n+1} - \mu_{2,n}'(\eta_{2,n+1} + \vartheta_{2,n}) - (\varkappa_{2,n}' - k_{2,n}\mu_{2,n}')\eta_1 = \varepsilon_{2,n}\eta_1 \leqq \eta_1. \tag{50''}$$

This means that

$$H_n'' = (\mu_{1,n}' - k_{2,n}\mu_{2,n}', \mu_{2,n}', \varkappa_{2,n}' - k_{2,n}\mu_{2,n}')$$

is the unique minimum triad for $Q_n''$ for which (49) holds and, therefore

$$\mu_{1,n}'' = \mu_{1,n}' - k_{2,n}\mu_{2,n}', \quad \mu_{2,n}'' = \mu_{2,n}', \quad \varkappa_{2,n}'' = \varkappa_{2,n}' - k_{2,n}\mu_{2,n}', \quad \varepsilon_{2,n}'' = \varepsilon_{2,n}'.$$

The matrix of this transformation is the transpose of the second factor of $\underline{\underline{M}}_{n,n+1}^{-1}$ in (42).

At last we substitute the expression $\vartheta_{2,n} = l_{2,n}\tau_{1,n+1} + \vartheta_{2,n+1}$ from (2d) into (50'') correspondingly to the transformation $Q_n'' \to Q_{n+1}$. We obtain the inequality

$$0 \leqq (\mu_{1,n}'' - l_{2,n}\mu_{2,n}'')\tau_{1,n+1} - \mu_{2,n}''\tau_{2,n+1} - \varkappa_{2,n}''\eta_1 = \varepsilon_{2,n}''\eta_1 \leqq \eta_1.$$

In consequence of the uniqueness of the transformation $Q_n'' \to Q_{n+1}$ and the minimum triads for their $R_{1,2}$-schedules we get

$$\mu_{1,n+1} = \mu_{1,n}'' - l_{2,n}\mu_{2,n}'', \quad \mu_{2,n+1} = \mu_{2,n}'', \quad \varkappa_{2,n+1} = \varkappa_{2,n}'', \quad \varepsilon_{2,n+1} = \varepsilon_{2,n}''$$

as the characteristics of $R_{n+1}$. The matrix of this transformation is the transpose of the third factor of $\underline{\underline{M}}_{n,n+1}^{-1}$ in (42). This proves the theorem. □

Fig. 3 illustrates the course of the proof.

Theorem 3 makes it possible to determine the characteristics $\Pi$ of $R=R_{1,2}(Q)$ from the characteristics $\Pi^*$ of $R^*=R_{1,2}(Q^*)$ if $Q$ is reducible, $R^*$ is periodic and $\Pi^*$ is known. The question of reducibility was discussed in the previous section. The characteristics of reduced configurations will be inspected in the next two sections.

## 4. Priority schedules of specific configurations

We saw in the proof of Theorem 3 that the periodicity of a priority schedule $R=R_{1,2}(Q)$ depends on the fact whether there exists a triad $H(i)$ satisfying (49). This is not equivalent to the existence of an integer solution of the inequality

$$0 \leqq \mu_1\tau_1-\mu_2\tau_2-\varkappa_2\eta_1 \leqq \eta_1 \tag{51}$$

because not every triple $(\mu_1, \mu_2, \varkappa_2)$ satisfying this inequality is a triad defined by (32), (46)—(49) on a schedule $R_{1,2}(Q)$. Unfortunately, we do not know analytic conditions for the triads instead of the fact that its elements represent the number of $C_1$-cycles, $C_2$-cycles and preemptions, respectively, until the $C_2$-cycle finishing points of $R_{1,2}(Q)$. The triads and (51) cannot be used, therefore, to decide the periodicity and determine the characteristics of a priority schedule $R_{1,2}(Q)$. This circumstance raises the significance of results on characteristics for some specific configurations $Q\in\mathcal{Q}$ including reduced ones.

The characteristics of $R_{1,2}(Q)$ were made clear for configurations for which $\vartheta_1\tau_2=0$ in Theorem 2. We suppose that

$$\vartheta_1\tau_2 > 0. \tag{52}$$

We can make clear the special cases in which (9), the condition $\eta_1\vartheta_2=0$ for $Q$ is true. Let first $\eta_1=0$. Since $Q^{(1)}$ do not delay the service of $Q^{(2)}$ in this case, we can determine the condition of periodicity of $R_{1,2}(Q)$ as $\vartheta_1$ and $\tau_2$ are rationally dependent. This is illustrated in Fig. 4a.

Independently of the value of $\eta_1$, we can easily determine the condition of $R_{1,2}(Q)$ to be periodic for $Q\in\mathcal{Q}$ with $\vartheta_2=0$ (but $\vartheta_1\tau_2>0$!). This condition is that

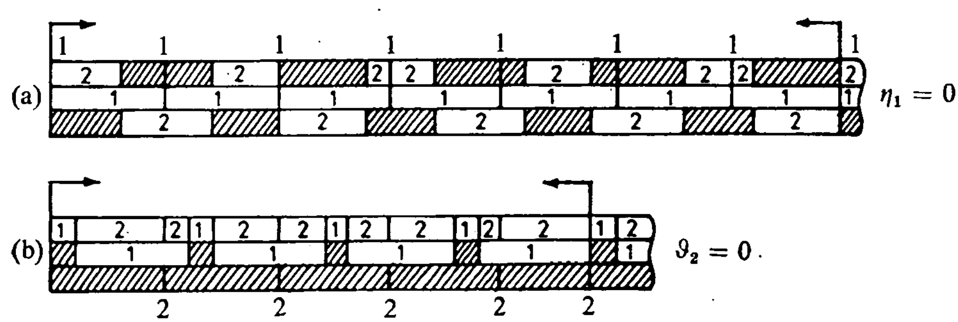



*Fig. 4*

$R_{1,2}(Q)$ schedules for specific configurations with $\vartheta_1\tau_2>0$, $\eta_1\vartheta_2=0$

$\vartheta_1$ and $\eta_2$ are rationally dependent, which is the same condition as in case $\eta_1=0$. The values of the characteristics of the periodic schedule $R_{1,2}(Q)$ are, obviously, determined by the relation of $\vartheta_1$ and $\tau_2$ according to

**Theorem 4.** *For the configurations $Q\in\mathcal{Q}$ with*

$$\vartheta_1\tau_2>0,\quad \eta_1\vartheta_2=0 \tag{53}$$

*the priority schedule $R=R_{1,2}(Q)$ is periodic iff $\vartheta_1$ and $\tau_2$ are rationally dependent. If*

$$\frac{\vartheta_1}{\tau_2}=\frac{A}{B}, \tag{54}$$

*with relatively prime integers $A, B>0$, then the characteristics of $R$ are*

$$\Pi=\left(B;\, A;\, f_<\left(\frac{\eta_2}{\tau_2}B\right);\, 1\right), \tag{55}$$

*where $f_<(x)$ is the greatest integer less than $x$.*

*Proof.* Fig. 4 shows that $\mu_1=B, \mu_2=A$ if (54) holds because $(B, A)$ is the least integer solution of the equation $x\vartheta_1-y\tau_2=0$. Since $\varrho(A)=0$, therefore, $\varepsilon_2=1$ from the relationship (49′) if $\eta_1>0$ and $\varepsilon_2=1$ can be considered as a convention if $\eta_1=0$. If $\vartheta_2=0$ then every $A_1$-task but the first in the period is a preempting one and, therefore, $\varkappa_2=B-1=\left[\frac{\eta_2}{\tau_2}B\right]-1$. In case $\eta_1=0$ the $A_{1,j}$ task is preempting if $i\tau_2<j\vartheta_1<i\tau_2+\eta_2$ for some integer $i\geqq 0$ (see Fig. 4a). This means that $i<j\vartheta_1/\tau_2<i+\eta_2/\tau_2$ and using (54) we get $i<jA/B<i+\eta_2/\tau_2$, i.e.

$$0<\left\{j\frac{A}{B}\right\}<\frac{\eta_2}{\tau_2}, \tag{$*$}$$

where $\{x\}$ denotes the fractional part of $x$. It is well known [4] that the numbers $\{jA/B\}$, $j=0, 1, \dots, B-1$, go through the points $k/B$, $k=0, 1, \dots, B-1$, of the interval $[0, 1)$ in some order. This means that for $j=1, 2, \dots, B$, the inequality takes place as many times as many of the points $k/B$ are in the interval $(0, \eta_2/\tau_2)$. This number is $[(\eta_2/\tau_2)/(1/B)]$ if $(\eta_2/\tau_2)/(1/B)$ is not an integer and is $(\eta_2/\tau_2)/(1/B)-1$ if this is an integer. This number is exactly $f_<((\eta_2/\tau_2)B)$. □

Lemma 3 establishes that every configuration $Q$ becomes reduced or defective with (53) after a finite number $\nu'\geqq 0$ of application of the operator $\Delta$ to it. Theorem 4 means that after finite $\nu'\geqq 0$ times application of $\Delta$ we can reduce $Q$ or decide whether its schedule $R_{1,2}(Q)$ is periodic. We show that $Q$ with (53) is reducible when $R_{1,2}(Q)$ is periodic, i.e. $\vartheta_1$ and $\tau_2$ are rationally dependent.

**Lemma 7.** *The configurations $Q\in\mathcal{Q}$ with* (53) *are reducible iff* (54) *is true except eventually the case $\eta_1=0$ in which $Q$ can be reducible with rationally independent $\vartheta_1$ and $\tau_2$ as well.*

*Proof.* If $\vartheta_2=0$ then the reduction procedure is equivalent to the regular continued fraction expansion of the number $\xi=\vartheta_1/\eta_2$ and is finite exactly when

$\xi$ is rational and so (54) holds (see also the proof of the Lemma 4). Let now $\vartheta_2>0$ and $\eta_1=0$. If $Q$ is not reducible then neither $\vartheta_{1,n}$ nor $\eta_{2,n}+\vartheta_{2,n}$ of $Q_n=\Delta^n Q$, $n\geqq 0$, is zero by Lemma 4. If $\eta_{2,n}\vartheta_{2,n}=0$ for some finite $n\geqq 0$ then the reducibility is equivalent to the validity of (54) by the same lemma.

Let, therefore, $\vartheta_{1,n}\eta_{2,n}\vartheta_{2,n}>0$, $n=0, 1, \ldots$. Suppose $Q$ is not reducible. This means that the series $(\lambda)$ has infinite length and has no zero element after $\lambda_0=l_{1,0}$. This means that $l_{1,n}>0, n\geqq 1$. From (2b) we conclude then that $0<\vartheta_{1,n+1}<$ $<\tau_{2,n}<\vartheta_{1,n}$, $n=1, 2, \ldots$, which means that

$$\xi_{2i}=\frac{\vartheta_{1,i}}{\tau_{2,i}}>1 \quad \text{if} \quad i>0, \quad \xi_{2i+1}=\frac{\tau_{2,i}}{\vartheta_{1,i+1}}>1 \quad \text{if} \quad i\geqq 0,$$

and (2) is equivalent to the definition of series

$$\xi_n=\lambda_n+\frac{1}{\xi_{n+1}}, \quad n=0, 1, \ldots,$$

where $0<1/\xi_{n+1}<1$ and, consequently, $\lambda_n=[\xi_n]$. This is, however, exactly the definition of the Euclidean algorithm of the regular continued fraction expansion of the number $\xi_0=\vartheta_{1,0}/\tau_{2,0}=\vartheta_1/\tau_2$. This algorithm is infinite exactly when $\xi_0$ is an irrational number, i.e. (54) does not hold [3]. If (54) is true, $Q$ must be reducible. If (54) does not hold but $\eta_1=0$ then $Q$ can be reducible as for instance $Q=(0;\ 1;\ \pi/2;\ \pi/2)$ shows for which $\xi_0$ is irrational but $\nu=1$ and $Q^*=$ $=(0;\ 1;\ \pi/2-1;\ \pi/2-1)$. □

From Lemma 7 we can conclude that the question of periodicity of $R_{1,2}(Q)$ remained unanswered in cases in which $Q$ is reducible and for its reduction $Q^*$

$$\vartheta_1^*\tau_2^*>0, \quad \eta_1\vartheta_2^*>0. \tag{56}$$

In all other cases reducibility and periodicity are equivalent except the case $\eta_1=0$, $\vartheta_1^*$ and $\tau_2^*$ are rationally independent, in which case the periodicity is not true.

