Subdirectly irreducible commutative automata

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M. Yoeli gave a characterization of finite subdirectly irreducible automata with a single input sign (cf. [9]). In [8] G. H. Wenzel generalized this result for the infinite case. In this paper we present another result along this line. Namely, we characterize all subdirectly irreducible commutative automata and hence all subdirectly irreducible commutative semigroups as well.

Notions and notations

An *automaton* is a system $\mathbf{A}=(A, X, \delta)$ where A is a nonempty set, the set of states, X is an arbitrary set, the set of input signs and, finally, $\delta\colon A\times X\to A$ is the transition function. As in general, we shall also use this transition function in the extended sense, i.e. as a mapping $\delta\colon A\times X^*\to A$. Here X^* denotes the free monoid generated by X. The identity of X^* is the empty word λ and $X^+=X^*\setminus\{\lambda\}$. We use the notation δ_p to denote the mapping induced by p: $\delta_p(a)=\delta(a, p)$ $(a\in A, p\in X^*)$. If a sign $x\in X$ induces a permutation of A then it is called a *permutation sign*. In this way we can divide X into two disjoint sets X_P and X_{NP}. X_P is the set of all permutation signs and $X_{NP}=X\setminus X_P$.

The mappings δ_p $(p\in X^*)$ form a monoid with respect to the composition of mappings. The identity of this monoid is the identity mapping on A, $\delta_\lambda=\mathrm{id}_A$. This monoid $S(\mathbf{A})$ is called the *characteristic semigroup* of $\mathbf{A}$. Sometimes another representation of the characteristic semigroup is useful in the literature. However, there is no essential difference among these definitions.

Each automaton $\mathbf{A}=(A, X, \delta)$ can be considered as a unoid, i.e. as a universal algebra equipped with unary operations only. Thus the notions such as subautomaton, homomorphism, congruence relation, quotient automaton, free automaton etc. can be introduced in a natural way. In connection with these notions we shall use the following notations: if $B\subseteq A$ then $[B]$ denotes the subautomaton generated by B, $C(\mathbf{A})$ denotes the lattice of all congruence relations of $\mathbf{A}$, if $\theta\in C(\mathbf{A})$ and $a\in A$ then $\theta(a)$ denotes the block containing a in the partition induced by θ, Δ_A is the equality relation of A, if $B\subseteq A$ then $\theta_{|B}=\theta\cap B\times B$, finally, if $\theta\in C(\mathbf{A})$ then the quotient automaton induced by θ is denoted by $\mathbf{A}/\theta=(A/\theta, X, \delta)$. Observe that we have used the same notation δ for the transition function of $\mathbf{A}/\theta$ as well. An automaton $\mathbf{A}$ is called *subdirectly irreducible* if either $\mathbf{A}$ has one state only, or $\Delta_A \neq \bigcap(\theta: \theta \in C(\mathbf{A}), \theta \neq \Delta_A)$.

Each subautomaton $\mathbf{B}=(B, X, \delta)$ of an automaton $\mathbf{A}=(A, X, \delta)$ can be viewed as a congruence relation $\sigma_{\mathbf{B}} \in C(\mathbf{A})$: $a \sigma_{\mathbf{B}} b$ if and only if $a, b \in B$ or $a=b$. And what is more, $C(\mathbf{B})$ can be embedded into $C(\mathbf{A})$ in a natural way, i.e. by the correspondence $\theta \to \theta'$ where $a\theta' b$ if and only if $a\theta b$ or $a=b$ for any $a, b \in A$. From this it follows that an automaton is subdirectly irreducible if and only if each of its subautomaton is subdirectly irreducible (cf. also [8]).

In the sequel we shall need a more general concept of subautomata, too. The automaton $\mathbf{B}=(B, Y, \delta')$ is an *X-subautomaton* of $\mathbf{A}=(A, X, \delta)$ if $B \subseteq A$, $Y \subseteq X$ and $\delta_{|B \times Y}=\delta'$. For the sake of simplicity we shall not make any distinction between δ and δ'. A special X-subautomaton of $\mathbf{A}$ is the X-subautomaton $\mathbf{B}=(A, X_P, \delta)$. It is called the *permutational subautomaton* of $\mathbf{A}$.

Various concepts of connectedness can be found in the literature. In what follows we shall use two of these concepts. An automaton $\mathbf{A}=(A, X, \delta)$ is called *strongly connected* if each state $a \in A$ is a generator of $\mathbf{A}$ and it is called *connected* if for arbitrary $a, b \in A$ $[a] \cap [b] \neq \emptyset$.

Our results pertain to commutative automata. An automaton $\mathbf{A}=(A, X, \delta)$ is said to be *commutative* if $\delta_{xy}=\delta_{yx}$ is satisfied for any $x, y \in X$, i.e. $xy=yx$ is an identity in $\mathbf{A}$. It is well-known that $\mathbf{A}$ is commutative if and only if δ_p is an endomorphism of $\mathbf{A}$ for every $p \in X^*$, and this is the reason why if $\mathbf{A}$ is generated by a state a then $\mathbf{A}$ is a free automaton with free generator a.

Thus a strongly connected commutative automaton is freely generated by any of its states. This implies that each input sign of a strongly connected commutative automaton $\mathbf{A}$ is a permutation sign, i.e. $S(\mathbf{A})$ is a commutative permutation group on A.

