On attributed tree transducers

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Introduction

The concept of attribute grammar was introduced by Knuth in [1] as a formal tool for defining the meaning of sentences generated by a context free grammar. Taking trees over some ranked alphabet instead of derivation trees of a context free grammar, and allowing the values of attributes to be only trees over another ranked alphabet, finally, restricting the semantic functions to tree-concatenation we obtain the notion of attributed tree translators.

In this paper we study some basic properties of attributed tree transformations. Namely, we point out that each completely defined top-down tree transformation can be induced by an attributed tree translator while the class of all completely defined bottom-up tree transformations and the class of all attributed tree transformations are incomparable. Finally, we prove some results concerning the composition of attributed tree transformations.

I. Notions and notations

Before turning to the discussion of attributed tree transducers we recall some fundamental notions and notations.

By a type, or ranked alphabet, we mean a finite nonempty set F of the form $F=F_0\cup F_1\cup\ldots\cup F_{\nu(F)}$, where the sets F_n $(n=0, \ldots, \nu(F))$ are pairwise disjoint. The elements of F_n are called n-ary operator symbols.

For arbitrary ranked alphabet F and set S the set of trees over S of type F is the smallest set $T_F(S)$ satisfying

(i) $F_0\cup S\subseteq T_F(S)$ and

(ii) if $f\in F_n$ $(n\geqq 0)$; $p_1, \ldots, p_n\in T_F(S)$ then $f(p_1, \ldots, p_n)\in T_F(S)$.

We can define the height $(\mathrm{ht}(p))$, rank $(\mathrm{rn}(p))$, root $(\mathrm{root}(p))$ and the set of subtrees $(\mathrm{sub}(p))$ of a tree $p(\in T_F(S))$ as follows: if $p\in F_0\cup S$ then $\mathrm{ht}(p)=0$, $\mathrm{rn}(p)=1$, $\mathrm{root}(p)=p$ and $\mathrm{sub}(p)=\{p\}$ else, if p is of form $f(p_1, \ldots, p_n)$ for some $n(\geqq 1)$ and $f(\in F_n)$, then $\mathrm{ht}(p)=\max\{\mathrm{ht}(p_j)\mid 1\leqq j\leqq n\}+1$, $\mathrm{rn}(p)=1+\sum_{i=1}^{n}\mathrm{rn}(p_i)$, $\mathrm{root}(p)=f$ and $\mathrm{sub}(p)=\left(\bigcup_{i=1}^{n}\mathrm{sub}(p_i)\right)\cup\{p\}$.

Next we define the set path (p) of paths being in p as a subset of $\mathbf{N}^*$ (where $\mathbf{N}^*$ is the free monoid generated by the natural numbers, with identity λ) in the following way:

$$\operatorname{path}(p)=\begin{cases}\{\lambda\} & \text{if} \quad p\in F_0\cup S\\ \left(\bigcup\limits_{j=1}^{n}\{jw\,|\,w\in\operatorname{path}(p_j)\}\right)\cup\{\lambda\} & \text{if} \quad p=f(p_1,\dots,p_n).\end{cases}$$

There is a corresponding label $\mathrm{lb}_p(w)$ and a subtree $\mathrm{str}_p(w)$ for each path w in a tree $p(\in T_F(S))$. They are defined as follows:

$$\mathrm{lb}_p(w)=\begin{cases}\operatorname{root}(p) & \text{if} \quad w=\lambda\\ \mathrm{lb}_{p_j}(v) & \text{if} \quad w=jv,\ p=f(p_1,\dots,p_n), \quad 1\leqq j\leqq n,\end{cases}$$

and

$$\mathrm{str}_p(w)=\begin{cases}p & \text{if} \quad w=\lambda\\ \mathrm{str}_{p_j}(v) & \text{if} \quad w=jv,\ p=f(p_1,\dots,p_n), \quad 1\leqq j\leqq n.\end{cases}$$

In the rest of the paper the pairwise disjoint sets of variables $X=\{x_1,x_2,\dots\}$, $Y=\{y_1,y_2,\dots\}$, $U=\{u_1,u_2,\dots\}$ and $Z=\{z_0,z_1,\dots\}$ are kept fix. The variables, $z_0,z_1,\dots$ are used as auxiliary variables. For an arbitrary integer $n(\geqq 0)$ the notations X_n, Y_n, U_n, Z_n are used to denote the sets $\{x_1,\dots,x_n\}$, $\{y_1,\dots,y_n\}$, $\{u_1,\dots,u_n\}$, $\{z_1,\dots,z_n\}$, respectively.

If at most the auxiliary variables $z_1,\dots,z_l$ $(l\geqq 0)$ appear in a tree p, then p is also denoted by $p(z_1,\dots,z_l)$. Substituting the elements $s_1,\dots,s_l$ of a set S for the auxiliary variables $z_1,\dots,z_l$ in a tree $p(z_1,\dots,z_l)$, respectively, we obtain another tree which is denoted by $p(s_1,\dots,s_l)$.

Sets of form $T(\subseteq T_F(X_n)\times T_G(Y_m))$ $(n,m\geqq 0)$ are called tree transformations and if $(p,q)\in T$ then q is called an image of p.

By a bottom-up tree transducer we mean a system $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', P)$ where $n,m\geqq 0$ are integers, A is a nonempty finite set, $A'\subseteq A$, finally, P is a finite set of rules (or rewriting rules) having one of the following two forms:

(a) $f(a_1z_1,\dots,a_kz_k)\to a\bar{q}(z_{i_1},\dots,z_{i_l})$ where $k,l\geqq 0$; $f\in F_k$; $a,a_1,\dots,a_k\in A$; $\bar{q}\in T_G(Y_m\cup Z_l)$; $1\leqq i_1,\dots,i_l\leqq k$, and

(b) $x_j\to a\bar{q}$ where $1\leqq j\leqq n$, $a\in A$, $\bar{q}\in T_G(Y_m)$.