We now show that in case (56) the schedule $R_{1,2}(Q)$ is periodic if $\tau_1^*\geqq\tau_2^*$.

**Theorem 5.** *If the configuration $Q\in\mathcal{Q}$ is reducible and for its reduction $Q^*=Q_\nu$ the relations*

$$\tau_1^*\geqq\tau_2^*>\vartheta_1^*>0 \tag{57}$$

*hold then the priority schedule $R_{1,2}(Q)$ of $Q$ is periodic with characteristics*

$$\Pi=\left(\mu_1;\ \mu_2;\ \varkappa_2;\ \frac{\tau_1^*-\tau_2^*}{\eta_1}\right) \tag{58}$$

*with*

$$\begin{aligned}\mu_1&=B_{2\nu-2}+B_{2\nu-1}\\ \mu_2&=A_{2\nu-2}+A_{2\nu-1}\\ \varkappa_2&=B'_{2\nu-2}+B'_{2\nu-1}\end{aligned} \tag{59}$$

*where $\nu$ is the degree of compositeness of $Q$. $\mu_1$ and $\mu_2$ are relatively prime integers.*

*Proof.* First of all $\eta_1 > 0$ follows from (57) because the reducedness of $Q^*$ implies $\vartheta_2^* < \tau_1^*$ if $\tau_1^* > 0$ by (5c). From $\vartheta_1^* > 0$ and (5b) it follows that $0 \leqq \eta_2^* \leqq \vartheta_1^*$ and, therefore, the characteristics of $R^* = R_{1,2}(Q^*)$ cannot be else than

$$\Pi^* = \left(1; 1; 0; \frac{\tau_1^* - \tau_2^*}{\eta_1}\right) \tag{58'}$$

which is the special case of (58) with $\nu = 0$ in (59). This fact can be verified most simply on the Gantt-chart of $R^*$ as in Fig. 5. (59) follows then from Theorem 3

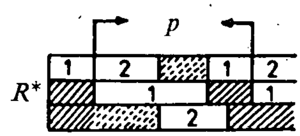



*Fig. 5*

The $R_{1,2}(Q^*)$ schedule for a reduced configuration with $\tau_1^* \geqq \tau_2^* > \vartheta_1^* > 0$

applied for $n = \nu$ and entities $x^* = x_\nu$. By the last relationship of (45), $\underline{\pi} = \underline{\underline{M}}_\nu^T \underline{\pi}^*$ and in detailed form

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \varkappa_2 \end{pmatrix} = \begin{pmatrix} B_{2\nu-2} & B_{2\nu-1} & 0 \\ A_{2\nu-2} & A_{2\nu-1} & 0 \\ B'_{2\nu-2} & B'_{2\nu-1} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

which is (59). $\varepsilon_2 = \varepsilon_2^*$ follows from (44).

Applying $\underline{\mu}^* = \underline{\underline{D}}_\nu^{-T} \underline{\mu}$ obtained from (45) for $n = \nu$, we get from (28) the relationships $1 = A_{2\nu-1}\mu_1 - B_{2\nu-1}\mu_2$ and $1 = -A_{2\nu-2}\mu_1 + B_{2\nu-2}\eta_2$ and from (21) that $\mu_1$ and $\mu_2$ cannot have common divisors other than $\pm 1$. □

After this theorem the only questionable case remained is the set of configurations reducible to $Q^*$ with

$$0 < \eta_1 < \tau_1^* < \tau_2^*. \tag{60}$$

The domain (60) of $\mathcal{Q}$ is the part of the domain ($\delta$) in Fig. 2d and is illustrated in Fig. 6. We will further investigate this case in the next section.

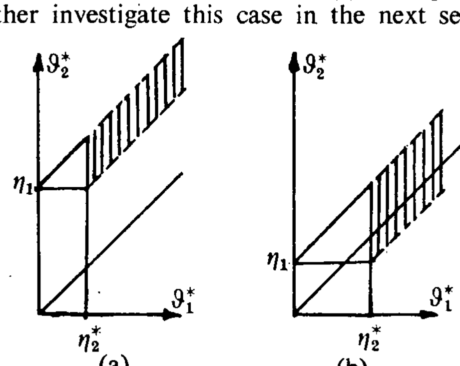



*Fig. 6*

The domain of reduced configurations with $0 < \eta_1 < \tau_1^* < \tau_2^*$

Supposing that $R^*$ is periodic, some relations among its characteristics can be stated. These follow from the following more general Lemma 8. We need some simple definitions. Let $s(X)$ and $f(X)$ denote the *start* and *finishing point* of the service of a task or cycle $X$, respectively. We say that task $A$ *starts during* task $B$ if $s(B) \leqq s(A) \leqq f(B)$ and task $A$ *runs during* task $B$ if $s(B) \leqq s(A)$ and $f(A) \leqq f(B)$. Let $u$ denote the number of task type $A_1$ in a period of $R_{1,2}(Q)$ which *do not preempt* task type $A_2$.

**Lemma 8.** *For the characteristics $\Pi$ and $u$ of a periodic priority schedule $R = R_{1,2}(Q)$ the following assertions are true:*

$$\mu_1 = u + \varkappa_2; \tag{61}$$

$$u = \mu_2, \quad \mu_1 = \mu_2 + \varkappa_2 \tag{62}$$

*iff exactly one $A_1$-task starts during every $B_2$-task;*

$$\begin{array}{lllll} \text{(a)} & u \leqq \mu_2, & \mu_1 \leqq \mu_2 + \varkappa_2 & \textit{if} & \vartheta_2 < \tau_1, \\ \text{(b)} & u \geqq \mu_2, & \mu_1 \geqq \mu_2 + \varkappa_2 & \textit{if} & \vartheta_1 \leqq \vartheta_2, \\ \text{(c)} & u = \mu_2, & \mu_1 = \mu_2 + \varkappa_2 & \textit{if} & \vartheta_1 \leqq \vartheta_2 < \tau_1; \end{array} \tag{63}$$

$$\begin{array}{llll} \text{(a)} & \mu_1 \geqq \mu_2 + 1 & \textit{if} & \tau_1 < \tau_2, \\ \text{(b)} & \mu_2 \geqq \varkappa_2 + 1 & \textit{if} & \eta_2 \leqq \vartheta_1, \quad \vartheta_1 > 0, \\ \text{(c)} & \mu_1 > \mu_2 > \varkappa_2 \geqq 0 & \textit{if} & \eta_2 \leqq \vartheta_1 \leqq \tau_1 < \tau_2, \quad \vartheta_1 > 0; \end{array} \tag{64}$$

$$\varkappa_2 \geqq 1 \quad \textit{if} \quad \vartheta_2 < \tau_1 < \tau_2, \quad \vartheta_1 > 0; \tag{65}$$

$$\mu_1 \geqq 3, \quad \mu_2 \geqq 2, \quad \varkappa_2 \geqq 1 \quad \textit{if} \quad \eta_2 \leqq \vartheta_1, \quad \vartheta_1 > 0, \quad \vartheta_2 < \tau_1 < \tau_2. \tag{66}$$

*Proof.* (61) follows from the definition of $u$ and $\varkappa_2$. $u = \mu_2$ in (62) is clearly true if exactly one $A_1$-task starts during every $B_2$-task because these $A_1$-tasks are those which do not cause preemption. The number of $B_2$-tasks in a period is $\mu_2$. Suppose $u = \mu_2$ and there exists a $B_2$-task during which more than one $A_1$-tasks start. This is possible only if $\tau_1 \leqq \vartheta_2$, and so $\vartheta_1 \leqq \vartheta_2$. But at least one $A_1$-task must start during every $B_2$-task if $\vartheta_1 \leqq \vartheta_2$ and, therefore, we get $u \geqq \mu_2 + 1$, which proves (63b) but contradicts $u = \mu_2$. If we suppose that no $A_1$-task starts during some $B_2$-task in the period of $R$, it follows that $\vartheta_2 < \vartheta_1$ must hold. But if $\vartheta_2 < \tau_1$ then no $B_2$-task during which more than one $A_1$-tasks start exists and, therefore, $u \leqq \mu_2 - 1$, proving (63a) but contradicting $u = \mu_2$. This proves (62), and (63a) and (63b) involve (63c).

To prove (64a) we use Theorem 2. From (37) $(\mu_1 - \mu_2)\tau_1 = \mu_2(\tau_2 - \tau_1) + (\varkappa_2 + \varepsilon_2)\eta_1$ and $\mu_1 > \mu_2$ follow if $\tau_2 > \tau_1$ and $\mu_2 > 0$. But $\mu_2 > 0$ follows from $\vartheta_1 > 0$ by (39). If $\vartheta_1 = 0$ then $\mu_1 = 1 > \mu_2 = 0$ by (39). If $\eta_2 \leqq \vartheta_1$ then no $A_2$-task can exist which is preempted more than once and, therefore, $\varkappa_2 \leqq \mu_2$. If $\vartheta_1 > 0$ then the first $A_{2,1}$ task is serviced without preemption as soon as $\eta_2 \leqq \vartheta_1$. Therefore, $\varkappa_2 \leqq \mu_2 - 1$, as (64b) asserts. (64a) and (64b) imply (64c).

To prove (65) we consider the last $B_2$-task in the first period of $R$ which precedes the recurrence point $T_1^*$ of the $\beta_1$-situation. This task finishes in the interval $[T_1^* - \eta_1, T_1^*]$ as Fig. 7 shows. The period ends with the service of an $A_1$-task. The last $B_2$-task cannot start before the preceding $A_1$-task because $\vartheta_2 \geqq \tau_1$ would follow

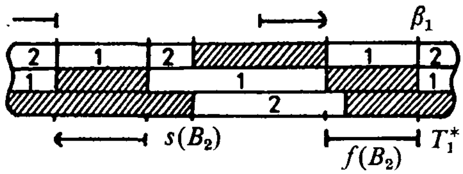



*Fig. 7*

Illicit intervals for the last $B_2$-task starting point $s(B_2)$ if $\vartheta_2<\tau_1<\tau_2$, $\vartheta_1>0$

in this case. This $B_2$-task cannot start, however, $\eta_2$ later than the preceding $A_1$-task finishing because $\vartheta_2\leqq\tau_1-\eta_2$ and $\tau_2\leqq\tau_1$ would follow. This means that $\vartheta_2<\tau_1<\tau_2$ implies that the last $B_2$-task starts after the preceding $A_1$-task but the previous $A_2$-task cannot be serviced without preemption and so $\varkappa_2\geqq 1$. (66) follows from (64c) and (65). □

Before we turn to the case (60), we prove two theorems which give the characteristics of $R_{1,2}(Q)$ for configurations not necessarily reduced but representing (58′) as their special case.

**Theorem 6.** *If for the configuration* $Q\in\mathcal{Q}$

$$\vartheta_1>0 \quad \text{and} \quad \vartheta_2<\eta_1 \tag{67}$$

*hold then $R_{1,2}(Q)$ is periodic. Its characteristics are*

$$\Pi=\left(A;B;A-1;1-\frac{\Delta\vartheta_1}{\eta_1}\right) \tag{68}$$

*where $\omega=(B,A)$ is the least solution of the coincidence problem*

$$0\leqq B\xi-A\leqq\alpha, \quad \omega\geqq(1,0) \tag{69}$$

*and*

$$\Delta=B\xi-A \tag{70}$$

*is its error, where*

$$\xi=\frac{\tau_2}{\vartheta_1}, \quad \alpha=\frac{\vartheta_2}{\vartheta_1}. \tag{71}$$

*The cycle numbers $\mu_1$ and $\mu_2$ are relatively prime integers.*

*Proof.* An $A_1$-task causing no preemption starts during a $B_2$-task. Since $\eta_1>\vartheta_2$, this $A_1$-task must finish later than the $B_2$-task and cause a recurrence of the $\beta_1$-situation. Only one such $A_1$-task can exist in every period. Therefore, $\varkappa_2=\mu_1-1$ if $R_{1,2}(Q)$ is periodic. The condition of the periodicity is the recurrence of the $\beta_1$-situation and the existence of $\mu_1$ and $\mu_2>0$ fulfilling the inequality

$$0\leqq\eta_1+\mu_2\tau_2+(\mu_1-1)\eta_1-\mu_1\tau_1\leqq\vartheta_2.$$

The cycle numbers represent the least solution of this inequality which is equivalent to the inequality $0\leqq\mu_2\tau_2-\mu_1\vartheta_1\leqq\vartheta_2$ and this to (69) with $\mu_2=B$, $\mu_1=A$ and (71). The coincidence problem (69) always has a unique least solution $(B,A)$ because $\alpha>0$ and this solution represents a pair of relatively prime integers [4]. □

In the special case $0<\eta_2^*\leqq\vartheta_1^*<\tau_2^*$ of (67) $\xi>\alpha$ but $0\leqq\xi-1\leqq\alpha$ and, therefore, the solution of (69) is $\omega=(1, 1)$ with $\Delta=\xi-1=\tau_2^*/\vartheta_1^*-1$ and $\Pi=\left(1;\ 1;\ 0;\ \frac{\tau_1^*-\tau_2^*}{\eta_1}\right)$ from (68), correspondingly to (58′).