We have proved in [2] (cf. Theorem 1) that if a finite commutative automaton $\mathbf{A}$ has a generator state then $C(\mathbf{A}) \cong C(S(\mathbf{A}))$ and $|A|=|S(\mathbf{A})|$, where $C(S(\mathbf{A}))$ denotes the lattice of all congruences of $S(\mathbf{A})$. However, we have not used the finiteness of $\mathbf{A}$ in proving this statement thus this remains valid for arbitrary commutative automaton as well. Consequently, if $\mathbf{A}$ is a singly generated commutative automaton then $\mathbf{A}$ is subdirectly irreducible if and only if $S(\mathbf{A})$ is subdirectly irreducible. This was also discovered by I. PEÁK in [5].

Strongly connected commutative automata

The previously mentioned fact helped us to prove in [2] that a finite strongly connected commutative automaton is subdirectly irreducible if and only if it is a cyclic automaton of prime-power order. In this section we extend this result to the infinite case.

According to [3, 6] Abelian groups Z_{p^k} and Z_{p^∞} — where p is a prime — are called *cocyclic*. An automaton $\mathbf{A}=(A, X, \delta)$ is cocyclic, if its input-reduced subautomaton is (A, X)-isomorphic[1] to a strongly connected X-subautomaton of

[1] An automaton $\mathbf{A}=(A, X, \delta)$ is said to be (A, X)-isomorphic to an automaton $\mathbf{B}=(B, Y, \delta')$ if there exist bijections $\mu: A \to B$ and $\nu: X \to Y$ such that $\mu(\delta(a, x))=\delta'(\mu(a), \nu(x))$ for any $a \in A$ and $x \in X$.

an automaton obtained by viewing a cocyclic group as an automaton. (By the input-reduced subautomaton of an automaton $\mathbf{A}=(A, X, \delta)$ we mean an X-subautomaton $\mathbf{B}=(A, Y, \delta)$ where Y is a maximal subset of X with the property that $y_1 \neq y_2 (\in Y)$ implies $\delta_{y_1} \neq \delta_{y_2}$. $\mathbf{B}$ is unique up to isomorphism.) Observe that a strongly connected commutative automaton $\mathbf{A}$ is cocyclic if and only if $S(\mathbf{A})$ is a cocyclic group. It is known that an Abelian group is subdirectly irreducible if and only if it is a cocyclic group (cf. [3, 6]). Thus, by our previous remarks we obtain the following

Statement. A strongly connected commutative automaton is subdirectly irreducible if and only if it is a cocyclic automaton.

The general case

In this section we shall characterize all subdirectly irreducible commutative automata. First we need some definitions.

Let $\mathbf{A}=(A, X, \delta)$ be an arbitrary commutative automaton and define the binary relation $\leqq$ on A as follows: $a \leqq b$ if and only if there is a word $p \in X^*$ satisfying $\delta(a, p)=b$. It is not difficult to see that this relation is a preorder on A and it has the substitution property. Thus the relation $\leqq$ determines a congruence relation $\theta \in C(\mathbf{A})$: $a \theta b$ if and only if $a \leqq b$ and $b \leqq a$. Furthermore, the system $(A/\theta, \leqq)$ — where $\theta(a) \leqq \theta(b)$ if and only if $a \leqq b$ — becomes a partially ordered set. It is obvious that if $\mathbf{B}=(B, X, \delta)$ is a subautomaton of $\mathbf{A}$ then $B=\bigcup(\theta(b): b \in B)$ and B/θ is an upper ideal in $(A/\theta, \leqq)$. Conversely, if B is an upper ideal in A/θ then $(\bigcup(\theta(b): \theta(b) \in B), X, \delta)$ is a subautomaton of $\mathbf{A}$.

The automaton $\mathbf{A}$ is called *quasi-nilpotent* if the following three conditions are satisfied by $\mathbf{A}$:

i) $(A/\theta, \leqq)$ has a greatest element $\theta(a_0)$ and $\theta(a_0)=\{a_0\}$ where a_0 is called the absorbent state,

ii) $A/\theta \setminus \theta(a_0)$ has a greatest element which will always be denoted by $\theta(a_1)$,

iii) $\theta(a) < \delta(\theta(a), x)$ holds for any $a \in A \setminus \{a_0\}$ and $x \in X$ provided that $\delta_x \neq \mathrm{id}_{A/\theta}$ holds in the factor automaton $\mathbf{A}/\theta$.

Observe that for a quasi-nilpotent automaton $\mathbf{A}=(A, X, \delta)$ the condition $\delta_x=\mathrm{id}_{A/\theta}$ is equivalent to the condition that x is a permutation sign of $\mathbf{A}$. Furthermore, if $\mathbf{A}$ is quasi-nilpotent and finite then $(A/\theta, X_{NP}, \delta)$ is nilpotent.

Let $\mathbf{A}=(A, X, \delta)$ be again an arbitrary commutative automaton and let $P(A/\theta)$ denote the power set of A/θ. Define the mapping f: $P(A/\theta) \to P(A/\theta)$ by $f(C)=C \cup \max \bar{C}$ where $\max \bar{C}$ denotes the set of all maximal elements in the complement of C. It is easy to verify that f is a monoton mapping, i.e. $f(C) \subseteq f(C')$ provided $C \subseteq C'$. Thus, by Tarski's fixpoint theorem, (cf. [7]) f has a least fixpoint M'. M' is the smallest subset of A/θ such that $\max \overline{M'}=\emptyset$. Let $M(\mathbf{A})=\bigcup(\theta(a): \theta(a) \in M')$.