If there is a rule of form (a) in P for each $k(\geqq 0)$, $f(\in F_k)$; $a_1,\dots,a_k$ $(\in A)$ as well as a rule of form (b) for each j $(1\leqq j\leqq n)$, then $\mathbf{A}$ is said to be completely defined. Furthermore, if different rules have different left sides, then $\mathbf{A}$ is called deterministic. Let $p,q\in T_F(X_n\cup(A\times T_G(Y_m)))$. We say that q is directly derived from p — written $p\underset{\mathbf{A}}{\Rightarrow}q$ — if q appears from p in one of the following two ways:

(i) the tree $a\bar{q}(p_{i_1},\dots,p_{i_l})$ is substituted for a subtree $f(a_1p_1,\dots,a_kp_k)$ of p and the rule (a) is in P;

(ii) the tree $a\bar{q}$ is substituted for a subtree x_j of p and the rule (b) is in P.

Let us denote by $\overset{*}{\underset{\mathbf{A}}{\Rightarrow}}$ the reflexive, transitive closure of the relation $\underset{\mathbf{A}}{\Rightarrow}$. Then the transformation $T(\mathbf{A})$ induced by $\mathbf{A}$ is:

$$T(\mathbf{A})=\{(p,q)\,|\,p\in T_F(X_n),\ q\in T_G(Y_m) \ \text{ and } \ p\overset{*}{\underset{\mathbf{A}}{\Rightarrow}}aq \ \text{ for } \ \text{some } a(\in A')\}.$$

Another type of tree-transducers is the top-down tree transducer. The system $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), A', P))$ is called a top-down tree transducer if A is a finite nonempty set, $A' \subseteq A$ and finally P is a finite set of rules of the following two forms:

(c) $af(z_1, \ldots, z_k) \to \bar{q}(a_1 z_{i_1}, \ldots, a_l z_{i_l})$ where $k, l \geqq 0$; $a, a_1, \ldots, a_l \in A$; $f \in F_k$; $1 \leqq i_1, \ldots, i_l \leqq k$; $\bar{q} \in T_G(Y_m \cup Z_l)$;

(d) $ax_j \to \bar{q}$ where $a \in A$, $1 \leqq j \leqq n$, $\bar{q} \in T_G(Y_m)$.

Consider the trees $p, q(\in T_G(Y_m \cup (A \times T_F(X_n))))$. The relation $\underset{\mathbf{A}}{\Rightarrow}$ is now defined as follows: $p \underset{\mathbf{A}}{\Rightarrow} q$ if q appears from p

(i) by substituting the tree $\bar{q}(a_1 p_{i_1}, \ldots, a_l p_{i_l})$ for a subtree $af(p_1, \ldots, p_k)$ of p if the rule (c) is in P, or

(ii) by substituting the tree $\bar{q}$ for a subtree ax_j of p if the rule (d) is in P.

Again, $\overset{*}{\underset{\mathbf{A}}{\Rightarrow}}$ denotes the reflexive, transitive closure of $\underset{\mathbf{A}}{\Rightarrow}$ and the transformation $T(\mathbf{A})$ induced by $\mathbf{A}$ is given by

$$T(\mathbf{A}) = \{(p, q) \mid p \in T_F(X_n),\ q \in T_G(Y_m) \quad \text{and} \quad ap \overset{*}{\underset{\mathbf{A}}{\Rightarrow}} q \quad \text{for some } a(\in A')\}.$$

If for all $a(\in A)$, $k(\geqq 0)$, $f(\in F_k)$ there is a rule of form (c) in P, moreover, for all $a(\in A)$, $j(=1, \ldots, n)$ there is a rule of form (d) in P then $\mathbf{A}$ is called completely defined. Finally, if different rules have different left sides and A' is a singleton set then $\mathbf{A}$ is called deterministic.

The cardinality of a set S is denoted by $|S|$ and we write s instead of the singleton $\{s\}$.

II. Attributed tree transducers

We now introduce the concept of attributed tree transducers.

Definition 2.1. The system $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), A_s', P, rt))$ where $n, m \geqq 0$ is called an attributed tree transducer — shortly, AT transduder — provided

(a) F and G are ranked alphabets;

(b) A is a finite set, the set of attributes which can be written in the form $A = A_s \cup A_i$ where A_s is the set of synthesized, A_i is the set of inherited attributes with $A_s \cap A_i = \emptyset$;

(c) $A_s' \subseteq A_s$;

(d) rt is a mapping of A_i into nonempty, finite subsets of $T_G(Y_m)$ (if $A_i = \emptyset$ then rt is not specified);

(e) the set of rules $P = (\bigcup_{f \in F} P_f) \cup \left(\bigcup_{j=1}^{n} P_{x_j}\right)$ is a finite subset of the set $(A \times \times (T_F(Z) \cup X_n)) \times T_G(Y_m \cup (A \times Z))$. For the sets P_f, for all $k(\geqq 0)$ and $f(\in F_k)$, it holds:

(i) for each $a(\in A_s)$ at least one rule of the form $af(z_1, \ldots, z_k) \leftarrow q(a_1 z_{i_1}, \ldots, a_l z_{i_l})$ $(l \geqq 0;\ 0 \leqq i_1, \ldots, i_l \leqq k;\ a_1, \ldots, a_l \in A;\ q \in T_G(Y_m \cup Z_l))$ is in P_f,

(ii) for each $a(\in A_i)$ and $1 \leqq j \leqq k$ at least one rule of the form $az_j \leftarrow q(a_1 z_{i_1}, \ldots, a_l z_{i_l})$ $(l \geqq 0;\ 0 \leqq i_1, \ldots, i_l \leqq k;\ a_1, \ldots, a_l \in A;\ q \in T_G(Y_m \cup Z_l))$ is in P_f,

(iii) P_f contains only rules of type (i) and (ii). For P_{x_j} (for each $j\ (=1, \ldots, n)$) it holds that for any $a(\in A_s)$ at least one rule of form $ax_j \leftarrow q(a_1 z_0, \ldots, a_l z_0)$ is in P_{x_j} and there is no other rule in P_{x_j}. (Observe that here, as well as in the rest of this paper, the elements (x, y) of P are written $x \leftarrow y$.)

If we write "one and only one" instead of "at least one" in (e) moreover require A_s' and $rt(a)$ (for each $a \in A_i$) to be a singleton then we obtain the concept of deterministic AT transducer.