**Theorem 7.** *If for the configuration* $Q\in\mathcal{Q}$

$$\eta_1\vartheta_1\vartheta_2>0,\quad \eta_2=0 \tag{72}$$

*holds then* $R_{1,2}(Q)$ *is periodic. Its characteristics are*

$$\Pi=\left(B;\, A;\, 0;\, \frac{\Delta\vartheta_2}{\eta_1}\right) \tag{73}$$

*where* $\omega=(B, A)$ *is the least solution of the coincidence problem* (69) *with error* (70) *where now*

$$\xi=\frac{\tau_1}{\vartheta_2},\quad \alpha=\frac{\eta_1}{\vartheta_2}. \tag{74}$$

*The cycle numbers* $\mu_1$ *and* $\mu_2$ *are relatively prime integers.*

*Proof.* Because of $\eta_2=0$, preemption cannot exist in $R_{1,2}(Q)$ and $R_{1,2}(Q)$ is periodic if and only if $B_2$-tasks finishing during $A_1$-tasks exist. This is the condition of the recurrence of the $\beta_1$-situation. Such a $B_2$-task exists iff integers $B>0$, $A>0$ exist such that

$$B\tau_1\leqq\eta_1+A\vartheta_2\leqq B\tau_1+\min(\eta_1, \vartheta_2)$$

holds. The least $\omega=(B, A)$ supplies $\mu_1$ and $\mu_2$, respectively. This inequality is equivalent to

$$\eta_1-\min(\eta_1, \vartheta_2)\leqq B\tau_1-A\vartheta_2\leqq\eta_1.$$

The left side is positive if $\eta_1>\vartheta_2$. In this case the least $\omega=(B, A)$ satisfying the inequality is $\omega=(1, f_{\geqq}(\vartheta_1/\vartheta_2))$ where $f_{\geqq}(x)$ is the least integer not less than $x$. Namely, from $x\leqq f_{\geqq}(x)<x+1$ the inequality $\eta_1-\vartheta_2<\tau_2-f_{\geqq}(\vartheta_1/\vartheta_2)\vartheta_2\leqq\leqq\tau_1-\vartheta_1=\eta_1$ follows. This $\omega$ is the least solution of (69) with (74) as well. (69) always has a solution because of $\alpha>0$, and the least solution is a relatively prime integer pair [4]. The values of $\mu_1, \mu_2$ and $\varkappa_2$ in (73) are proved. Obviously, $\varepsilon_2\eta_1==\Delta\vartheta_2$ from which the value of $\varepsilon_2$ in (73) follows. □

If (57) holds, i.e. $0<\vartheta_1^*<\vartheta_2^*\leqq\tau_1^*$ is true then the least solution of (69) with (74) is $\omega=(1, 1)$ and $\Delta\vartheta_2^*=\tau_1^*-\vartheta_2^*=\tau_1^*-\tau_2^*$. (73) gives (58′) as a special case.

## 5. The case $0<\tau_1^*<\tau_2^*$

We did not find conditions for a reduced configuration $Q^*$ with (60) to have a periodic schedule $R^*=R_{1,2}(Q^*)$. This case requires further investigation. By (60) and condition (5) we can write

$$0<\eta_1^*<\tau_1^*<\tau_2^*,\quad \eta_2^*\leqq\vartheta_1^*,\quad \vartheta_2^*<\tau_1^*. \tag{75}$$

This is equivalent to the two series of inequalities

$$0 < \eta_2^* \leqq \vartheta_1^* < \tau_1^* < \tau_2^* < \eta_2^* + \tau_1^*$$
$$0 < \eta_1^* < \vartheta_2^* < \tau_1^* < \tau_2^* \leqq \vartheta_1^* + \vartheta_2^* < \vartheta_1^* + \tau_1^*. \tag{76}$$

These relations do not determine the relations between $\eta_i^*$ and $\vartheta_i^*$, $\eta_1^*$ and $\eta_2^*$, or $\vartheta_1^*$ and $\vartheta_2^*$ if $\eta_2^* > \eta_1^*$ (Fig. 6b). These latter relations are, however, not independent of each other. E.g. the following series of implications is right:

$$\vartheta_1^* \leqq \eta_1^* \Rightarrow \eta_2^* \leqq \eta_1^* \Rightarrow \vartheta_1^* < \vartheta_2^* \Rightarrow \vartheta_1^* \leqq \vartheta_2^*. \tag{77}$$

From Lemma 8 we can obtain relations among the characteristics of $R^*$ if it is periodic. From (63a) we get

$$\mu_1^* \leqq \mu_2^* + \varkappa_2^* \tag{78}$$

but from (63c) we get $\mu_1^* = \mu_2^* + \varkappa_2^*$ if any member of the series of implications (77) is true. From (64c) and (65)

$$\mu_1^* \geqq \mu_2^* + 1 \geqq \varkappa_2^* + 2 \geqq 3. \tag{79}$$

Before we further investigate some special cases of (75) we introduce an algo rithm to generate some entities and the characteristics $\Pi^*$ of $R^*$ if $R^*$ is periodic.

In the schedule $R^*$ the sequence $C_{21}, C_{22}, \ldots$ of $C_2$-cycles can be grouped into subsequences in which all cycles are either preempted or not preempted. Denote by $M_i$, $i=1, 2, \ldots,$ the sequence of the subsequences of the preempted and $N_i$, $i=1, 2, \ldots,$ the sequence of the subsequences of the non-preempted $C_2$-cycles. The first subsequence will be the $N_1$ with at least one $C_2$-cycle since $A_{2,1}$ is a non-preempted task because of $\eta_2^* \leqq \vartheta_1^*$. We call an *M-section* or an *N-section* of $R^*$ the section from the last cycle-finishing point of the previous subsequence until the last cycle-finishing point of the current subsequence $M_i$ or $N_i$, respectively. This definition will be modified slightly below by dividing some $M$-sections defined now into more $M$-sections and inserting empty $N$-sections in between them.

Define

$$f(0) = \eta_1^*, \quad f(i) = \eta_1^* + i\tau_2^* + \chi(i)\eta_1^* \tag{80}$$

as $C_2$-cycle finishing points,

$$\varphi(0) = 0, \quad \varrho(0) = \eta_1^*, \quad \varphi(i) = \left[\frac{f(i)}{\tau_1^*}\right], \quad \varrho(i) = f(i) - \varphi(i)\tau_1^*, \tag{81}$$

$i=1, 2, \ldots,$ as moduli and residua of the cycle-finishing points and

$$H(i) = (\varphi(i), i, \chi(i)), \quad i = 0, 1, \ldots \tag{82}$$

as triads according to (32) and proof of Theorem 3. (80)—(82) are only valid until the first recurrence point $T_1^*$ of the $\beta_1$-situation which occurs exactly when the residuum $\varrho(i)$ is not greater than $\eta_1^*$, i.e.

$$0 \leqq \varrho(i) \leqq \eta_1^*. \tag{83}$$

After $\varrho(0) = \eta_1^*$ the next such residuum and the corresponding triad determine the characteristics of $R^*$ which is periodic if such a residuum exists. Otherwise

$R^*$ is not periodic. The value of the residuum $\varrho(i)$ determines whether the next $A_2$-task $A_{2,i+1}$ is preempted or not. If

$$\eta_1^* \leqq \varrho(i) \leqq \tau_1^* - \eta_2^* \tag{83'}$$

then $A_{2,i+1}$ will be serviced without preemption and if

$$\tau_1^* - \eta_2^* < \varrho(i) < \tau_1^* \tag{83''}$$

then $A_{2,i+1}$ will be preempted.

Without preemption $f(i+1) = f(i) + \tau_2^*$ and

$$\varrho(i+1) = \varrho(i) + \tau_2^* - \tau_1^* > \varrho(i) \tag{84}$$

because from (83′) we obtain $\eta_1^* < \tau_2^* - \vartheta_1^* \leqq \varrho(i+1) \leqq \vartheta_2^* < \tau_1^*$.

With preemption $f(i+1) = f(i) + \tau_2^* + \eta_1^*$. In this case we get

$$\varrho(i+1) = \begin{cases} \varrho(i) + \tau_2^* - \vartheta_1^* > \varrho(i) & \text{if} \quad \vartheta_2^* < \vartheta_1^* \quad \text{and} \quad \tau_1^* - \eta_2^* < \varrho(i) < \tau_1^* + \vartheta_1^* - \tau_2^* \\ \varrho(i) + \tau_2^* - \vartheta_1^* - \tau_1^* < \varrho(i) & \text{if} \quad \tau_1^* - \min(\eta_2^*, \tau_2^* - \vartheta_1^*) \lessdot \varrho(i) < \tau_1^* \end{cases} \tag{85}$$

where the symbol $\lessdot$ denotes a relation sign by

$$\lessdot = \begin{cases} < & \text{if} \quad \vartheta_1^* \leqq \vartheta_2^* \\ \leqq & \text{if} \quad \vartheta_2^* < \vartheta_1^*. \end{cases} \tag{86}$$

(85) holds because $\vartheta_2^* + \eta_1^* < \varrho(i) + \tau_2^* - \vartheta_1^* < \tau_1^*$ if $\tau_1^* - \eta_2^* < \tau_1^* + \vartheta_1^* - \tau_2^*$, i.e. $\vartheta_2^* < \vartheta_1^*$, and $\tau_1^* - \eta_2^* < \varrho(i) < \tau_1^* + \vartheta_1^* - \tau_2^*$ and $0 \leqq \tau_2^* - \vartheta_1^* - \min(\eta_2^*, \tau_2^* - \vartheta_1^*) \lessdot \varrho(i) + \tau_2^* - \vartheta_1^* - \tau_1^* < \tau_2^* - \vartheta_1^* < \tau_1^*$ if $\tau_1^* - \min(\eta_2^*, \tau_2^* - \vartheta_1^*) \lessdot \varrho(i) < \tau_1^*$.

Since $\varrho(0) = \eta_1^* \leqq \tau_1^* - \eta_2^*$ by (75), $R^*$ starts with a non-preempted $A_2$-task and $\varrho(i)$ is monoton increasing until (83″) results and preempted $A_2$-task follows. $\varrho(i)$ can increase further until a decrease because of $\tau_1^* - \min(\eta_2^*, \tau_2^* - \vartheta_1^*) \lessdot \varrho(i)$ follows. If the $\varrho(i+1)$ obtained by (85) satisfies (83′), a non-preempted $C_2$-cycle follows, otherwise the following $C_2$-cycle is preempted as well. In both cases we regard the situation as the end of an $M$-section and beginning of an $N$-section. In the second case in which the following $C_2$-cycle is preempted as well, the $N$-section is empty and begins a new $M$-section simultaneously.

The schedule $R^*$ consists of a sequence $(N_1, M_1), (N_2, M_2), \ldots$ of $(N, M)$-*section pairs* in which $N_1$ cannot but $N_i, i > 1$, can be empty, too. Let the numbers of $C_2$-cycles in the sections $N_i$ and $M_i$ be $n_i'$ and $m_i'$, respectively. These are called the *lengths* of the sections.