On the other hand it is well-known that the least fixpoint of a monoton mapping on a complete lattice can be obtained as the least upper bound of a chain constructed from the least element of the lattice. Applying this construction to f we get $M'=\bigcup_{\alpha} M'_\alpha$ — or equivalently $M'=\bigcup_{\alpha<\beta} M'_\alpha$ — where for an arbitrary ordinal α the set M'_α is defined by transfinite induction as follows:

i) $M'_0=\max A/\theta$,

ii) $M'_\alpha=M'_{\alpha_1}\cup\max \overline{M'_{\alpha_1}}$ if $\alpha=\alpha_1+1$,

iii) $M'_\alpha=\bigcup_{\alpha_1<\alpha} M'_{\alpha_1}$ if $\alpha\neq 0$ is a limit ordinal.

It is obvious — by transfinite induction on α — that M'_α is an upper ideal in $(A/\theta, \leqq)$ and M'_α does not contain ω-chains. (By an ω-chain in a partially ordered set $(R, \leqq)$ we mean a subset $Q=\{q_0, q_1, \ldots\}\subseteq R$ such that $q_0<q_1<\ldots$. ω^{op}-chains are similarly defined just require $q_0>q_1>\ldots$ instead of the above condition.) As M'_α is always an upper ideal in $(A/\theta, \leqq)$ the system $M_\alpha(\mathbf{A})=(\bigcup(\theta(a)\colon \theta(a)\in M'_\alpha), X, \delta)$ is a subautomaton of $\mathbf{A}$. Observe that if $\mathbf{A}$ was a quasi-nilpotent automaton then $M_0(\mathbf{A})=\{a_0\}$ and $M_1(\mathbf{A})=\{a_0\}\cup\theta(a_1)$. If there is no danger of confusion we shall omit $\mathbf{A}$ in $M_\alpha(\mathbf{A})$ and $M(\mathbf{A})$.

A quasi-nilpotent automaton $\mathbf{A}=(A, X, \delta)$ will be called *separable* if for arbitrary states $a\neq b\in A$ such that $\{a, b\}\nsubseteq M_1$ there is a word $p\in X^+_{NP}$ satisfying both $\{\delta(a, p), \delta(b, p)\}\cap M\neq\emptyset$ and $\delta(a, p)\neq\delta(b, p)$.

We are now ready to state our main result.

Theorem. A commutative automaton $\mathbf{A}=(A, X, \delta)$ is subdirectly irreducible if and only if one of the following three conditions is satisfied by $\mathbf{A}$:

(a) $\mathbf{A}$ is a cocyclic automaton,

(b) $\mathbf{A}$ is a separable quasi-nilpotent automaton and the X-subautomaton $(A\setminus\{a_0\}, X_P, \delta)$, i.e. its permutational subautomaton without the absorbent state a_0, is the disjoint sum of pairwise isomorphic cocyclic automata,

(c) $\mathbf{A}$ is the disjoint sum of a cocyclic automaton and an automaton of one state.

Proof. In order to prove the necessity of our Theorem assume that $\mathbf{A}$ is subdirectly irreducible. First we shall consider the case when $\mathbf{A}$ is connected and show that $(A/\theta, \leqq)$ has a greatest element.

As $\mathbf{A}$ is connected there is at most one maximal element in A/θ. Therefore, it is enough to show that each element of A/θ has an upper bound which is maximal. Assume to the contrary that there is no maximal element in the upper ideal B' generated by an element $\theta(a)\in A/\theta$. Let $B=\bigcup(\theta(b)\colon \theta(b)\in B')$. (B, X, δ) is exactly the subautomaton generated by a, i.e. $B=[a]$. Let $b\in B$ be arbitrary. There is a state $b'\in B$ such that $\theta(b)<\theta(b')$, thus $\sigma_{[b]}\neq\Delta_A$. We shall show that $\bigcap(\sigma_{[b]}\colon b\in B)=\Delta_A$.

Suppose that $c\neq d$ and $c\sigma_{[b]}d$ holds for any $b\in B$. Of course we have $c, d\in B$. There is a state $\bar b\in B$ such that $\{c, d\}\nsubseteq[\bar b]$. Indeed, if $\theta(c)=\theta(d)$ then we may choose $\bar b$ such that $\theta(c)<\theta(\bar b)$ if $\theta(c)<\theta(d)$ or $\theta(c)$ and $\theta(d)$ are incomparable then let $\bar b=d$. We supposed that $c\sigma_{[\bar b]}d$. But this is possible only if $c=d$, a contradiction. Therefore, $\bigcap(\sigma_{[b]}\colon b\in B)=\Delta_A$.

Let $\theta(a_0)$ denote the greatest element of A/θ. Since $\theta(a_0)$ is maximal in A/θ $(\theta(a_0), X, \delta)$ is a subautomaton of $\mathbf{A}$, furthermore, by the definition of θ, it is strongly connected. On the other hand we know that $(\theta(a_0), X, \delta)$ has to be a subdirectly irreducible automaton, thus, by the previous statement, it is a cocyclic automaton.