Now let **A** be an AT transducer defined in 2.1 and take the trees $p(\in T_F(X_n \cup Z))$, $q, r(\in T_G(Y_m \cup (A \times \text{path}(p)) \cup (A \times Z)))$. We say that r is directly derived from q in p — and write $q \underset{p,\mathbf{A}}{\Leftarrow} r$ — if r appears from q by one of the following manners:

(a) substituting the tree $\bar{q}((a_1, v_1), \ldots, (a_l, v_l))$ for some leaf $(a, w)(\in A \times \text{path}(p))$ of q if the following conditions hold:

(i) $a \in A_s$,

(ii) $\text{lb}_p(w) = f(\in F_k$ for some $k \geqq 0)$,

(iii) $af(z_1, \ldots, z_k) \leftarrow \bar{q}(a_1 z_{i_1}, \ldots, a_l z_{i_l}) \in P_f$,

(iv) $v_j = \begin{cases} w & \text{if } i_j = 0 \\ wi_j & \text{if } 1 \leqq i_j \leqq k \quad (j = 1, \ldots, l); \end{cases}$

(b) substituting the tree $\bar{q}((a_1, w), \ldots, (a_l, w))$ for some leaf (a, w) of q if

(i) $a \in A_s$,

(ii) $\text{lb}_p(w) = x_j\ (\in X_n)$,

(iii) $ax_j \leftarrow \bar{q}(a_1 z_0, \ldots, a_l z_0) \in P_{x_j}$ hold;

(c) substituting the tree $\bar{q}((a_1, v_1), \ldots, (a_l, v_l))$ for some leaf (a, w) of q if the following conditions hold:

(i) $a \in A_i$,

(ii) $w = vj$ ($v \in \mathbf{N}^*$, $j \in \mathbf{N}$ where $\mathbf{N}$ is the set of natural numbers),

(iii) $\text{lb}_p(v) = f \quad (\in F_k$ for some $k \geqq 1)$,

(iv) $1 \leqq j \leqq k$,

(v) $az_j \leftarrow \bar{q}(a_1 z_{i_1}, \ldots, a_l z_{i_l}) \in P_f$,

(vi) $v_t = \begin{cases} v & \text{if } i_t = 0 \\ vi_t & \text{if } 1 \leqq i_t \leqq k \quad (t = 1, \ldots, l); \end{cases}$

(d) substituting a tree in $rt(a)$ for some leaf (a, w) of q if

(i) $w = \lambda$,

(ii) $a \in A_i$ hold;

(e) substituting the tree (a, z_j) for some leaf (a, w) if $\text{lb}_p(w) = z_j(\in Z)$ holds.

Let $\overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}}$ denote the reflexive and transitive closure of the relation $\underset{p,\mathbf{A}}{\Leftarrow}$. (Sometimes, if **A** is clear, instead of the notations $\underset{p,\mathbf{A}}{\Leftarrow}, \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}}$ we simply write $\underset{p}{\Leftarrow}, \overset{*}{\underset{p}{\Leftarrow}}$, respectively.)

Definition 2.2. Let **A** be the AT transducer defined in 2.1. By the transformation induced by **A** we mean the set

$$T(\mathbf{A}) = \{(p, q) \mid p \in T_F(X_n),\ q \in T_G(Y_m),\ (s_0, \lambda) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} q \text{ for some } s_0(\in A_s')\}.$$

Observe that, in order to define the transformation induced by an AT transducer, it would have been enough to introduce the concept of derivation in a simpler way. Namely, it would have been enough to take p from $T_F(X_n)$ and the trees q, r from $T_G(Y_m(\cup A\times \text{path}(p)))$ — hence, (e) would have disappeared. The previously given more general notion of derivation will be needed only in Section IV.

Definition 2.3. Let **A** be an AT transducer. We say that **A** is circular if there exist $p(\in T_F(X_n))$, $q(\in T_G(Y_m\cup A\times \text{path}(p)))$ and $(a, w)(\in A\times \text{path}(p))$ such that $(a, w)\overset{+}{\underset{p,\mathbf{A}}{\Leftarrow}} q$ holds and (a, w) occurs in q as a leaf (where $\overset{+}{\underset{p,\mathbf{A}}{\Leftarrow}}$ is the transitive closure of $\underset{p,\mathbf{A}}{\Leftarrow}$). D. E. KNUTH has pointed out in [1] that the circularity problem of attribute grammars is decidable. The algorithm presented by Knuth, with a small modification, is suitable to decide whether an AT transducer is circular or not. In the rest of this paper we shall always confine ourselves to noncircular AT transducers.

Therefore, it is clear that for an arbitrary AT transducer $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), A'_s, P, rt))$, and for each $p(\in T_F(X_n))$ and $(a, w)(\in A\times \text{path}(p))$ there exists a tree $q(\in T_G(Y_m))$ (if **A** is deterministic then only one) for which $(a, w)\overset{*}{\underset{p}{\Leftarrow}} q$ holds. Thus we may say that **A** is completely defined and this way of speaking is in accordance with the discussion of bottom-up and top-down tree transducers. Since **A** is completely defined it is clear that the domain of $T(\mathbf{A})$ is the set $T_F(X_n)$. Furthermore, if **A** is deterministic then $T(\mathbf{A})$ is a mapping of $T_F(X_n)$ into $T_G(Y_m)$.

Definition 2.4. The AT transducer (defined in 2.1) is called reduced if the following two conditions are satisfied by any leaf $(a, z)\in(A\times Z)$ appearing on the right side of a rule in P:

(i) if $z=z_0$ then $a\in A_i$,

(ii) if $z\in Z-\{z_0\}$ then $a\in A_s$.

Concerning attribute grammars the property being reduced means that no semantic rule may depend on a synthesyzed attribute of the left side or an inherited attribute of a nonterminal appearing in the right side of the corresponding context-free rewritting rule.

It is easy to show that for every AT transducer $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), A'_s, P, rt))$ there exists an AT transducer $\mathbf{A}'(=(T_F(X_n), A, T_G(Y_m), A'_s, P', rt))$ which is reduced and equivalent to **A** in the sense that $T(\mathbf{A})=T(\mathbf{A}')$. P' can be obtained from P by a suitable substituting of rules in P in each other, and this process will terminate because **A** is noncircular.