The bounds obtained for $\varrho(i+1)$ show that

$$0 \leqq \varrho(i+1) \leqq \eta_1^* \tag{87}$$

can only come to pass if $\varrho(i+1) < \varrho(i)$ i.e. at the end of an $M$-section. With the purpose of finding the first $\varrho(i+1)$, $i \geqq 0$, for which (87) comes true, the residua at the end of $M$-sections are enough to consider. These residua are the local minima in the series $\varrho(0), \varrho(1), \ldots$. The next minimum comes after the $i$th local minimum $\varrho_{i-1}$, when in the series $\varrho_{i-1}, \varrho_{i-1} + \tau_2^* - \tau_1^*, \ldots, \varrho_{i-1} + n_i'(\tau_2^* - \tau_1^*)$, $\varrho_{i-1} +$

$+n_i'(\tau_2^*-\tau_1^*)+\tau_2^*-\vartheta_1^*, \ldots, \varrho_{i-1}+n_i'(\tau_2^*-\tau_1^*)+j(\tau_2^*-\vartheta_1^*), \ldots$ the first $j=m_i'$ occurs for which

$$\varrho_{i-1}+n_i'(\tau_2^*-\tau_1^*)+m_i'(\tau_2^*-\vartheta_1^*) \geqq \tau_1^*$$

and, therefore,

$$\varrho_i = \varrho_{i-1}+n_i'(\tau_2^*-\tau_1^*)+m_i'(\tau_2^*-\vartheta_1^*)-\tau_1^*.$$

This condition determines $m_i'$ and $\varrho_i$ by $\varrho_{i-1}$ and $n_i'$. $n_i'$ is determined by $\varrho_{i-1}$ as the first $j=n_i'\geqq 0$ for which

$$\varrho_{i-1}+n_i'(\tau_2^*-\tau_1^*) > \tau_1^*-\eta_2^*.$$

This means that $n_i', m_i', \varrho_i$ are uniquely determined by $\varrho_{i-1}$ as

$$n_i' = \left[\frac{\tau_1^*-\eta_2^*-\varrho_{i-1}}{\tau_2^*-\tau_1^*}\right]+1 = \left[\frac{\vartheta_2^*-\varrho_{i-1}}{\tau_2^*-\tau_1^*}\right] \tag{88}$$

$$m_i' = f_{\geqq}\left(\frac{\tau_1^*-\varrho_{i-1}-n_i'(\tau_2^*-\tau_1^*)}{\tau_2^*-\vartheta_1^*}\right) = [\zeta]+\operatorname{sgn}\{\zeta\} \tag{89}$$

$$\varrho_i = \varrho_{i-1}+n_i'(\tau_2^*-\tau_1^*)+m_i'(\tau_2^*-\vartheta_1^*)-\tau_1^*, \tag{90}$$

where

$$\zeta = \frac{\tau_1^*-\varrho_{i-1}-n_i'(\tau_2^*-\tau_1^*)}{\tau_2^*-\vartheta_1^*} \tag{91}$$

and $f_{\geqq}(x)$ is the least integer not less than $x$.

Let us use the notations

$$n_0 = m_0 = k_0 = 0, \quad n_i = \sum_{j=1}^{i} n_j', \quad m_i = \sum_{j=1}^{i} m_j', \quad \psi_i = n_i+m_i, \quad i = 1, 2, \ldots. \tag{92}$$

The integers $n_i, m_i$ and $\psi_i$ give the number of $C_2$-cycles serviced without preemption, with preemption and totally until the end of the $(N_i, M_i)$ section pair, respectively.

Denote by

$$H_i = (\varphi_i, \psi_i, \chi_i), \quad i = 1, 2, \ldots,$$

the triads at the ends of the $(N, M)$-section pairs. We call $H_i$, $i=1, 2, \ldots$, $R_{12}$-*triples*. Clearly $H_i=H(n_i+m_i)$ and

$$\varphi_i = n_i+m_i+i, \quad \psi_i = n_i+m_i, \quad \chi_i = m_i, \quad i = 1, 2, \ldots. \tag{93}$$

The residuum at the end of the $(N_i, M_i)$ section pair can be written from the recursion (90) and $\varrho_0=\varrho(0)=\eta_1^*$ as

$$\varrho_i = \eta_1^*+n_i(\tau_2^*-\tau_1^*)+m_i(\tau_2^*-\vartheta_1^*)-i\tau_1^* \tag{94}$$

or with (93) as

$$\varrho_i = \eta_1^*+\psi_i\tau_2^*+\chi_i\eta_1^*-\varphi_i\tau_1^*. \tag{95}$$

The end of the first period of $R^*$, if such one exists, is determined by the entities at the end of the first $(N, M)$-section pair with $\varrho_i$ satisfying (83). If such a section-pair exists, it can be determined recursively by the formulae (88)—(91). If for $i=I>0$ the relation (83) comes to pass first, the characteristics of $R^*$ will be

$$\Pi^* = (\varphi_I;\, \psi_I;\, \chi_I;\; 1-\varrho_I/\eta_1^*)$$

by (49′), i.e.

$$\begin{aligned} \mu_1^* &= \varphi_I = n_I+m_I+I, & \varkappa_2^* &= \chi_I = m_I \\ \mu_2^* &= \psi_I = n_I+m_I, & \varepsilon_2^* &= 1-\varrho_I/\eta_1^*. \end{aligned} \tag{96}$$

From (93) we can express $i, n_i, m_i$ by the elements of the $R_{12}$-triple $H_i$ as

$$i = \varphi_i-\psi_i, \quad n_i = \psi_i-\chi_i, \quad m_i = \chi_i \tag{97}$$

and from (96) we can express $I, n_I, m_I, \varrho_I$ by the characteristics $\Pi^*$ of $R^*$ as

$$I = \mu_1^*-\mu_2^*, \quad n_I = \mu_2^*-\varkappa_2^*, \quad m_I = \varkappa_2^*, \quad \varrho_I = \eta_1^*(1-\varepsilon_2^*). \tag{97′}$$

These quantities are the number of $(N, M)$-section pairs, the number of $C_2$-cycles serviced without and with preemption and the last residuum, respectively, in a period of $R^*$.

We phrase our main results in

**Theorem 8.** *The priority schedule* $R^*=R_{1,2}(Q^*)$ *of a reduced configuration* $Q^*$ *satisfying*

$$0 < \eta_1 < \tau_1^* < \tau_2^* \tag{98}$$

*is periodic exactly when such a residuum* $\varrho(i)$, $i>0$, *does exist which fulfils* (83). *This condition is equivalent to the fact that* $R^*$ *has an M-section* $M_I$, $I>0$, *the last residuum* $\varrho_I$ *of which fulfils the inequality*

$$\max(0,\, \vartheta_2^*-\vartheta_1^*) \prec \varrho_I \leqq \eta_1^*. \tag{99}$$

*The characteristics are determined then by the* $R_{12}$*-triple* $H_I$ *and the residuum* $\varrho_I$ *as*

$$\Pi^* = (\varphi_I;\, \psi_I;\, \chi_I;\; 1-\varrho_I/\eta_1^*). \tag{100}$$

*Proof.* The only assertion to be proved is that (83) is equivalent to (99) with regard to $\varrho_I$. This follows, however, from the fact that if $\varrho(i)$ is the last residuum of an $M$-section then $\varrho(i)=\varrho(i-1)+\tau_2^*-\vartheta_1^*-\tau_1^*$ and, since $\tau_1^*-\eta_2^*<\varrho(i-1)$ by (83″) because of the preemption of the last $C_2$-cycle, $\varrho(i)>\vartheta_2^*-\vartheta_1^*$ and $\vartheta_2^*-\vartheta_1^*<$ $<\varrho(i)\leqq\eta_1^*$ must stand instead of (83) in the case $\vartheta_2^*-\vartheta_1^*\geqq 0$. Using the definition (86) of $\prec$ we obtain the inequality (99) for $\varrho(i)$ and consequently for $\varrho_I$. □

We now define the formal algorithm to determine the characteristics $\Pi^*$ of $R^*$ if $R^*$ is periodic. As we do not have finite method to decide whether $R^*$ is periodic, we have to choose an integer $L$ as the tolerable number of $(N, M)$-section pairs for which the criterium (99) is allowed to be tested. If $R^*$ is not periodic or the number $I$ of the $(N, M)$-section pairs in a period is greater than $L$ the algorithm finishes without giving the characteristics $\Pi^*$. Nevertheless, the algoritm gives the values of the $R_{12}$-triple $H_L$ and residuum $\varrho_L$ also in this case. The output for $\Pi^*$ is as its input $(0;\ 0;\ 0;\ 0)$ in this case.

**Algorithm** $R_{12}^*$. *Input data:* $Q^* = (\eta_1^*; \vartheta_1^*; \eta_2^*; \vartheta_2^*), L$;
*Output data:* $\Pi^* = (\mu_1^*; \mu_2^*; \varkappa_2^*; \varepsilon_2^*)$, $H_L = (\varphi_L, \psi_L, \chi_L)$, $\varrho_L$;
*Step 0*: $\tau_1^* := \eta_1^* + \vartheta_1^*$; $\tau_2^* := \eta_2^* + \vartheta_2^*$;
If $0 < \eta_2^* \leqq \vartheta_1^* < \tau_1^* < \tau_2^* < \eta_2^* + \tau_1^*$ does not hold then ERROR and go to *End*;
$\varrho := \eta_1^*$; $n := m := i := 0$;
*Step 1*: $n' := \left[\dfrac{\vartheta_2^* - \varrho}{\tau_2^* - \tau_1^*}\right]$; $n := n + n'$; $\varrho := \varrho + n'(\tau_2^* - \tau_1^*)$; $\zeta := \dfrac{\tau_1^* - \varrho}{\tau_2^* - \vartheta_1^*}$;
$m' := [\zeta] + \operatorname{sgn}\{\zeta\}$; $m := m + m'$; $\varrho := \varrho + m'(\tau_2^* - \vartheta_1^*) - \tau_1^*$; $i := i + 1$;
*Step 2*: If $\varrho \leqq \eta_1^*$ then $\mu_1^* := n + m + i$, $\mu_2^* := n + m$, $\varkappa_2^* := m$, $\varepsilon_2^* := 1 - \varrho/\eta_1^*$ and go to *End*;
If $i = L$ then $\varphi_L := n + m + i$, $\psi_L := n + m$, $\chi_L := m$, $\varrho_L := \varrho$ and go to *End*;
Go to *Step 1*;
*End*.

We say that the Algorithm $R_{12}^*$ finishes *normally* if it gives $\Pi^*$ and *abnormally* if it does not give $\Pi^*$ but gives $H_L$ and $\varrho_L$. The algorithm does not put out the data of all $(N, M)$-section pairs but only those of the last. After minimal modification it would furnish these data as well. Independently of the algorithm it is worth to analyse the data the algorithm is dealing with because we can obtain further inferences from this analysis.

First we show bounds on the lengths $n_i', m_i'$ of the $N$- and $M$-sections. Let us use the quantities

$$\underline{n} = \frac{\vartheta_1^* - \eta_2^*}{\tau_2^* - \tau_1^*} - 1, \quad \bar{n} = \frac{\vartheta_1^* - \eta_2^*}{\tau_2^* - \tau_1^*} + 1, \quad \underline{m} = \frac{\eta_1^* + \eta_2^*}{\tau_2^* - \vartheta_1^*} - 1, \quad \bar{m} = \frac{\eta_2^*}{\tau_2^* - \vartheta_1^*} + 1. \tag{101}$$

Let $I$ be the number of the $(N, M)$-section pairs in a period of $R^*$ if $R^*$ is periodic and $I = \infty$ otherwise. The formulae (88)—(91) define $n_i', m_i', \varrho_i$ for $i = 1, 2, \ldots$ ($I$, if $I$ is finite).

**Lemma 9.** *For the lengths $n_i', m_i'$, $i = 1, 2, \ldots (I)$ the following bounds are valid:*

$$n_1' = [\bar{n}], \quad \underline{n} < n_i' < \bar{n}, \quad 1 < i \leqq I, \tag{102}$$

$$\underline{m} < m_i' < \bar{m}, \quad 1 \leqq i < I, \quad \underline{m} \lessdot m_I' < \bar{m}, \tag{103}$$

*where the symbol $\lessdot$ is defined by* (86).