Suppose that $|\theta(a_0)|>1$. We show that in this case $\theta(a_0)=A$, i.e. **A** satisfies condition (a) of our theorem.

Assume that $a\in A$ and $a\notin\theta(a_0)$. Because of $\theta(a)<\theta(a_0)$ there is a word $p\in X^*$ such that $\delta(a,p)=a_0$. Let $\varrho\in C(\mathbf{A})$ be the congruence relation induced by the endomorphism δ_p. As $\delta_{p|\theta(a_0)}$ is a permutation of $\theta(a_0)$ we have $\varrho|_{\theta(a_0)}=$ $=\Delta_{\theta(a_0)}$ and $\varrho\neq\Delta_A$. Thus $\varrho\cap\sigma_{\theta(a_0)}=\Delta_A$. This, by $|\theta(a_0)|>1$ yields that **A** is subdirectly reducible, which is a contradiction.

Now consider the case $\theta(a_0)=\{a_0\}$ and $A\neq\{a_0\}$. By the same order of ideas as we have shown that A/θ has a greatest element one can easily prove that every element of $A/\theta\setminus\theta(a_0)$ has a maximal upper bound in $A/\theta\setminus\theta(a_0)$. But $A/\theta\setminus\theta(a_0)$ can not have two distinct maximal elements, consequently, there exists a greatest element $\theta(a_1)$ in $A/\theta\setminus\theta(a_0)$. Indeed, if both $\theta(a)$ and $\theta(b)$ are maximal in $A/\theta\setminus\theta(a_0)$ then $\sigma_{[a]}\cap\sigma_{[b]}=\Delta_A$ and $\sigma_{[a]},\sigma_{[b]}\neq\Delta_A$ are satisfied, contrary to the subdirect irreducibility of **A**.

Let $\theta(a_1)$ be the greatest element of $A/\theta\setminus\theta(a_0)$. Let us divide X into two disjoint sets X_1 and X_2: $X_1=\{x:\ x\in X, \delta(a_1,x)\in\theta(a_1)\}$, $X_2=\{x:\ x\in X, \delta(a_1,x)=a_0\}$. Since θ is a congruence relation we have $\delta(\theta(a_1),x)\subseteq\theta(a_1)$ if $x\in X_1$ and $\delta(\theta(a_1),x)=\theta(a_0)$ if $x\in X_2$. Hence $\mathbf{A}_1=(\theta(a_1),X_1,\delta)$ is a strongly connected X-subautomaton of **A**. We now show that $\mathbf{A}_1$ is a cocyclic automaton.

Assume that $\mathbf{A}_1$ is subdirectly reducible, i.e. there exist congruence relations $\{\varrho_i\in C(\mathbf{A}_1):\ i\in I\}$ with $\bigcap(\varrho_i:\ i\in I)=\Delta_{\theta(a_1)}$ and $\varrho_i\neq\Delta_{\theta(a_1)}$ $(i\in I)$. Define the congruence relations $\Psi_i\in C(\mathbf{A})$ $(i\in I)$ by the equivalence $a\Psi_i b$ if and only if $a\varrho_i b$ or $a=b$ $(a,b\in A)$. It can be immediately seen that $\bigcap(\Psi_i:\ i\in I)=\Delta_A$ and $\Psi_i\neq\Delta_A$ $(i\in I)$ are satisfied. This contradicts the subdirect irreducibility of **A**. Therefore, $\mathbf{A}_1$ is subdirectly irreducible and thus, by our Statement, it is a cocyclic automaton.

Next we show that δ_x is a permutation of A and $\delta(\theta(a),x)\subseteq\theta(a)$ holds for any $x\in X_1$ and $a\in A$. Indeed, δ_x is injective since otherwise we would have $\sigma_{\theta(a_0)\cup\theta(a_1)}\cap\varrho=\Delta_A$ and $\sigma_{\theta(a_0)\cup\theta(a_1)},\varrho\neq\Delta_A$ where $\varrho\in C(\mathbf{A})$ is the congruence relation induced by the endomorphism δ_x. Now let $a\in A$ be arbitrary and let r^k be the order of δ_x in $S(\mathbf{A}_1)$. Define $\varrho\subseteq A\times A$ by $c\varrho d$ if and only if there is a non-negative integer n such that either $\delta(c,x^{nr^k})=d$ or $\delta(d,x^{nr^k})=c$. It is obvious that ϱ is reflexive and simmetric and has the substitution property, i.e. it is an invariant tolerance relation of **A**. By the injectivity of δ_x, it can be seen that it is transitive as well. Thus $\varrho\in C(\mathbf{A})$. It is not difficult to see that $\varrho\cap\sigma_{\theta(a_0)\cup\theta(a_1)}=\Delta_A$ while $\sigma_{\theta(a_0)\cup\theta(a_1)}\neq\Delta_A$. On the other hand $\varrho\neq\Delta_A$ holds if $a\notin\{\delta_{x^m}(a):\ m\geqq 1\}$. Therefore, for every $x\in X_1$ and $a\in A$ there is an integer $n\geqq 1$ such that $a=\delta(a,x^n)$. Consequently, $\delta(\theta(a),x)\subseteq\theta(a)$ and $x^n=\lambda$ is an identity in $[a]$ implying that δ_x is a permutation of A.