Similary to the concept of dependency graph introduced by D. E. KNUTH in [1], for every AT transducer **A**, each derivation $(a, w)\overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} q$ can be represented by a directed graph. The nodes of this graph are the elements of $A\times \text{path}(p)$, moreover, if, in the derivation mentioned above some leaf (b, v) is substituted by some tree $\bar{q}((b_1, v_1), \ldots, (b_l, v_l))$, then there are directed arcs from nodes $(b_1, v_1), \ldots, (b_l, v_l)$ to the node (b, v). This representation of derivation makes the notions and proofs clearer. E.g. the notion of circularity means that the dependency directed graph corresponding to some derivation contains a directed circle.

We shall name the elements of $A \times \text{path}\,(p)$ attribute occurences in accordance with the above representation.

Further on we shall not always study the properties of AT transducers on the whole input set $T_F(X_n)$. The restriction of $T(\mathbf{A})$ to some $R(\subseteq T_F(X_n))$ will be denoted by $T(\mathbf{A})|_R$.

EXAMPLE 2.1. Let $n=m=3$ and $\mathbf{A}=(T_F(X_3), A, T_F(X_3), s_0, P, rt)$ where

(i) $F = F_1 \cup F_2, \quad F_1 = \{g\}, \qquad F_2 = \{f\};$

(ii) $A = A_s \cup A_i, \quad A_s = \{s_0, s_1\}, \quad A_i = \{i\};$

(iii) $P = P_g \cup P_f \cup \left(\bigcup_{j=1}^{3} P_{x_j} \right),$

$$P_g = \{s_0 g(z_1) \leftarrow g(s_0 z_1),\ s_1 g(z_1) \leftarrow \text{arbitrary tree},\ iz_1 \leftarrow s_1 z_1\},$$

$$P_f = \{s_0 f(z_1, z_2) \leftarrow s_0 z_2,\ s_1 f(z_1, z_2) \leftarrow s_1 z_1,\ iz_1 \leftarrow \text{arbitrary tree},\ iz_2 \leftarrow f(iz_0, s_1 z_2)\},$$

$$P_{x_j} = \{s_0 x_j \leftarrow iz_0,\ s_1 x_j \leftarrow x_j\}, \quad j = 1, 2, 3;$$

(iv) rt is an arbitrary mapping.

It is obvious that $\mathbf{A}$ is a deterministic and reduced AT transducer. Take the tree $p = g(f(x_2, f(x_1, x_3)))(\in T_F(X_3))$. The derivation $(s_0, \lambda) \underset{p}{\Leftarrow} g((s_0, 1)) \underset{p}{\Leftarrow} g((s_0, 12)) \underset{p}{\Leftarrow} g((s_0, 122)) \underset{p}{\Leftarrow} g((i, 122)) \underset{p}{\Leftarrow} g(f((i, 12), (s_1, 122))) \underset{p}{\Leftarrow} g(f(f((i, 1), (s_1, 12)), (s_1, 122))) \underset{p}{\Leftarrow} g(f(f((s_1, 1), (s_1, 12)), (s_1, 122))) \overset{*}{\underset{p}{\Leftarrow}} g(f(f(x_2, x_1), x_3)) = q$ holds, consequently $(p, q) \in T(\mathbf{A})$ (see Figure 1).

We can see the directed graph corresponding to this derivation in Figure 2. The path components of the elements of $A \times \text{path}\,(p)$ are left for the sake of clarity.

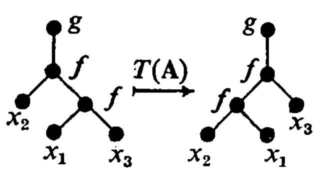

Fig. 1

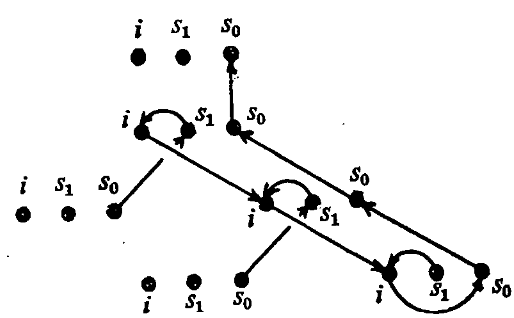

Fig. 2

Let us introduce the notation

$$R = \{g(f(x_{i_1}, f(x_{i_2}, \ldots, f(x_{i_{n-1}}, x_{i_n})\ldots)))| n \geqq 2,\ 1 \leqq i_1, \ldots, i_n \leqq 3\}.$$

One can show that

$$T(\mathbf{A})|_R = \{(g(f(x_{i_1}, \ldots, f(x_{i_{n-1}}, x_{i_n})\ldots)), g(f(f\ldots f(x_{i_1}, x_{i_2}), \ldots, x_{i_{n-1}}), x_{i_n}))| n \geqq 2\},$$

hence $\mathbf{A}$ does not change the frontier of trees of R.

III. Comparing between AT transducers and classical tree transducers

Let us denote the class of all tree transformations induced by
(i) AT transducers,
(ii) AT transducers having only synthesized attributes,
(iii) deterministic AT transducers,
(iv) deterministic AT transducers having only synthesized attributes,
(v) completely defined top-down tree transducers,
(vi) completely defined deterministic top-down tree transducers,
(vii) completely defined bottom-up tree transducers,
(viii) completely defined deterministic bottom-up tree transducers

by
(i) $\mathcal{TA}$,
(ii) $\mathcal{TA}_s$,
(iii) $\mathcal{TDA}$,
(iv) $\mathcal{TDA}_s$,
(v) $\mathcal{TT}$,
(vi) $\mathcal{TDT}$,
(vii) $\mathcal{TB}$,
(viii) $\mathcal{TDB}$.