*Proof.* From (88) with $\varrho_0 = \eta_1^*$ we get $n_1' > \dfrac{\vartheta_2^* - \eta_1^*}{\tau_2^* - \tau_1^*} - 1 = \bar{n} - 1$ and $n_1' \leqq \dfrac{\vartheta_2^* - \eta_1^*}{\tau_2^* - \tau_1^*} = \bar{n}$ and so $n_1' = [\bar{n}]$. Using the inequalities $\varrho_{i-1} > \eta_1^*$ and $\varrho_{i-1} < \tau_2^* - \vartheta_1^*$, obtainable from (89) and (90), we get from (88) for $i > 1$ that $n_i' > \dfrac{\vartheta_2^* - \varrho_{i-1}}{\tau_2^* - \tau_1^*} - 1 > \underline{n}$ and $n_i' \leqq \dfrac{\vartheta_2^* - \varrho_{i-1}}{\tau_2^* - \tau_1^*} < \bar{n}$.

If $\zeta$ would be integer by (91) for $i < I$ then we would get $m_i' = \zeta$ and $\varrho_i = 0$ which contradicts the definition of $I$. For $i = I$, $\varrho_I = 0$ is only possible by (99) if $\vartheta_2^* < \vartheta_1^*$. This means that $\zeta < m_i' < \zeta + 1$ if $1 \leqq i < I$ and if $i = I$ and $\vartheta_2^* \geqq \vartheta_1^*$. By this fact and $\tau_1^* - \eta_2^* < \varrho_{i-1} + n_i'(\tau_2^* - \tau_1^*) \leqq \vartheta_2^*$ obtainable from (88) we get $m_i' >$

$>\zeta\geqq\frac{\tau_1^*-\vartheta_2^*}{\tau_2^*-\vartheta_1^*}=\underline{m}$ and $m_i'<\zeta+1<\frac{\eta_2^*}{\tau_2^*-\vartheta_1^*}+1=\overline{m}$ for $i<I$ and $i=I$, $\vartheta_2^*\geqq\vartheta_1^*$, and we get $m_I'\geqq\zeta\geqq\underline{m}$ and $m_I'<\zeta+1<\overline{m}$ for $i=I$ and $\vartheta_2^*<\vartheta_1^*$. □

This lemma shows that the series $n_i'$, $i=1, 2, \dots,$ and $m_i'$, $i=1, 2, \dots,$ of lengths have only small fluctuations, if any. The bandwidth of the variations are

$$\bar{n}-\underline{n}=2 \quad\text{and}\quad 1<\overline{m}-\underline{m}=2-\frac{\eta_1^*}{\tau_2^*-\vartheta_1^*}<2 \quad\text{if}\quad \eta_1^*>0. \tag{104}$$

These show that both the $n_i'$ and $m_i'$ values can always vary at most on two adjacent integers.

From the conditions (78), definitions (101) and estimations (102) and (103) we easily get

$$n_1'\geqq 1, \quad n_i'\geqq 0, \quad 1<i\leqq I, \tag{105}$$

$$m_i'\geqq 1, \quad 1\leqq i\leqq I. \tag{106}$$

Simple regularity conditions can be given for the series of lengths by the parameters of $Q^*$ which further limit their fluctuations. To simplify writing we use the quantities

$$x_j=\vartheta_j^*-\eta_{3-j}^*, \quad j=1, 2. \tag{107}$$

**Lemma 10.** *For the lengths $n_i'$ and $m_i'$ of the $(N, M)$-section pairs the following assertions hold.*

(a) *If*

$$n'<\frac{x_1}{x_2-x_1}<n'+1 \tag{108a}$$

*for some integer* $n'\geqq 0$, *then*

$$n_1'=n'+1 \quad\textit{and}\quad n'\leqq n_i'\leqq n'+1 \tag{109a}$$

*for* $1<i\leqq I$. *Especially*

$$\begin{aligned} &n_1'=1 \quad\textit{and}\quad 0\leqq n_i'\leqq 1,\ 1<i\leqq I \quad\textit{if}\quad 0<\vartheta_1^*-\eta_2^*<\tau_2^*-\tau_1^*\\ &n_1'=2 \quad\textit{and}\quad 1\leqq n_i'\leqq 2,\ 1<i\leqq I \quad\textit{if}\quad \tau_2^*-\tau_1^*<\vartheta_1^*-\eta_2^*<2(\tau_2^*-\tau_1^*)\end{aligned} \tag{109'a}$$

(b) *If*

$$\frac{x_1}{x_2-x_1}=n' \tag{108b}$$

*for some integer* $n'\geqq 0$, *then*

$$n_1'=n'+1 \quad\textit{and}\quad n_i'=n' \tag{109b}$$

*for* $1<i\leqq I$. *Especially*

$$\begin{aligned} &n_1'=1, \quad n_i'=0, \quad 1<i\leqq I, \quad\textit{if}\quad \vartheta_1^*=\eta_2^*\\ &n_1'=2, \quad n_i'=1, \quad 1<i\leqq I, \quad\textit{if}\quad \vartheta_1^*-\eta_2^*=\tau_2^*-\tau_1^*.\end{aligned} \tag{109'b}$$

(c) *If*

$$\frac{m'}{\eta^*} < \frac{1}{x_2 - x_1 + \eta_1^*} \leqq \frac{m'}{\eta_2^*} \tag{108c}$$

*for some integer* $m' \geqq 1$, *then*

$$m_i' = m' \tag{109c}$$

*for all* $1 \leqq i \leqq I$. *Especially*

$$\begin{aligned} &m_i' = 1, \quad 1 \leqq i \leqq I, \quad \textit{if} \quad \vartheta_2^* \geqq \vartheta_1^* \\ &m_i' = 2, \quad 1 \leqq i \leqq I, \quad \textit{if} \quad \tau_2^* - \tau_1^* < \vartheta_1^* - \vartheta_2^* \leqq \tau_2^* - \vartheta_1^*. \end{aligned} \tag{109'c}$$

(d) *If*

$$\frac{\eta^*}{x_2 - x_1 + \eta_1^*} = m' \tag{108d}$$

*for some integer* $m' > 1$, *then*

$$m_i' = m' \quad \textit{for} \quad 1 \leqq i < I \quad \textit{and} \quad m' - 1 \leqq m_I' \leqq m'. \tag{109d}$$

*Especially*

$$m_i' = 2 \quad \textit{for} \quad 1 \leqq i < I \quad \textit{and} \quad 1 \leqq m_I' \leqq 2, \quad \textit{if} \quad \tau_2^* - \tau_1^* = \vartheta_1^* - \vartheta_2^*. \tag{109'd}$$

COMMENT. (108d) cannot be true for $m'=1$ because $\vartheta_2^* = \tau_1^*$ would follow which contradicts (75). (108d) is equivalent to $(m'-1)(\tau_2^* - \vartheta_1^*) + \vartheta_2^* - \vartheta_1^* = \eta_1^*$ from which $\vartheta_1^* - \vartheta_2^* = (m'-1)(\tau_2^* - \vartheta_1^*) - \eta_1^* \geqq \tau_2^* - \tau_1^* > 0$ if $m' > 1$ and, therefore, $\vartheta_2^* < \vartheta_1^*$ follows. In case of $\vartheta_2^* \geqq \vartheta_1^*$ the condition (108d) is impossible.

*Proof.* The method of proof is to relate the bounds (101) to the parameter $n'$ or $m'$ of the condition (108). (101) is equivalent to $\underline{n} = x_1/(x_2 - x_1) - 1$, $\bar{n} = x_1/(x_2 - x_1) + 1$, $\underline{m} = \eta^*/(x_2 - x_1 + \eta_1^*) - 1$, $\overline{m} = \eta_2^*/(x_2 - x_1 + \eta_1^*) + 1$. From (108a) we get $n' - 1 < \underline{n} < n'$ and $n' + 1 < \bar{n} < n' + 2$ and, therefore, the interval $(\underline{n}, \bar{n})$ contains the integers $n'$ and $n'+1$ and (102) is equivalent to (109a). We get (109'a) from (109a) for $n'=0$ and $n'=1$. From (108b) we get $\underline{n} = n' - 1$ and $\bar{n} = n' + 1$ and the relations (102) make possible only (109b). (109'b) follows from (109b) for $n'=0$ and $n'=1$. From (108c) we obtain $m' - 1 < \underline{m}$ and $\overline{m} \leqq m' + 1$ and, therefore, the interval $[\underline{m}, \overline{m})$ contains the only integer $m'$ and (109c) follows from (103). (109'c) follows from (109c) for $m'=1$ and $m'=2$. From (108d) we get $\underline{m} = m' - 1$ as an integer. The interval $[\underline{m}, \overline{m})$ contains now the integers $m'-1$ and $m'$ and (109d) follows from (103) and (86) because (108d) is possible only if $\vartheta_2^* < \vartheta_1^*$ (see Comment) and $\prec \, = \, \leqq$ by (86) in this case. (109'd) follows from (109d) for $m'=2$. □

The conditions (108) are only sufficient but not necessary for (109) to be valid. One of the conditions (108a) and (108b) is always true and (109a) is valid because (109b) implies (109a). Lemma 10 is valid also for $I = \infty$ ($R^*$ is not periodic) if the assertions with $i = I$ are neglected.

From Lemma 10 we can deduce some relationships among the $R_{12}$-triples which can reduce the problem of existence and determination of the least $R_{12}$-triple satisfying (99) to the problem of solution of a coincidence problem [4]. This problem is generally solved and leads to the regular continued fraction expansion of a number depending on the parameters of $Q^*$ [4]. The coincidence problems encountering have the form of the determination of the least solution $\omega^*=(B^*, A^*)$ of an inequality pair

$$0 \leqq B\xi - A \prec \alpha, \quad \omega \geqq \omega_0 \tag{110}$$

for the unknown integers $\omega \doteq (B, A)$ where reals $\xi, \alpha \geqq 0$, sign $\prec$ and integers $\omega_0=(B_0, A_0)$ are given. $\omega^*$ exists and is unique if $\alpha>0$ or $\prec = \leqq$, $\alpha=0$ and $\xi$ is rational. $\omega^*$ does not exist otherwise. $B^*$ and $A^*$ are relatively prime [4].

The following lemma is necessary to prove the periodicity of $R^*$ if $0<\vartheta_1^* \leqq \vartheta_2^*$ in addition to (75).

**Lemma 11.** *For the schedule* $R^*=R_{1,2}(Q^*)$ *of any configuration* $Q^* \in \mathcal{Q}$ *fulfilling* (75) *the following assertions hold.*

(I) *The following three facts are equivalent:*

$$\begin{aligned} &\text{(a)} \quad \varphi_i = \psi_i + \chi_i, \quad 1 \leqq i \leqq I,\\ &\text{(b)} \quad m_i' = 1, \quad\quad\ \ 1 \leqq i \leqq I,\\ &\text{(c)} \quad R^* \textit{ is periodic and } \mu_1^* = \mu_2^* + \varkappa_2^*; \end{aligned} \tag{111}$$

(II) *If any of* (111a—c) *holds, the characteristics* $\Pi^*$ *of* $R^*$ *are determined by the least solution* $\omega^*=(B^*, A^*)$ *and its error* $\Delta^*=B^*\xi^*-A^*$ *of a coincidence problem*

$$0 \leqq B\xi^* - A \prec \alpha^*, \quad \omega \geqq (1, 0) \tag{112}$$

*where* $\xi^*$, $\alpha^*>0$ *are determined by* $Q^*$ *and* $\prec$ *is defined by* (86); $\mu_1^*, \mu_2^*, \varkappa_2^*$ *are pairwise relatively prime integers;*

(III) $\xi^*$ *and* $\alpha^*$ *in* (112) *and the characteristics* $\Pi^*$ *have the alternative values by the three rows of the following table:*

| | $\xi^*$ | $\alpha^*$ | $\mu_1^*$ | $\mu_2^*$ | $\varkappa_2^*$ | $\varepsilon_2^*$ |
|---|---|---|---|---|---|---|
| (a) | $\dfrac{\vartheta_1^*}{\tau_2^*-\tau_1^*}$ | $\dfrac{\eta_1^*-r}{\tau_2^*-\tau_1^*}$ | $A^*+B^*$ | $A^*$ | $B^*$ | $\dfrac{\Delta^*(\tau_2^*-\tau_1^*)}{\eta_1^*}$ |
| (b) | $\dfrac{\tau_2^*-\eta_1^*}{\tau_2^*-\tau_1^*}$ | $\dfrac{\eta_1^*-r}{\tau_2^*-\tau_1^*}$ | $A^*$ | $A^*-B^*$ | $B^*$ | $\dfrac{\Delta^*(\tau_2^*-\tau_1^*)}{\eta_1^*}$ |
| (c) | $\dfrac{\vartheta_1^*}{\tau_2^*-\eta_1^*}$ | $\dfrac{\eta_1^*-r}{\tau_2^*-\eta_1^*}$ | $B^*$ | $A^*$ | $B^*-A^*$ | $\dfrac{\Delta^*(\tau_2^*-\eta_1^*)}{\eta_1^*}$ |

(113)

*where*

$$r = \max(0, \vartheta_2^*-\vartheta_1^*).$$

*Proof.* We begin with the assertions (I). From $m_i' \equiv 1$ we get $\varphi_i=n_i+2i$, $\psi_i=n_i+i$, $\chi_i=i$ from (93), and (111a) is true. From (111a) and (97) we get $i=\varphi_i-\psi_i=\chi_i=m_i$, and (106) and definition (92) prove $m_i'=1$. If $R^*$ is periodic, exactly one $A_1$-task starts during every $B_2$-task by (111c) and (62). This means that the number $\varphi_i-\chi_i$ of $A_1$-tasks causing no preemption is equal to $\psi_i$, the number of $C_2$-cycles. This proves (111a). From the assertion (I) only the periodicity of $R^*$ if (111a) is true, remained to be proved. This will be done together with (II) and (III).