As $X_1\subseteq X_P$ and $X_2\subseteq X_{NP}$ we get $X_1=X_P$ and $X_2=X_{NP}$. We have shown that if $x\in X_P$ then $\delta(\theta(a),x)\subseteq\theta(a)$ holds for each $a\in A$. Conversely, if $\delta(\theta(a),x)\subseteq\theta(a)$ holds for some $a\in A\setminus\theta(a_0)$ then also $\delta(\theta(a_1),x)\subseteq\theta(a_1)$ i.e. $x\in X_P$. This can be seen immediately as follows. As $\delta(\theta(a),x)=\theta(a)$ holds in $\mathbf{A}/\theta$ we obtain that $x=\lambda$ is an identity in $[\theta(a)]$. But $\theta(a_1)\in[\theta(a)]$, thus, $\delta(\theta((a_1),x)=\theta(a_1)$ in $\mathbf{A}/\theta$, i.e. $\delta(\theta(a_1),x)\subseteq\theta(a_1)$ in **A**.

So far we have proved that if **A** is subdirectly irreducible, connected, moreover, $\theta(a_0)=\{a_0\}$ and $A\neq\theta(a_0)$ then it is a quasi-nilpotent automaton. Next we show that in this case $(\theta(a),X_P,\delta)\cong(\theta(a_1),X_P,\delta)$ for any $a\in A\setminus\theta(a_0)$, hence the permutational subautomaton of $\mathbf{A}$ without the absorbent state is the disjoint sum of pairwise isomorphic cocyclic automata.

Indeed, if $a\in A\setminus\theta(a_0)$ then there exists a word $p\in X^*$ such that $\delta(a,p)=a_1$. By commutativity, the mapping $\delta_{p|\theta(a)}$: $\theta(a)\to\theta(a_1)$ is a homomorphism of $(\theta(a), X_P, \delta)$ into $(\theta(a_1), X_P, \delta)$. As $(\theta(a_1), X_p, \delta)$ is strongly connected $\delta_{p|\theta(a)}$ is an epimorphism. Now we shall show that $\delta_{p|\theta(a)}$ is an isomorphism. Assume that $b, c\in\theta(a)$ satisfy the condition $\delta(b,p)=\delta(c,p)=d$. Since $(\theta(a), X_P, \delta)$ is strongly connected there is a word $q\in X_P^*$ such that $\delta(b,q)=c$. By commutativity, $\delta(d,q)=d$, thus, $q=\lambda$ is an identity in $(\theta(a_1), X_P, \delta)$. In other words $\delta_{q|\theta(a_1)}=\mathrm{id}_{\theta(a_1)}$. Let us define the relation $\varrho\in C(\mathbf{A})$ by $u\varrho v$ if and only if there is an integer $n\geqq 0$ such that either $\delta(u,q^n)=v$ or $\delta(v,q^n)=u$. Obviously, $\varrho\cap\sigma_{\theta(a_0)\cup\theta(a_1)}=\Delta_A$, and hence, by the subdirect irreducibility of $\mathbf{A}$, from this it follows that $\varrho=\Delta_A$. Thus $b=c$ and $\delta_{p|\theta(a)}$ is an isomorphism.

It remained to prove that $\mathbf{A}$ is separable. Consider the set Z of all pairs (a,b) $(a\neq b\in A)$ such that $\{a,b\}\subsetneqq\theta(a_0)\cup\theta(a_1)$ and for every word $p\in X_{NP}^+$ if $\delta(a,p)\in M$ then $\delta(a,p)=\delta(b,p)$. We shall show that if $(a,b)\in Z$ and $x\in X_P$ then also $(\delta(a,x),\delta(b,x))\in Z$. Assume to the contrary $(\delta(a,x),\delta(b,x))\notin Z$. There are two cases. Either there is a word $p\in X_{NP}^+$ with $\delta(\delta(a,x),p)\in M$ and $\delta(\delta(b,x),p)\notin M$ or $\delta(a,xp), \delta(b,xp)\in M$ and $\delta(a,xp)\neq\delta(b,xp)$. In the first case, by commutativity and the facts $\delta_x(M)\subseteq M$ and $\delta_x(\overline{M})\subseteqq\overline{M}$ it follows that $\delta(a,p)\in M$ and $\delta(b,p)\notin M$. This contradicts $(a,b)\in Z$. One can get a similar contradiction in the other case, too.

Suppose now that $\mathbf{A}$ is not separable, i.e. $Z\neq\emptyset$. Let $(a,b)\in Z$ and denote by $\varrho\in C(\mathbf{A})$ the congruence relation generated by the pair (a,b). By Malcev's lemma (cf. Theorem 10.3 in [4]), ϱ is the transitive closure of the relation Ψ given by $c\Psi d$ if and only if there is a word $p\in X^*$ with $\{c,d\}\subseteqq\{\delta(a,p),\delta(b,p)\}$ or $c=d$. As $(a,b)\in Z$ and $(\delta(a,p),\delta(b,p))\in Z$ holds for every $p\in X_P^*$ it is not difficult to see that if $\theta(u)>\theta(a)$ and $u\Psi v$ are valid for some states $u, v\in M$ then $u=v$. Consequently, $\varrho_{|([a]\setminus\theta(a))\cap M}=\Delta_{([a]\setminus\theta(a))\cap M}$. If $a\in\theta(a_0)\cup\theta(a_1)$ then $\delta(a,p)=a_0$ holds for each $p\in X_{NP}^+$. Thus $\delta(b,p)=a_0$ is also valid for each $p\in X_{NP}^+$. But this is possible only if $b\in\theta(a_0)\cup\theta(a_1)$ contradicting $\{a,b\}\subsetneqq\theta(a_0)\cup\theta(a_1)$. Therefore $\theta(a)<\theta(a_1)$ and hence $\varrho_{|\theta(a_0)\cup\theta(a_1)}=\Delta_{\theta(a_0)\cup\theta(a_1)}$. Thus $\varrho\cap\sigma_{\theta(a_0)\cup\theta(a_1)}=\Delta_A$, a contradiction.