Before we shall go further let us introduce the concept of length of a derivation. Let $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), A'_s, P, rt))$ be an AT transducer and let $p(\in T_F(X_n))$, $(a, w)(\in A\times \text{path}(p))$ and $q(\in T_G(Y_m))$ satisfy the derivation $d=(a, w)\overset{*}{\underset{p}{\Leftarrow}} q$. The length $\text{lt}(d)$ of the derivation d is the least integer $n(\geqq 1)$ such that $(a, w)\overset{n}{\underset{p}{\Leftarrow}} q$, where $\overset{n}{\underset{p}{\Leftarrow}}$ denotes the n-th power of the relation $\underset{p}{\Leftarrow}$.

By induction on the length of derivation it is easy to prove:

Lemma 3.1. Let $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), A'_s, P))$ be a reduced AT transducer satisfying $A_i=\emptyset$. Then the following equivalence holds for each $p(\in T_F(X_n))$, $(a, w)(\in A\times\text{path}(p))$, $q(\in T_G(Y_m))$ and partition $w=uv$

$$(a, w)\overset{*}{\underset{p}{\Leftarrow}} q \quad \text{if and only if} \quad (a, v)\overset{*}{\underset{\text{str}_p(u)}{\Longleftarrow}} q. \quad \square$$

The next theorem has essentially appeared in [2] but we mention it for the sake of completness.

Theorem 3.1. $\mathcal{TT}=\mathcal{TA}_s$

Proof. First we are going to show that $\mathcal{TT}\subseteq\mathcal{TA}_s$. Indeed, let $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), A', P))$ be a completely defined top-down tree transducer. Consider the AT transducer $\mathbf{B}(=(T_F(X_n), B, T_G(Y_m), B'_s, P'))$ where
(i) $B=B_s=A$,
(ii) $B'_s=A'$,
(iii) $P'=P$.

It is easy to show by induction on $\text{ht}(p)$, and making use of Lemma 3.1, that for any $p(\in T_F(X_n))$, $a(\in A)$ and $q(\in T_G(Y_m))$

$$ap\overset{*}{\underset{\mathbf{A}}{\Rightarrow}} q \quad \text{if and only if} \quad (a, \lambda)\overset{*}{\underset{p,\mathbf{B}}{\Leftarrow}} q,$$

consequently, $T(\mathbf{A})=T(\mathbf{B})$. Conversely, take an arbitrary completely defined AT transducer $\mathbf{B}(=(T_F(X_n), B, T_G(Y_m), B'_s, P'))$ with $B_i=\emptyset$. We may assume without loss of the generality that $\mathbf{B}$ is reduced. Define $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), A', P))$ by the equalities from (i) to (iii). Then, as earlier, we have $T(\mathbf{A})=T(\mathbf{B})$. This proves $\mathcal{TA}_s \subseteq \mathcal{TT}$. □

It is obvious that if $\mathbf{A}$ was deterministic in Theorem 3.1 then $\mathbf{B}$ would have been deterministic and conversely, hence we have

Corollary 3.1. $\mathcal{TDT}=\mathcal{TDA}_s$. □

However, it is easy to see that the tree transformation given in Example 2.1. can not be induced by a (deterministic) top-down tree transducer. Therefore, it is valid

Corollary 3.2. $\mathcal{TT}\subset\mathcal{TA}$ and $\mathcal{TDT}\subset\mathcal{TDA}$. □

Now we are going to see that these inclusions are not true in the bottom-up case.

Theorem 3.2. The class $\mathcal{TDB}$ and $\mathcal{TDA}$ are incomparable.

Proof. The tree transformation given in Example 2.1. is in $\mathcal{TDA}$ but it can not be induced by deterministic bottom-up tree transducers.

On the other hand the following deterministic bottom-up tree transformation will not be in $\mathcal{TDA}$.

Let $\mathbf{A}(=T_F(X_2), A, T_{F'}(X_2), A', P))$ be the bottom-up tree transducer where

(i) $F=F_1=\{f, g\}$, $F'=F'_1=\{f_1, f_2, g\}$;

(ii) $A=A'=\{a_1, a_2\}$;

(iii) P consists of the following rules:

$$x_1 \to a_1 x_1, \qquad x_2 \to a_2 x_2,$$
$$g(a_1 z_1) \to a_1 g(z_1), \quad g(a_2 z_1) \to a_2 g(z_1),$$
$$f(a_1 z_1) \to a_1 f_1(z_1), \quad f(a_2 z_1) \to a_2 f_2(z_1).$$

It is obvious that $\mathbf{A}$ is completely defined and deterministic. Consider $T(\mathbf{A})|_R$ where

$$R = \{f^n g^m(x_1)|n, m \geqq 0\} \cup \{f^n g^m(x_2)|n, m \geqq 0\}.$$

It is easy to see that

$$T(\mathbf{A})|_R = \{(f^n g^m(x_1), f_1^n g^m(x_1))|n, m \geqq 0\} \cup \{(f^n g^m(x_2), f_2^n g^m(x_2))|n, m \geqq 0\}.$$

Suppose that $T(\mathbf{A})$ is in $\mathcal{TDA}$ i.e. $T(\mathbf{A})$ can be induced by a deterministic AT transducer $\mathbf{B}(=(T_F(X_2), B, T_{F'}(X_2), b_0, P', rt))$ and suppose that $\mathbf{B}$ is reduced. Then $T(\mathbf{A})|_R=T(\mathbf{B})|_R$, necessarily. Let

$$K = |B_s|, \quad L = |B_i| \quad (\text{where } B = B_s \cup B_i),$$

$N=\max\{\mathrm{ht}(q)|q$ is the right side of some rule of $P'\}$, let $n>2NL(K+L)$ be fixed, and consider the trees $p_j^{(1)}=f^n g^j(x_1)$, $q_j^{(1)}=f_1^n g^j(x_1)$, $p_j^{(2)}=f^n g^j(x_2)$, $q_j^{(2)}=f_2^n g^j(x_2)$ for all $j\ (=0, 1, \ldots, L)$. (In the special case when the operator symbols appearing in some tree p are of arity 0 or 1, p is called unary. If p is unary then the elements of path (p) are of the form 1^l, further on simply written l.)