Consider the Gantt-chart of $R^*$ until the first recurrence point $T_1^*$ of the $\beta_1$-situation (not supposed finite). Carve out the $A_1$-tasks from it and denote the resulting chart by $R''$. Since exactly one $A_1$-task starts during every $B_2$-task and the $\beta_1$-situation occurs if the $A_1$-task does not finish during the $B_2$-task, it follows that exactly one $A_1$-task runs during every $B_2$-task except the last before the $\beta_1$-situation, where the $A_1$-task can finish after the $B_2$-task as well. Therefore, chart $R''$ will agree with the schedule $R'=R_{1,2}(Q')$ of the configuration $Q'=(0;\ \vartheta_1^*;\ \eta_2^*;\ \vartheta_2^*-\eta_1^*)$ except eventually the last $B_2$-task which has the length $\vartheta_2''=\vartheta_2^*-\eta_1^*+\varepsilon_2^*$ instead of $\vartheta_2'=\vartheta_2^*-\eta_1^*$. As $\eta_1'=0$, the preempting $A_1$-tasks in $R'$ do not cause delays and, therefore, the cycle-finishing points are

$$f'(C_{2,i}) = i(\tau_2^*-\eta_1^*), \quad i = 1, 2, \ldots .$$

The periodicity of $R^*$ is equivalent to the finiteness of $T_1^*$ and this to the fact that the last $B_2$-task in the first period (if such one exists) of $R'$ would run during a $B_1$-task and finish not more than $\eta_1^*$ earlier than the $B_1$-task (see Fig. 8). This corre-

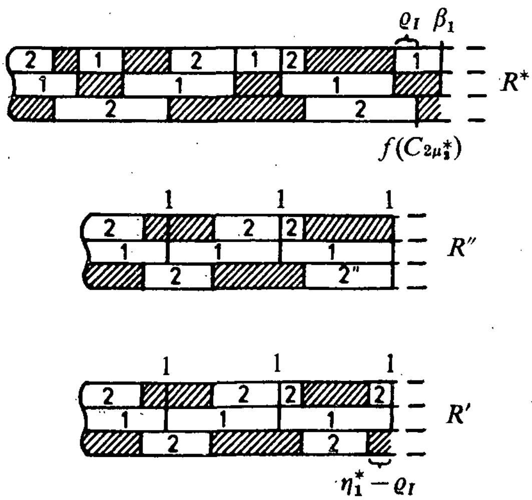



*Fig. 8*
The transformation $R^*\to R''$ and the schedule $R'$

sponds to the first situation in $R'$ in which the inequalities $\vartheta_2^*-\eta_1^*<i(\tau_2^*-\eta_1^*)-(j-1)\vartheta_1^*\leqq\vartheta_1^*$ and $0\leqq j\vartheta_1^*-i(\tau_2^*-\eta_1^*)\leqq\eta_1^*$ for some positive integers $i$, $j$, result: The values of $i$ and $j$ correspond to the characteristics $\Pi^*$ of $R^*$ as $i=\mu_2^*$, $j=\mu_1^*$. The two inequalities are equivalent to the inequality

$$0 \leqq \mu_1^*\vartheta_1^*-\mu_2^*(\tau_2^*-\eta_1^*) \lessdot \eta_1^*-\max(0, \vartheta_2^*-\vartheta_1^*)$$

in which the sign $\lessdot$ is defined by (86). This shows that the periodicity of $R^*$ is equivalent to the existence of positive integers $\omega=(B, A)$ for which the inequalities

(112) with $\xi^*$ and $\alpha^*$ of (113c) hold. The least such pair determines $\mu_1^*$ and $\mu_2^*$ by (113c). $\varkappa_2^*=B^*-A^*$ follows from (111a) and the expression of $\varepsilon_2^*$ from the relationships $\varepsilon_2^*=(\eta_1^*-\varrho_I)/\eta_1^*$ and $\varrho_I=\eta_1^*+\mu_2^*\tau_2^*+\varkappa_2^*\eta_1^*-\mu_1^*\tau_1^*=\eta_1^*+A^*\tau_2^*+(B^*-A^*)\eta_1^*-B^*\tau_1^*=\eta_1^*-\Delta^*(\tau_2^*-\eta_1^*)$. The existence of $\omega^*$ is garanteed by $\alpha^*>0$ and this by (75).

We have to prove that (113a)—(113c) are equivalent. The inequality $0\leqq B^*\vartheta_1^*-A^*(\tau_2^*-\eta_1^*)<\eta_1^*-r$ is equivalent to the inequality $0\leqq B'(\tau_2^*-\eta_1^*)-A'(\tau_2^*-\tau_1^*)<\eta_1^*-r$ if $B^*=A'$ and $A^*=A'-B'$. The least solutions of the two inequalities with the condition $(B, A)\geqq(1,0)$ correspond to each other by this transformation. This proves (113b). By the transformation $B^*=A'+B'$, $A^*=A'$ we can similarly prove the equivalence of (113c) and (113a). If $B^*$ and $A^*$ are relatively prime, such are the transformed values as well. This completes our proof. □

Lemma 10 and 11 enable us to solve the evaluation problem of $R^*$ for configurations $Q^*$ satisfying (75) and any of the relations (77).

**Theorem 9.** *If the configuration $Q^*\in\mathcal{Q}$ is reduced,*

$$\tau_1^*<\tau_2^* \quad \text{and} \quad 0<\vartheta_1^*\leqq\vartheta_2^* \tag{114}$$

*then $R^*=R_{1,2}(Q)$ is periodic and its characteristics $\Pi^*$ are obtainable by* (113) *and $\mu_1^*, \mu_2^*, \varkappa_2^*$ are pairwise relatively prime integers.*

*Proof.* In $R^*$ we obtain $m_i'\equiv 1$ from (109'c) and $R^*$ is periodic with $\mu_1^*=\mu_2^*+\varkappa_2^*$ by (111c). The assertions (II)—(III) of the Lemma 11 corresponds to the statement of the theorem. □

With this theorem the only case not solved is the configuration $Q\in\mathcal{Q}$ which is reducible and its reduction $Q^*$ satisfies the relations

$$\tau_1^*<\tau_2^*, \quad \vartheta_1^*>\vartheta_2^*. \tag{115}$$

If we know that $R^*=R_{1,2}(Q^*)$ is periodic, the Algorithm $R_{12}^*$ can be used to determine the characteristics $\Pi^*$. This method does not answer the question whether $\mu_1^*$, $\mu_2^*$ and $\varkappa_2^*$ are relatively prime integers wich fact was shown in all other cases. In fact, $\mu_1^*$ and $\mu_2^*$ are relatively prime in every known periodicity case. Some further specific cases of (115) can be solved by using Lemma 10. For example, it can be proved that $m_I=m'-1$ if (108d) hold and, under the conditions (115), $R^*$ is periodic if and only if $\vartheta_1^*-\eta_2^*$ and $\tau_2^*-\tau_1^*$ are rationally dependent. If

$$\xi=\frac{\vartheta_1^*-\eta_2^*}{\tau_2^*-\tau_1^*}=\frac{A}{B},$$

$A, B>0$ are relatively prime integers then the characteristics of $R^*$ are

$$\Pi^*=\big((m'+1)B+A;\, m'B+A;\, m'B-1;\, 1\big)$$

with relatively prime $\mu_1^*$ and $\mu_2^*$ [4]. This assertion will not be proved here. This result is interesting because it shows that $R^*$ *can be non-periodic for non-defective $Q^*$ as well.* By another assertion [4], $R^*$ is always periodic and its characteristics $\Pi^*$ is determined by a given coincidence problem type (110) if (108c) holds. $\mu_1^*$

and $\mu_2^*$ are relatively prime again. Similar assertions hold for non-defective configurations $Q \in \mathcal{Q}$ (not necessarily reduced) with $\eta_2 = \vartheta_1$ [4]. The proofs of these assertions are lengthy and, therefore, we do not show them here.

For any $Q \in \mathcal{Q}$, independently of its periodicity, the efficiency $\gamma_{1,2}$ of the priority schedule $R_{1,2}(Q)$ can be approximated by the $P_A$-utility $\gamma_{1,2}(\eta_1, t)$ of its section $\eta_1 \leqq s \leqq t$ defined by

$$\gamma_{1,2}(\eta_1, t) = \frac{\lambda(t) - \lambda(\eta_1)}{t - \eta_1} \tag{116}$$

as $t$ grows (see (1)). It can be proved [4] that

$$\gamma_{1,2}(\eta_1, t) \sim \gamma^{(1)} + \gamma^{(2)} - \frac{\varkappa_2(t)}{\mu_1(t)} \gamma^{(1)} \gamma^{(2)} \sim \gamma_{1,2} \tag{117}$$

if $t$ is big enough, where $\mu_1(t)$ is the number of the completed and $\varkappa_2(t)$ the number of preempting $A_1$-tasks until $t$ in the schedule $R_{1,2}(Q)$. If $R_{1,2}(Q)$ is periodic with characteristics $\Pi = (\mu_1;\ \mu_2;\ \varkappa_2;\ \varepsilon_2)$ then

$$\gamma_{1,2} = \gamma^{(1)} + \gamma^{(2)} - \frac{\varkappa_2 + \varepsilon_2}{\mu_1} \gamma^{(1)} \gamma^{(2)} \tag{118}$$

(Theorem 5.10 in [4]). The proof of these facts we omit as well.

## 6. Some comments on the reduction methods

Theorem 3 in section 3 establishes relationships between the characteristics of the priority schedule of $Q$ and of any transform $Q_n = \Delta^n Q$ of it. The reduction operator $\Delta$ defined in section 2 is actually the $\Delta_1$ from the two operators $\Delta_1$ and $\Delta_2$ defined for $Q$ symmetrically in the job-flows $Q^{(1)}$ and $Q^{(2)}$. The operator $\Delta_1$ is only usable in the investigation of the priority schedules $R_{1,2}(Q)$ and we know nothing about the connections between the characteristics of $R_{2,1}(Q)$ and $R_{2,1}(Q_n)$, for instance. In the investigation of $R_{2,1}(Q)$ we can use the operator $\Delta_2$. The $\tilde{Q} = \Delta_2 Q$ can be defined as the $\Delta_1 Q$ by (2) but the role of $Q^{(1)}$ and $Q^{(2)}$ (the indices 1 and 2) must be changed. The operation $\Delta_2 Q$ is, therefore, equivalent to the operation $\Delta_1 \bar{Q} = \Delta \bar{Q}$ with the conjugate configuration $\bar{Q}$ of $Q$ defined in section 1.