We have already proved that if $\mathbf{A}$ is a subdirectly irreducible connected commutative automaton then $\mathbf{A}$ satisfies condition (a) or (b) of our Theorem. Assume now that $\mathbf{A}$ is not connected. Then $\mathbf{A}$ is the disjoint sum of its connected subautomata $\mathbf{B}_i=(B_i, X, \delta)$ $(i\in I, |I|\geqq 2)$. We have $\cap(\sigma_{A\setminus B_i}: i\in I)=\Delta_A$ while if $|I|\geqq 3$ or $|I|=2$ and $|B_i|\geqq 2$ $(i\in I)$ then $\sigma_{A\setminus B_i}\neq\Delta_A$ $(i\in I)$. Therefore, $|I|=2$ — say $I=\{1,2\}$ — and $|B_2|=1$. As $\mathbf{B}_1$ has to be a subdirectly irreducible automaton and it is connected, one can show that $\mathbf{B}_1$ is a cocyclic automaton, i.e. $\mathbf{A}$ satisfies condition (c) of our Theorem. This ends the proof of necessity.

Conversely, by our Statement, it is obvious that if $\mathbf{A}$ contents condition (a) or (c) of the Theorem then $\mathbf{A}$ is subdirectly irreducible. Hence assume that condition (b) is satisfied by $\mathbf{A}$.

We shall show that $\varrho_{|\theta(a_0)\cup\theta(a_1)}\neq\Delta_{\theta(a_0)\cup\theta(a_1)}$ holds for each congruence relation $\varrho\in C(\mathbf{A})$ generated by two distinct states $a, b\in A$.

This is quite obvious if $a, b\in\theta(a_0)\cup\theta(a_1)$. Hence suppose that $\{a,b\}\subsetneqq\theta(a_0)\cup\theta(a_1)$ and set $Z=\{\varrho(c): c\in M, |\varrho(c)|>1\}$. Since $\mathbf{A}$ is separable there is a word $p\in X_{NP}^{+}$ such that — say — $\delta(a,p)\in M$ and $\delta(a,p)\neq\delta(b,p)$. Thus $Z\neq\emptyset$. Since M/θ does not contain ω-chains there is a state $c_0\in M$ such that $\varrho(c_0)\in Z$ and $\theta(c_0)\not<\theta(c)$ holds for any $\varrho(c)\in Z$.

Let us distinguish three cases and let $d_0\in\varrho(c_0)$, $d_0\neq c_0$. First assume that $c_0=a_0$. If $d_0\in\theta(a_1)$ we are ready. If $d_0\notin\theta(a_1)$ then there is a word $p\in X_{NP}^{+}$ with $\delta(d_0,p)\in\theta(a_1)$. At the same time $\delta(a_0,p)=a_0$ thus we get $a_0\varrho\delta(d_0,p)$, i.e. $\varrho_{|\theta(a_0)\cup\theta(a_1)}\neq\Delta_{\theta(a_0)\cup\theta(a_1)}$. Secondly assume that $c_0\in\theta(a_1)$. If $d_0\in\theta(a_1)$ then we are again ready. If $\theta(d_0)<\theta(a_1)$ then there is word $p\in X_{NP}^{+}$ such that $\delta(d_0,p)\in\theta(a_1)$. But $\delta(c_0,p)=a_0$ thus, $a_0\varrho\delta(d_0,p)$. Finally, let $c_0\notin\theta(a_0)\cup\theta(a_1)$. By separability, there is a word $p\in X_{NP}^{+}$ with $\delta(c_0,p)\neq\delta(d_0,p)$. But $\delta(c_0,p)\in M$ because (M,X,δ) is a subautomaton of **A** and $\theta(\delta(c_0,p))>\theta(c_0)$ since **A** is quasi-nilpotent. Consequently, $(\delta(c_0,p),\delta(d_0,p))\in Z$ contradicting the maximality of $\theta(c_0)$.

We have proved that every congruence relation $\varrho\in C(\mathbf{A})$ generated by two distinct elements of **A** satisfies $\varrho_{|\theta(a_0)\cup\theta(a_1)}\neq\Delta_{\theta(a_0)\cup\theta(a_1)}$. Therefore, **A** is subdirectly irreducible if and only if $(\theta(a_0)\cup\theta(a_1),X,\delta)$ is subdirectly irreducible. On the other hand $(\theta(a_0)\cup\theta(a_1),X,\delta)$ is subdirectly irreducible. This ends the proof of the Theorem.