Now let us fix an arbitrary index $j(=0, 1, \ldots, L)$ and denote $p_j^{(1)}, q_j^{(1)}, p_j^{(2)}, q_j^{(2)}$ by $p^{(1)}, q^{(1)}, p^{(2)}, q^{(2)}$, respectively. Then $(p^{(1)}, q^{(1)}) \in T(\mathbf{B})$, i.e. $(b_0, 0) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)}$. This derivation can be written as

$$(b_0, 0) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q((s, n+j)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)} \tag{1}$$

for some $q(\in T_{F'}(Z_1))$ and $s(\in B_s)$, since otherwise the derivation $(b_0, 0) \overset{*}{\underset{p^{(2)}}{\Leftarrow}} q^{(1)}$ would be true, and it is obviously a contradiction. (1) means that the derivation is to depend on some synthesized attribute of x_1. Furthermore, as $\mathbf{B}$ was reduced, we may suppose that for any tree $\bar{q}((b, w))$ $(\bar{q} \in T_{F'}(Z_1), (b, w) \in B \times \text{path}(p^{(1)}))$ if

$$(b_0, 0) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} \bar{q}((b, w)) \overset{+}{\underset{p^{(1)}}{\Leftarrow}} q((s, n+j)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)}$$

then $w < n+j$ is true. On the other hand we must have $q = z_1$ i.e.

$$(b_0, 0) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} (s, n+j) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)}. \tag{2}$$

If (2) would not be true then, by $(b_0, 0) \overset{*}{\underset{p^{(2)}}{\Leftarrow}} q((s, n+j))$, we would have trees $\bar{q}^{(1)}, \bar{q}^{(2)} (\in T_{F'}(X_2))$ with $q(\bar{q}^{(1)}) = q^{(1)}$ and $q(\bar{q}^{(2)}) = q^{(2)}$, yielding a contradiction.

In Fig. 3, a heavy line views of the directed graph corresponding to the derivation $(b_0, 0) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} (s, n+j)$. Take into account that in case of unary input and output trees the directed graph corresponding to any derivation is a directed "line".

Now we are going to study the derivation $(s, n+j) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)}$. Since there are n operator symbols f_1 in $q^{(1)}$ and $n > 2NL(K+L)$, for some $c(\in B_i)$ and $r(\in T_{F'}(Z_1))$ we have

$$(s, n+j) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} r((c, n-L)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)} \tag{3}$$

and

(i) $r = z_1$ or r contains operator symbols f_1 only,
(ii) if for some tree $\bar{q}((b, w))$ the derivation

$$(s, n+j) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} \bar{q}((b, w)) \overset{+}{\underset{p^{(1)}}{\Leftarrow}} r((c, n-L)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)}$$

holds then $w > n-L$.

This follows easily by taking into account that each attribute occurence may appear at most once in a derivation and at most $2L(K+L)$ attribute occurences are in the top $n-L$-th level, moreover, that $\mathbf{B}$ is reduced.

As $\mathbf{B}$ is reduced there exist trees $r_l(\in T_{F'}(Z_1))$ and attributes $i_l(\in B_i)$ $l(=0, 1, \ldots, L)$ such that

$$\begin{aligned}(s, n+j) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} r_0((i_0, n)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} r_1((i_1, n-1)) \ldots \overset{*}{\underset{p^{(1)}}{\Leftarrow}} r_L((i_L, n-L)) = \\ = r((c, n-L)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)},\end{aligned} \tag{4}$$

furthermore, if for some tree $\bar{q}((b, w))$ $(s, n+j) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} \bar{q}((b, w)) \overset{+}{\underset{p^{(1)}}{\Leftarrow}} r_0((i_0, n))$ then $w > n$.

Consider the attributes $i_0, i_1, \ldots, i_L(\in B_i)$. Since $L = |B_i|$ there exist indices k, l $(0 \leqq) k < l (\leqq L)$ with $i_k = i_l$. Let $i = i_k (= i_l)$, then (4) can be written as