In a previous article [5] we defined other operators $\mathcal{D}_1$ and $\mathcal{D}_2$ for $Q$ as reductions utilized in the investigations of non-preemptive schedulings. In the operation $\mathcal{D}Q = \mathcal{D}_1 Q$ only the parameters $\vartheta_1$ and $\vartheta_2$ are reduced versus operation $\Delta Q$ in which also $\eta_2$ is reduced. The $\mathcal{D}$-reduction is much simpler than the $\Delta$-reduction and is defined by (2b) and (2d) replaced (2c) by the instruction $\tilde{\eta}_2 = \eta_2$. $Q^*$ is reduced by $\mathcal{D}$ if [5]

$$\vartheta_1^* < \tau_2^* \quad \text{or} \quad \tau_2^* = 0 \quad \text{and} \quad \vartheta_2^* < \tau_1^* \quad \text{or} \quad \tau_1^* = 0$$

which are exactly the conditions (5a) and (5c) as part of conditions $Q^*$ to be reduced by $\Delta$. This means $Q^*$ reduced by $\Delta$ is always reduced by $\mathcal{D}$ as well. The opposite is not true, of course. The conditions (5a) and (5c) show that a configuration $Q^*$ is reduced simultaneously by both $\mathcal{D}_1$ and $\mathcal{D}_2$. This is not true in respect to $\Delta_1$ and

$\Delta_2$. Fig. 9 shows the domains of reduced configurations $Q$ by the operators $\mathcal{D}_i$ and $\Delta_i$, $i=1, 2$ (refer also to Fig. 2). We distinguish the following domains:

($\alpha$) $\tau_1\tau_2 = 0$; $Q$ is reduced by all operators

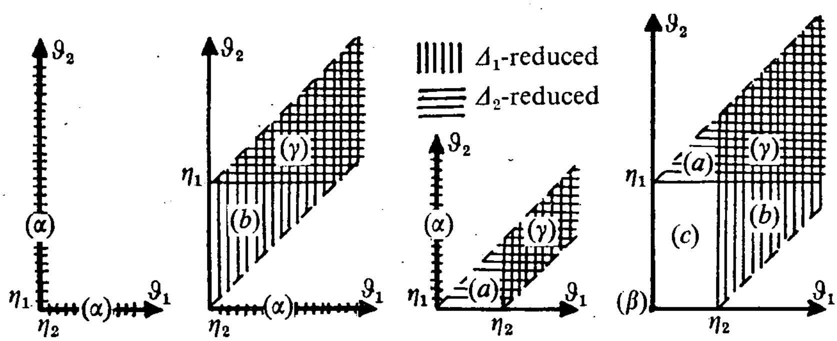
($\beta$) $\eta_1\eta_2 > 0$, $\vartheta_1 = \vartheta_2 = 0$; $Q$ is reduced by all operators

($\gamma$) $\eta > 0$, $0 \leqq \eta_1 \leqq \vartheta_2 < \tau_1$, $0 \leqq \eta_2 \leqq \vartheta_1 < \tau_2$; $Q$ is reduced by all operators

(a) $\eta_2 > 0$, $0 \leqq \eta_1 \leqq \vartheta_2 < \tau_1 < \eta$; $Q$ is not reduced by $\Delta_1$ but it is reduced by the other operators

(b) $\eta_1 > 0$, $0 \leqq \eta_2 \leqq \vartheta_1 < \tau_2 < \eta$; $Q$ is not reduced by $\Delta_2$ but it is reduced by the other operators

(c) $\eta_1\eta_2 > 0$, $\vartheta > 0$, $0 \leqq \vartheta_i < \eta_{3-i}$, $i = 1, 2$; $Q$ is not reduced by $\Delta_i$, $i = 1, 2$, but it is reduced by $\mathcal{D}_i$, $i = 1, 2$.




(a) (b) (c) (d)

*Fig. 9*
Domains of reduced configurations

Let us introduce two simple *operators* $\delta_1$ *and* $\delta_2$ defined by $\tilde{Q}=\delta_i Q$ as of parameters

$$\tilde{\eta}_{3-i} = \begin{cases} \eta_{3-i} - f_<\left(\dfrac{\eta_{3-i}}{\vartheta_i}\right)\vartheta_i & \text{if} \quad \vartheta_i > 0 \\ \eta_{3-i} & \text{otherwise} \end{cases} \tag{119}$$

where $f_<(x)$ is the greatest integer less than $x$. Let $\delta=\delta_1$. It is clear that $f_<(\eta_2/\vartheta_1)= =k_2$ in (2c) if $\vartheta_1>0$. The operator $\delta_i$ is *effective* for $Q$ if $\eta_{3-i}>\vartheta_i>0$ and *ineffective* for $Q$ if $\vartheta_i\eta_{3-i}=0$ or $\eta_{3-i}\leqq\vartheta_i$. Since the order of steps (2c) and (2d) in the operation $\Delta Q$ is indifferent, the operator $\Delta$ can be represented as the operators $\mathcal{D}$ and $\delta$ in succession:

$$\Delta = \delta\mathcal{D}.$$

As $\vartheta_1\geqq\tau_2$ implies $\eta_2\leqq\vartheta_1$, the operator $\delta$ will be ineffective until $Q$ is not reduced by $\mathcal{D}$ and $\mathcal{D}$ is effective on $Q$. This means that the manifestation of $\Delta$ for $Q$ is $\mathcal{D}$

until $\mathscr{D}Q$ will not be $\mathscr{D}$-reduced, i.e. $\Delta Q = \mathscr{D}Q$. If $\mathscr{D}Q$ is $\mathscr{D}$-reduced, but not $\Delta$-reduced, then $\Delta Q = \delta\mathscr{D}Q \neq \mathscr{D}Q$. This means that the manifestation of $\Delta^n Q$, $n>0$, is the alternate series of operator-powers $\mathscr{D}^v$ and the operator $\delta$.

The manifestation is determined by the series $(L)$ of quotients, or rather, by the subseries $(k)$ of $(L)$, defined in section 2. The operator $\delta$ in $\Delta = \delta\mathscr{D}$ is ineffective whenever $k_{2,n}=0$.

Define $v_0'=-1$ and for $i>0$, $v_i'=r$ if $k_{2,r}>0$ is the $i$th positive member in the series $(k)$, if such one exists, and $v_i'$ is undefined if less than $i$ positive members in $(k)$ exist. It can easily be seen that

$$-1 \leqq v_0' < v_1' < \dots \quad \text{and} \quad v_i' \geqq i-1$$

and for any integer $r \geqq 0$ there exists a greatest $v_i'$ for which $v_i' < r$. Let this be $v_{h(r)}'$, i.e.

$$h(r) = \max_{v_i' < r} i, \quad r = 0, 1, \dots.$$

$h(r)$ is the number of positive members in the series $k_{2,0}, k_{2,1}, \dots, k_{2,r-1}$ and $v_{h(r)}'$ is the index of the last positive member if such one exists, and $v_{h(r)}' = -1$, otherwise. This means that

$$v_{h(0)}' = -1, \quad -1 \leqq v_{h(r)}' \leqq r-1, \quad r \geqq 0.$$

By means of the series $(v')$ and function $h(r)$ the manifestation of $\Delta^r$ on $Q$ can be written as

$$\Delta^r Q = \mathscr{D}^{r-1-v'h(r)} \left( \prod_{j=h(r)}^{1} \delta\mathscr{D}^{v'_j - v'_{j-1}} \right) Q, \quad r \geqq 0, \tag{120}$$

and if the degree of compositeness $v$ of $Q$ is finite,

$$\Delta^r Q = \mathscr{D}^{v-1-v'h(v)} \left( \prod_{j=h(v)}^{1} \delta\mathscr{D}^{v'_j - v'_{j-1}} \right) Q, \quad r \geqq v. \tag{120'}$$

Here $\prod_{j=h(r)}^{1} x_j \doteq x_{h(r)} x_{h(r)-1} \dots x_1$ and $\prod_{j=0}^{1} x_j = \emptyset$ is the identity operator. The factorizations (120) and (120′) depend, of course, on $Q$ and, directly, on the series $(L)$. If $v < \infty$, the series $(v')$ is finite and, with $J = |(v')|$, the last positive member of it is $v'_{J-1}$. Let us supplement $(v')$ with the last member $v'_J = v-1$. Define the series of integers

$$v_j = v_j' - v_{j-1}', \quad j = 1, 2, \dots, J.$$

The $\mathscr{D}$-reduction of $Q$ is then

$$Q^{(*)} = \mathscr{D}^{v_1} Q = \mathscr{D}^{v_1'+1} Q = Q_{v_1'+1}$$

and the $\Delta$-reduction of $Q$ is

$$Q^* = \Delta^v Q = \mathscr{D}^{v_J} \left( \prod_{j=J-1}^{1} \delta\mathscr{D}^{v_j} \right) Q = Q_v. \tag{121}$$

The factorization (121) shows that the $\Delta$-reduction of any configuration $Q \in \mathscr{Q}$ is equivalent to some alternate series of $\mathscr{D}$-reductions and $\delta$-operations. This fact clearly shows the connection between the two kinds of reduction.

The reduction operators $\Delta_1$ and $\Delta_2$ differ in both of their factors, $\mathscr{D}_i$ and $\delta_i$:

$$\Delta_1 = \delta_1 \mathscr{D}_1, \quad \Delta_2 = \delta_2 \mathscr{D}_2 \tag{122}$$

but the manifestations (121) of the $\Delta_1$- and $\Delta_2$-reductions, if finite, are of similar factorizations in structure. In the analogous to (121) of the $\Delta_2$-reduction of $Q$ the same operator $\mathscr{D}$ can be applied because a configuration $Q^{(*)}$ is reduced by both of $\mathscr{D}_1$ and $\mathscr{D}_2$ at once and the degrees of compositeness by $\mathscr{D}_1$ and $\mathscr{D}_2$ have a known connection [4]. Nevertheless, the series $(L)$ by $\Delta_1$ and $\Delta_2$ are different and, consequently, the series $(\nu)$ playing the central role in (121) are also different. Though the data of $\Delta_1$- and $\Delta_2$-reduction are not independent of each other, the interrelationships are likewise complicated and hardly provide a useful basis in practice to avoid evaluation of one of the two schedules $R_{1,2}(Q)$ and $R_{2,1}(Q)$. To inspect the relationships between both schedules the two reductions $\Delta_1$ and $\Delta_2$ seem to be a usable basis. The results given here can provide a grounding to this inspection by revealing the nature of the priority schedules in themselves. The method of $\Delta$-reduction is a useful tool to this.

We mention the connection of the $\Delta$-reduction with the regular continued fraction expansion. The Euclidean algorithm of the expansion of the number $\xi=\tau_1/\tau_2$ can be defined as the iteration [2]:

$\tau_{1,0} = \tau_1, \quad \tau_{2,0} = \tau_2$ and for $n = 1, 2, \ldots$

$\tau_{1,n-1} = b_{2n-2}\tau_{2,n-1} + \tau_{1,n}$ where

$b_{2n-2} \geqq 0$ is an integer and $0 \leqq \tau_{1,n} < \tau_{2,n-1}$ if $\tau_{2,n-1} > 0$,

$b_{2n-2}$ and $\tau_{1,n}$ are not defined otherwise

$\tau_{2,n-1} = b_{2n-1}\tau_{1,n} + \tau_{2,n}$ where

$b_{2n-1} \geqq 0$ is an integer and $0 \leqq \tau_{2,n} < \tau_{1,n}$ if $\tau_{1,n} > 0$,

$b_{2n-1}$ and $\tau_{2,n}$ are not defined otherwise.

Both components of the pair $(\tau_{1,n-1}, \tau_{2,n-1})$ are reduced by the step. This iteration ends with a $\tau_{i,n}=0$, $i=1$ or 2, $n\geqq 0$ if $\xi$ is a rational number and is infinite if $\xi$ is irrational.

The definition (2) of the $\Delta$-reduction differs from this iteration by $\tau_1$ and $\tau_2$ being decomposed into two parts: $\tau_i=\eta_i+\vartheta_i$, $i=1, 2$, and this parts are reduced separately except $\eta_1$ which is not reduced at all. The iteration can end not only with a zero component but with conditions (5) of the reducedness. We have seen that the $\Delta$-reduction becomes continued fraction expansion if one of the parts $\eta_2$ and $\vartheta_2$ is zero. If, however, $\vartheta_2=0$, the reduction becomes the expansion of $\vartheta_1/\eta_2$ and not of $\tau_1/\vartheta_2$.