Commutative automata with a finite set of input signs

In this section we shall point out that there is a somewhat simpler characterization of subdirect irreducibility in case of commutative automata with a finite set of input signs. Actually, we prove

Corollary 1. Let $\mathbf{A}=(A,X,\delta)$ be a commutative automaton with finite X. Then **A** is subdirectly irreducible if and only if one of the following three conditions are satisfied by **A**:

(a) **A** is a cyclic automaton of prime-power order,

(b) **A** is a quasi-nilpotent automaton and its permutational subautomaton without the absorbent state is the disjoint sum of pairwise isomorphic cyclic automaton of prime-power order, furthermore, for any $a\neq b\in A$ such that $\{a,b\}\nsubseteq\theta(a_0)\cup\theta(a_1)$ there is a sign $x\in X_{NP}$ with $\delta(a,x)\neq\delta(b,x)$,

(c) **A** is the disjoint sum of a cyclic automaton of prime-power order and an automaton of one state.

Proof. The proof follows by our Theorem and the fact that if **A** is quasi-nilpotent then we have $A=M(\mathbf{A})$. This latter equality can be seen by showing that if **A** is quasi-nilpotent then $\mathbf{A}/\theta$ can not contain an ω-chain.

Assume to the contrary **A** is quasi-nilpotent and $\theta(b_0)<\theta(b_1)<\dots$ is an ω-chain in $(A/\theta,\leqq)$. Let $X=\{x_1,\dots,x_r\}$. As $\theta(a_1)$ is the greatest element of $A/\theta\setminus\theta(a_0)$ there is a word $q_n=x_1^{\alpha_1^{(n)}}\dots x_r^{\alpha_r^{(n)}}$ with $\delta(b_n,q_n)=a_1$ for any $n\geqq 0$. Let $\alpha^{(n)}$ denote the vector consisting of the exponents occuring in q_n, i.e. $\alpha^{(n)}=(\alpha_1^{(n)},\dots,\alpha_r^{(n)})$ $(n\geqq 0)$. By induction on t $(t=0,\dots,r)$ we show that there is an infinite sequence of indices $I_t\subseteq\{0,1,\dots\}$ such that $\alpha_s^{(i)}\leqq\alpha_s^{(j)}$ holds if $s\leqq t$ and $i<j\in I_t$. If $t=0$ then let $I_t=\{0,1,\dots\}$. Assume that we have already constructed the set I_{t-1} $(t\geqq 1)$ and consider $\Gamma=\{(\alpha_t^{(i)},\dots,\alpha_r^{(i)})\colon i\in I_{t-1}\}$. Supposing Γ is finite we obtain integers $i<j$ $(i,j\in I_{t-1})$ with $(\alpha_t^{(i)},\dots,\alpha_r^{(i)})=(\alpha_t^{(j)},\dots,\alpha_r^{(j)})$. Let $w=x_1^{\alpha_1^{(j)}-\alpha_1^{(i)}}\dots x_{t-1}^{\alpha_{t-1}^{(j)}-\alpha_{t-1}^{(i)}}$. By commutativity, $\delta(b_i,q_i)=\delta(b_j,q_j)=\delta(b_j,wq_i)=a_1$.

On the other hand, by $\theta(b_i)<\theta(b_j)$, there is a word $p\in X_{NP}X^*$ with $\delta(\theta(b_i), p)=\theta(b_j)$. Or even, we may choose p in such a way that $\delta(b_i, p)=b_j$. Thus $a_0=\delta(\delta(b_i, q_i), p)=\delta(\delta(b_i, p)q_i)=\delta(b_j, q_i)\equiv\delta(b_j, q_i w)=\delta(b_j, wq_i)=a_1$, i.e. $a_0\equiv a_1$ yielding a contradiction. We have shown that Γ is infinite from which the existence of I_t follows.

Now let $I=I_r$ and $i<j$ $(i, j\in I)$. Applying the same sequence of ideas for the corresponding states b_i and b_j one can get a similar contradiction. This ends the proof of Corollary 1.

It is interesting to note that if $\mathbf{A}=(A, X, \delta)$ is a subdirectly irreducible commutative automaton and X is finite then $A=M_\omega(\mathbf{A})$. This can be seen as follows. We have proved that $A=M$ and one can prove in a similar way that there is no commutative automaton $\mathbf{B}$ with a finite set of input signs which is generated by one state such that $(B/\theta, \leqq)$ contains an ω^{op}-chain. Now, to see that $A=M_\omega(\mathbf{A})$ assume to the contrary $\max \overline{M'_\omega}\neq\emptyset$ and let $\theta(a)\in\max\overline{M'_\omega}$. Set $Z=\{\theta(b)\colon \theta(a)<\theta(b)\}$ and let Z_0 consist of all minimal elements of Z (with respect to the ordering $\leqq$). Of course $Z\subseteq M'_\omega$. For every $\theta(b)\in Z_0$ there exists a sign $x\in X$ with $\delta(a, x)\in\theta(b)$. Thus Z_0 is finite, $Z_0=\{\theta(b_1), \dots, \theta(b_n)\}$. On the other hand Z can not contain ω^{op}-chains since otherwise $[a]/\theta$ would contain ω^{op}-chains. Thus, together with the fact that M'_ω is an upper ideal, $Z=\{\theta(b)\colon (\exists i)(i\in\{1, \dots, n\}, \theta(b_i)<\theta(b))\}$. As $M'_\omega=\bigcup_{k<\omega} M'_k$, there corresponds an integer k_i to each $i\in\{1, \dots, n\}$ such that $\theta(b_i)\in M'_{k_i}$. Let $k=\max_{i=1,\dots,n} k_i$. Obviously, $Z_0\subseteq M'_k$ and, since M'_k is an upper ideal as well, $Z\subseteq M'_k$. But in this case if $\theta(b)$ is such that $\theta(a)<\theta(b)$ then $\theta(b)\in M'_k$, therefore, $\theta(a)$ is maximal in $\overline{M'_k}$, too. This results that $\theta(a)\in M'_{k+1}\subseteq M'_\omega$ contradicting our assumption $\theta(a)\in\overline{M'_\omega}$.