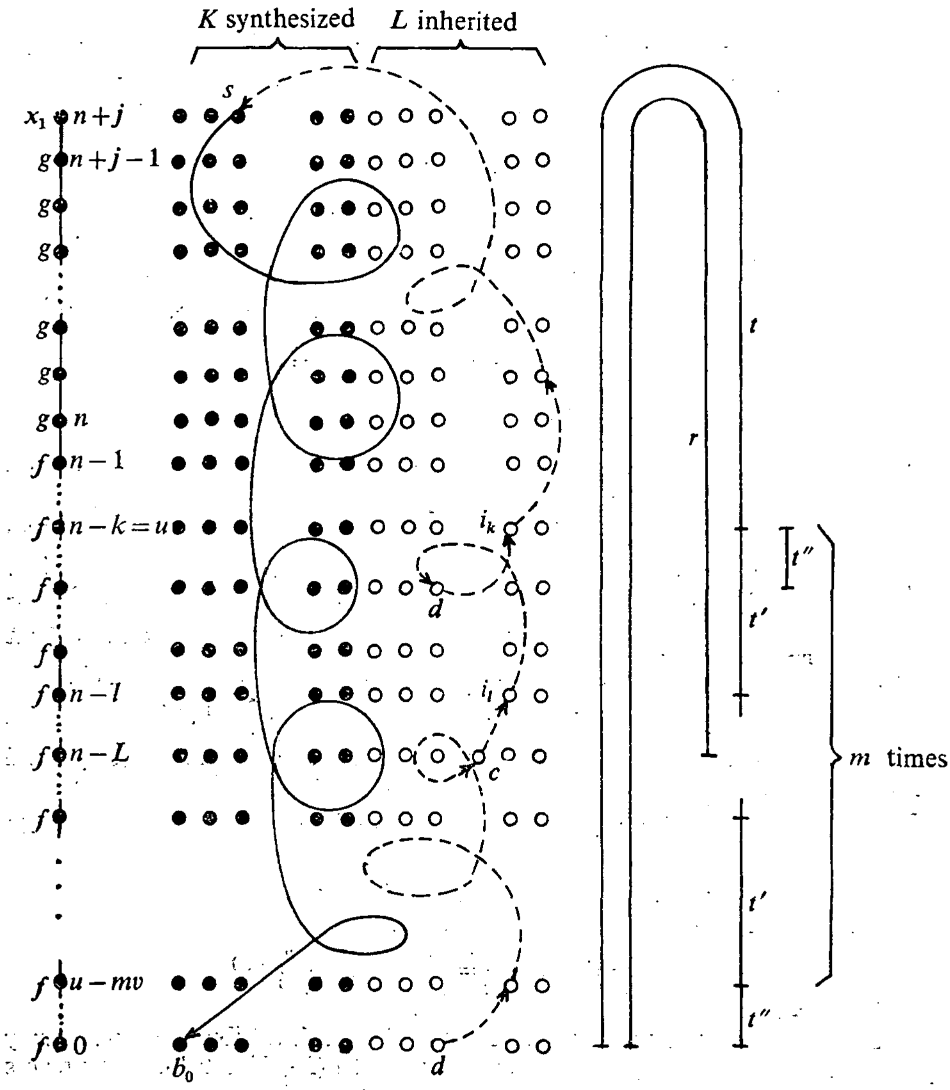

Fig. 3

$$(s, n+j) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} r_k((i, n-k)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} r_l((i, n-l)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} r((c, n-L)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)}. \tag{5}$$

Let us introduce the notations $u=n-k, v=l-k$ and $t=r_k$. Then there exists a tree $t'(\in T_{F'}(Z_1))$ such that $r_l=tt'$.[1] Thus (5) can be written as

$$(s, n+j) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} t((i, u)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} tt'((i, u-v)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} r((c, n-L)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)}. \tag{6}$$

[1] If t and t' are trees we often write tt' instead of $t(t')$.

Observe, that both t and t' may contain operator symbols f_1 only. Let m be the greatest integer number satisfying $u-mv\geqq 0$. It follows from (6) that

$$(s, n+j) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} t(t')^m((i, u-mv)) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} q^{(1)\,2} \tag{7}$$

(see Fig. 3). Finally, introduce the notation $y=u-mv$. Consider the tree t'' for which $(i, u) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} t''((d, u-y))$ holds for some $d(\in B_i)$ and which satisfies that if $(i, u) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} \bar{q}((b, w)) \overset{+}{\underset{p^{(1)}}{\Leftarrow}} t''((d, u-y))$ is valid for a tree $\bar{q}(\in T_{F'}(Z_1))$ and $(b, w)(\in B\times \text{path}\,(p^{(1)}))$ then $w>u-y$ holds. It follows from the definition of t'' that

$$(i, u-mv) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} t''((d, 0)) \tag{8}$$

and $t''=z_1$ or t'' contains operator symbols f_1 only, as t' does.

We have from (2), (7) and (8) that

$$(b_0, 0) \overset{*}{\underset{p^{(1)}}{\Leftarrow}} t(t')^m t''((d, 0)) \underset{p^{(1)}}{\Leftarrow} q^{(1)}. \tag{9}$$

Do not we forget that we have fixed j, therefore, t, t', t'', m and d depend on j.

But from (9), we can read that for each $j(=0, 1, \ldots, L)$ there exists a tree t_j as well as an inherited attribute d_j such that

$$(b_0, 0) \overset{*}{\underset{p_j^{(1)}}{\Leftarrow}} t_j((d_j, 0)) \underset{p_j^{(1)}}{\Leftarrow} q_j^{(1)}$$

moreover, t_j contains operator symbols f_1 only. Consider the inherited attributes $d_0, \ldots, d_L$. Then there exist indices k', l' such that $k'\neq l'$ and $d_{k'}=d_{l'}$. This implies that the trees $q_{k'}^{(1)}$ and $q_{l'}^{(1)}$ are of form $t_{k'}(s)$ and $t_{l'}(s)$, respectively, where $s=rt(d_{k'})=rt(d_{l'})$. But it is a contradiction which arises from $T(\mathbf{A})=T(\mathbf{B})$. Therefore $T(\mathbf{A})$ is not in $\mathcal{TDA}$. □

By a slight modification of the preceeding proof we get that $T(\mathbf{A})$ can not be induced by nondeterministic AT transducers. It is clear, besides, that the tree transformation given in Example 2.1 can not be induced by (nondeterministic) bottom-up tree transducers. Thus we obtain

Corollary 3.3. The classes $\mathcal{TB}$ and $\mathcal{TA}$ are incomparable. □

IV. Compositions of attributed tree transformations

First of all we are going to enter some notions. For any tree transformations $T_1(\subseteqq T_F(X_n)\times T_G(Y_m))$, $T_2(\subseteqq T_G(Y_m)\times T_H(U_r))$, the composition of T_1 and T_2 is the following transformation:

$$T_1\circ T_2 = \{(p, q)|(p, r)\in T_1 \text{ and } (r, q)\in T_2 \text{ for some } r\}.$$

[2] If t is a tree then $(t)^n$ means $\overbrace{tt\ldots t}^{n \text{ times}}$.

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be classes of tree transformations. The composition of $\mathscr{C}_1$ and $\mathscr{C}_2$ is the class

$$\mathscr{C}_1 \circ \mathscr{C}_2 = \{T_1 \circ T_2 | T_1 \in \mathscr{C}_1 \text{ and } T_2 \in \mathscr{C}_2\},$$

and for any class $\mathscr{C}$ and nonnegative integer n $\mathscr{C}^n$ is defined by induction

$$\mathscr{C}^1 = \mathscr{C},$$

$$\mathscr{C}^{n+1} = \mathscr{C}^n \circ \mathscr{C} \quad \text{if} \quad n \geqq 1.$$

We shall need the next Lemma.

Lemma 4.1. Let $n, m \geqq 0$ and let $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), A'_s, P, rt))$ be an AT transducer. Then there exists a constant N such that $\mathrm{rn}(q) \leqq N^{\mathrm{rn}(p)}$ holds for all $(p, q)(\in T(\mathbf{A}))$.