The entities defined in section 2 in connection with $\Delta$-reduction remind us of those in connection with the regular continued fraction expansion [3]. The special case of $\eta=0$ corresponds to the expansion of $\xi=\tau_1/\tau_2$.

## 7. Summary

We review below the points $Q$ of the configuration space $\mathcal{Q}$ by our theorems proved from the point of view of whether the Question of periodicity and evaluation of the priority schedules $R_{1,2}$ and $R_{2,1}$ of $Q$ is answered. See Fig. 10 as an illustration. Tx refers to the Theorem $x$ in the Fig. 10.

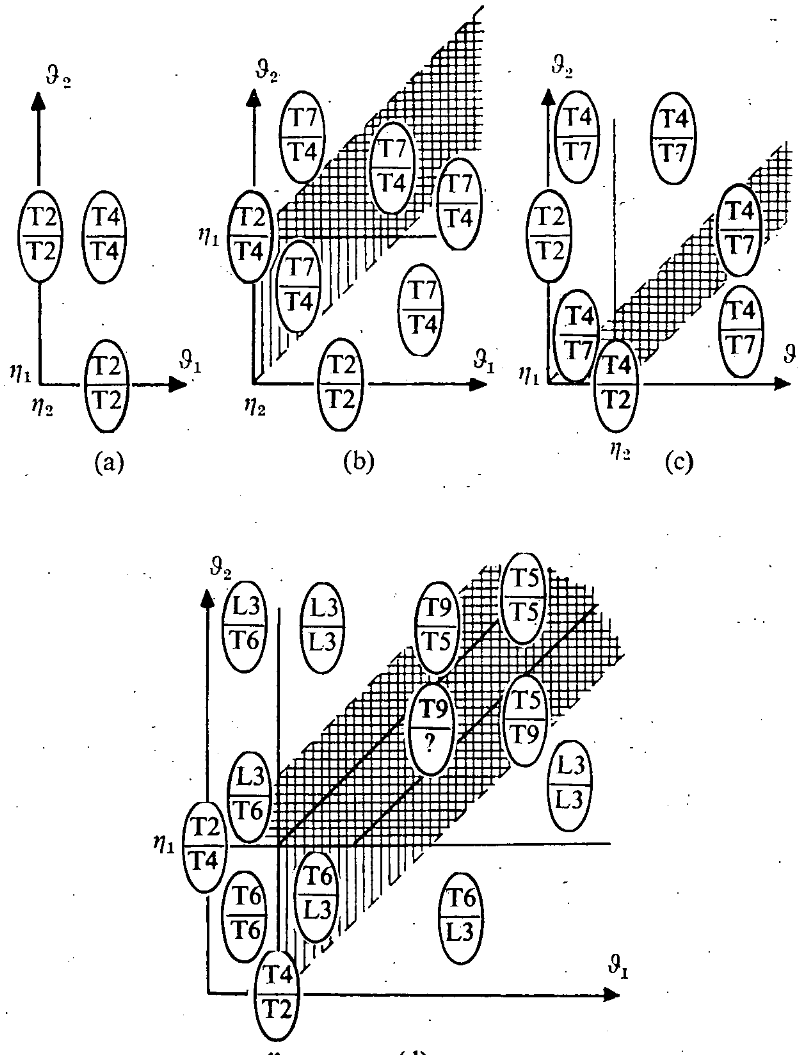



*Fig. 10*

The domains of $\mathcal{Q}$ where theorems answer the question of periodicity of $R_{1,2}(Q)$ (Tx) and $R_{2,1}(Q)$ (Ty) as $\frac{\text{Tx}}{\text{Ty}}$

By Lemma 3 any configuration $Q$ is reducible to a $\Delta_1$-reduced configuration $Q^*$ or a defective configuration $Q'$ with $\eta_1'\vartheta_2'=0$. This means that the questionable part of $\mathcal{Q}$ is reduced to the three-dimensional subspaces $\eta_1=0$, $\eta_2=0$, $\vartheta_1=0$, $\vartheta_2=0$ and to the four-dimensional domain of $\mathcal{Q}$ the two-dimensional cuts by fixing $(\eta_1, \eta_2)$ of which are the domains ($\alpha$), ($b$), and ($\gamma$) in Fig. 9d. Lemma 3 (L3) is used in Fig. 10 only when no other theorem answering the Question directly exists. In the three-dimensional subspaces $\eta_1\vartheta_2=0$ the Question of $R_{1,2}$ is solved by Theorem 2 if $\vartheta_1\tau_2=0$ and by Theorem 4 if $\vartheta_1\tau_2>0$. These solve the Question of $R_{2,1}$ in the subspaces $\eta_2\vartheta_1=0$. The Question of $R_{1,2}$ in the space $\vartheta_1=0$ and of $R_{2,1}$ in $\vartheta_2=0$ is solved by Theorem 2 independently of $\eta_i$ and $\tau_{3-i}$.

If $\eta_2=0$ but $\eta_1\vartheta_1\vartheta_2>0$ the Question of $R_{1,2}$ is answered by Theorem 7 and this answers the Question of $R_{2,1}$ if $\eta_1=0$ but $\eta_2\vartheta_1\vartheta_2>0$, too.

The Question is answered so for every defective configuration and, by Theorem 3, for every configuration reducible to a defective one by any of the operators $\Delta_1$ and $\Delta_2$. By Lemma 3 all other configurations are reducible by both of $\Delta_1$ and $\Delta_2$ to configurations $Q^*$ and $Q^{**}$, respectively, which are in the domains ($b$) and ($\gamma$) and domains ($a$) and ($\gamma$), respectively, in Fig. 9d. Theorem 6 answers the Question of $R_{1,2}$ in the domain $\vartheta_2<\eta_1$ and of $R_{2,1}$ in the domain $\vartheta_1<\eta_2$ without reduction.

As far as the configurations $Q$ reduced by both of $\Delta_1$ and $\Delta_2$ the Question of $R_{1,2}$ is answered by Theorem 5 in the domain $\tau_1\geqq\tau_2$ and the Question of $R_{2,1}$ in the domain $\tau_1\leqq\tau_2$. Theorem 9 answers the Question of $R_{1,2}$ in the domain $\vartheta_1\leqq\vartheta_2$ and the Question of $R_{2,1}$ in the domain $\vartheta_1\geqq\vartheta_2$.

In Fig. 10d the only questionable domain remained for $R_{2,1}$ is

$$\eta_2 \leqq \tau_2-\eta_1 < \vartheta_1 < \vartheta_2.$$

This contains "absolutely" (by both of $\Delta_1$ and $\Delta_2$) reduced configurations for which $\eta_1\leqq\vartheta_2<\tau_1$ and $\eta_2\leqq\vartheta_1<\tau_2$. In general, the unanswered domain of $\mathcal{Q}$, remaining only if $\eta_1\neq\eta_2$, is

$$0 < \eta_i \leqq \tau_i-\eta_{3-i} < \vartheta_{3-i} < \vartheta_i \quad \text{for} \quad R_{i,3-i} \quad \text{if} \quad \eta_i < \eta_{3-i}. \tag{123}$$

Further parts from the domain (123) are answered by results based upon the Lemma 10 and mentioned after (115) but not proved here. These are found in [4]. A direct answer is given by Theorem 6 for $R_{1,2}$ in the domain $\vartheta_2<\eta_1$ and for $R_{2,1}$ in the domain $\vartheta_1<\eta_2$ which is the answer for both schedules in the domain $0<\vartheta_i<$ $<\eta_{3-i}$, $i=1, 2$.

The flow of evaluation of the priority schedules $R_{1,2}$ and $R_{2,1}$ for a configuration $Q$ is illustrated on the flow-chart in Fig. 11. Tx refers to the Theorem $x$ and in $\boxed{x_1;\, y_1;\, x_2;\, y_2}$ $x_i, y_i$ refer to the schedule $R_{i,3-i}$. $x_i=p$ means periodicity, $x_i=?$ refers to unanswered Question and $x_i=$other refers to the rationality of $x_i$ as the condition of periodicity. $y_i=$number gives the efficiency value of $R_{i,3-i}$, $y_i=?$ refers to the undefinedness of the efficiency or unanswered Question and $y_i=$Tx refers to the Theorem $x$ as means of determination of the efficiency. $(x_i, y_i)=\Delta_i$ refers to the application of the operator $\Delta_i$ iteratively until a configuration results which is in a domain where the schedule $R_{i,3-i}$ is directly evaluable by one of the Theorems 2, 4, 5, 6, 7, 9.

KEYWORDS: steady job-flow pairs, priority schedules, reduction method

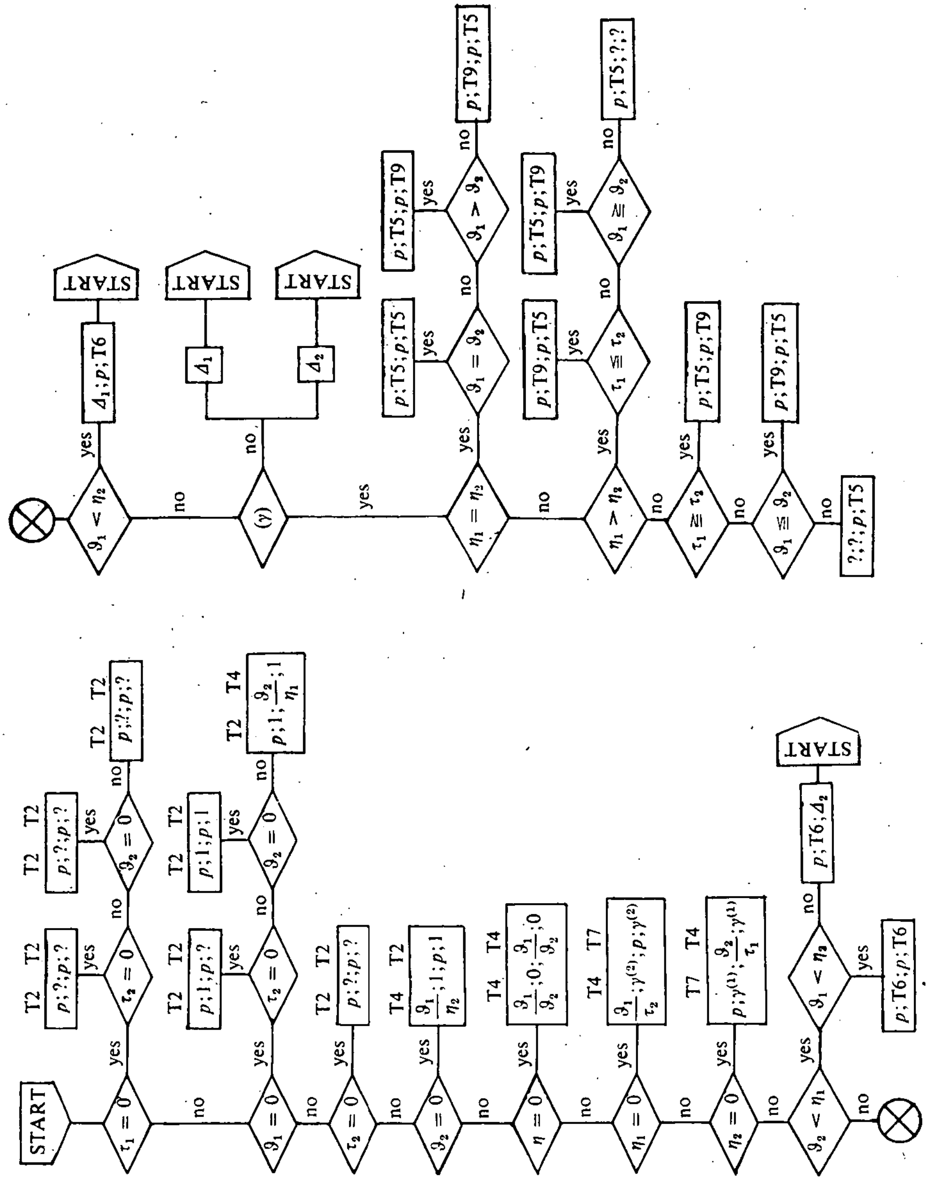


*Fig. 11*

The flow-chart of the evaluation of the priority schedules $R_{1,2}$ and $R_{2,1}$

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