Also observe that if $\mathbf{A}=(A, X, \delta)$ is a subdirectly irreducible commutative automaton with finite X and if $\mathbf{A}$ is generated by one state then A is finite, too. Indeed, we know that $A=M_\omega$ holds, thus, $a_0\in M_\omega$ where a_0 denotes an arbitrary generator of $\mathbf{A}$. But $M_\omega=\bigcup_{n<\omega} M_n$, therefore, there is an integer n such that $a_0\in M_n$ and hence, $A=M_n$. On the other hand the finiteness of X implies the finiteness of M_n.

The following simple example shows that the equality $A=M(\mathbf{A})$ does not hold in general for arbitrary subdirectly irreducible commutative automata. Indeed, let $A=\{a_i, b_i\colon i\geqq 0\}$, $X=\{x\}\cup\{y_i\colon i\geqq 0\}$ and let $\delta\colon A\times X\to A$ be defined by:

(a) $$\delta(a_i, x)=\begin{cases} a_{i-1}, & \text{if } i>0\\ a_0, & \text{if } i=0,\end{cases}$$

(b) $$\delta(a_i, y_j)=a_0 \quad (i, j\geqq 0),$$

(c) $$\delta(b_i, x)=b_{i+1},$$

(d) $$\delta(b_i, y_j)=\begin{cases} a_{j-i}, & \text{if } j\geqq i\\ a_0, & \text{if } j<i.\end{cases}$$

It can be seen by an easy computation that $\mathbf{A}=(A, X, \delta)$ is a subdirectly irreducible commutative automaton with $M(\mathbf{A})=\{a_0, a_1, \dots\}$.

Subdirectly irreducible commutative semigroups

Our Theorem makes possible for us to describe all subdirectly irreducible commutative semigroups.

First we note that if S is a commutative semigroup which has no identity then S is subdirectly irreducible if and only if S^1 is subdirectly irreducible where S^1 is S equipped with a new element 1, the identity of S^1. The sufficiency of this statement is obvious and does not require the commutativity of S. Conversely, assume that S^1 is subdirectly reducible, i.e. there are congruence relations $\varrho_i \neq \Delta_{S^1}$ $(i\in I)$ of S^1 such that $\bigcap(\varrho_i;\ i\in I)=\Delta_{S^1}$. We shall show that $\varrho_i|_S \neq \Delta_S$ is satisfied for each $i\in I$. Suppose that $\varrho_i|_S=\Delta_S$. There is exactly one element $s\in S$ with $s\varrho_i 1$. Let $s'\in S$ be arbitrary. As ϱ_i is a congruence relation of $S^1 ss'\varrho_i s'$. As S is closed under composition and $\varrho_{i|S}=\Delta_S$ from this we obtain $ss'=s'$. This means that s is a left identity, and by commutativity, an identity. This contradicts our assumption on S.

In the next corollary we use the notations in accordance with [1]. Observe that the congruence relations θ of the previous section corresponds to the Green's congruence relations $\mathcal{J}$ of commutative semigroups.

Corollary 2. A commutative semigroup S is subdirectly irreducible if and only if one of the following conditions is satisfied by S:

(i) S is a cocyclic group,

(ii) S is a commutative monoid with zero element and

(a) there is a least 0-minimal ideal R in S,

(b) J_1 is a cocyclic group, $(J_s, J_1|J_s)\cong J_1$ under the correspondence $\alpha\to\alpha|J_s$ $(\alpha\in J_1)$ if $s\neq 0$ furthermore, $J_s\neq J_sJ_{s'}$, for arbitrary $s\in S\setminus\{0\}$ and $s'\in S\setminus J_1$,

(c) for any $\{s_1, s_2\}\subsetneqq R$ $(s_1\neq s_2)$ there is an element $s\in S\setminus J_1$ with $\{s_1 s, s_2 s\}\cap$ $\cap M\neq\emptyset$ and $s_1 s\neq s_2 s$ where M denotes the least ideal in S such that $\overline{M}/\mathcal{J}$ does not contain maximal elements with respect to the ordering $J_s\leqq J_{s'}$ if and only if $s|s'$ $(s, s'\in S)$,

(iii) S does not contain identity element and S^1 satisfies condition (ii) with $J_1=\{1\}$.

Every finitely generated subdirectly irreducible commutative semigroup is finite.

Proof. By our Theorem, Theorem 1 in [2], the representation theorem of semigroups and our previous remarks.

This Corollary implies Corollaries IV.7.4. and IV.7.5. in [6].

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