Proof. Let us enter the notations:

$$K = |A_s|,$$

$$L = |A_i| \quad \text{where} \quad A = A_s \cup A_i,$$

$$M = \max\{\mathrm{ht}(q) | q \text{ is the right side of some rule of } P\}.$$

Let $(p, q) \in T(\mathbf{A})$, i.e. assume that $d = (s_0, \lambda) \overset{*}{\underset{p,\mathbf{A}}{\Longleftarrow}} q$ for some $s_0(\in A'_s)$. Since $\mathbf{A}$ is noncircular $\mathrm{ht}(q) \leqq (K+L) M \mathrm{rn}(p)$ follows. It is obvious that there exists a constant R such that $\mathrm{rn}(q) \leqq R^{\mathrm{ht}(q)}$ for all $q(\in T_G(Y_m))$. It follows from the two latter inequality that the choice $N = R^{(K+L)M}$ will be right for our purposes. □

Theorem 4.1. $\mathscr{T}\mathscr{D}\mathscr{A}^n \subset \mathscr{T}\mathscr{D}\mathscr{A}_s \circ \mathscr{T}\mathscr{D}\mathscr{A}^n$

Proof. The inclusion $\mathscr{T}\mathscr{D}\mathscr{A}^n \subseteq \mathscr{T}\mathscr{D}\mathscr{A}_s \circ \mathscr{T}\mathscr{D}\mathscr{A}^n$ is obvious. In order to show that the inclusion is proper consider the transformation T in the class $\mathscr{T}\mathscr{D}\mathscr{A}_s \circ \mathscr{T}\mathscr{D}\mathscr{A}^n$ defined in the following way.

Let $\mathbf{A}(=(T_F(X_1), A, T_G(X_1), s, P))$ be a deterministic AT transducer with

(i) $F = F_1 = \{f\}, \quad G = G_2 = \{g\};$

(ii) $A = A_s = \{s\};$

(iii) $P = P_f \cup P_{x_1}$, where

$$P_f = \{s f(z_1) \leftarrow g(s z_1, s z_1)\},$$

$$P_{x_1} = \{s z_1 \leftarrow x_1\}.$$

If we denote by q_m the balanced tree over X_1 of type G, the height of which is m, then it is obvious that

$$T(\mathbf{A}) = \{(f^m(x_1), q_m) | m \geqq 0\}. \tag{10}$$

Moreover, let $\mathbf{B}\big(=(T_G(X_1), B, T_G(X_1), b, P', rt)\big)$ be the deterministic AT transducer, where

(i) $G = G_2 = \{g\}$;

(ii) $B = B_s \cup B_i$ and $B_s = \{b\}$, $B_i = \{i\}$;

(iii) $P' = P'_g \cup P'_{x_1}$, where

$P'_g = \{bg(z_1, z_2) \leftarrow g(bz_1, bz_1),\ iz_1 \leftarrow bz_2,\ iz_2 \leftarrow iz_0\}$,

$P'_{x_1} = \{bx_1 \leftarrow g(iz_0, iz_0)\}$;

(iv) $rt(i) = x_1$.

Figure 4 shows the effect of $\mathbf{B}$ on balanced trees of type G. Let $R=\{q_m \mid m \geqq 0\}$. If we take into account, that for each $m(\geqq 0)$ the rank of q_m is $2^{m+1}-1$ then we can easily prove that

$$T(\mathbf{B})|_R =$$

$$= \{(q_m, q_{m'}) \mid m \geqq 0,\ m' = 2^{m+1}-1\}.$$

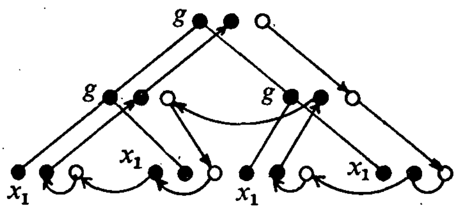

Fig. 4

Now let $T = T(\mathbf{A}) \circ \underbrace{T(\mathbf{B}) \circ \ldots \circ T(\mathbf{B})}_{n \text{ times}}$,

hence $T \in \mathscr{TDA}_s \circ \mathscr{TDA}^n$. It follows from (10) and (11) that

$$T = \Big\{(f^m(x_1), q_{m'}) \mid m \geqq 0,\ m' = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{m+1}}}}}}}_{n \text{ times}} - 1\Big\} \tag{11}$$

and this means that the rank of the image of the tree $f^m(x_1)$ at T is

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{m+1}}}}}}}_{n+1 \text{ times}} - 1 \tag{12}$$

for each $m(\geqq 0)$.

We show that $T \notin \mathscr{TDA}^n$. Indeed, in the opposite case we would have a decomposition $T = T'_1 \circ \ldots \circ T'_n$ where $T'_j = T(\mathbf{A}'_j)$ for some deterministic AT transducers $\mathbf{A}'_j$ $(j=1, \ldots, n)$. Thus, for each $j(=1, \ldots, n)$, Lemma 4.1 would give a constant N_j belonging to $\mathbf{A}'_j$ such that $\mathrm{rn}\,(q) \leqq N_j^{\mathrm{rn}\,(p)}$ if $(p, q) \in T'_j$ holds. From it would follow that the rank of the image of $f^m(x_1)$ at T would be at least

$$N_n^{N_{n-1}^{\cdot^{\cdot^{\cdot^{N_1^{m+1}}}}}}$$

for each $m(\geqq 0)$. This contradicts to (12). □

Taking into consideration the inclusion $\mathscr{TDA}_s \subset \mathscr{TDA}$ and the fact that Lemma 4.1 is true for nondeterministic AT transducers, we have

Corollary 4.1. $\mathscr{TDA}^n \subset \mathscr{TDA}^{n+1}$ and $\mathscr{TA}^n \subset \mathscr{TA}^{n+1}$. □

As we have seen, the proof of Theorem 4.1 depended on the fact that the AT transducers can "greatly" augment the rank of the trees. The question arises, whether the above inclusions will be true if we study a smaller class of AT transducers which can not do it. For this purpose we introduce the concept of linear AT transducer.

We say that an AT transducer $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), A'_s, P, rt))$ is linear (where $n, m \geqq 0$) if there exists a constant K such that from $(p, q) \in T(\mathbf{A})$ it follows that $\mathrm{rn}(q) \leqq K \mathrm{rn}(p)$. Let us denote the class of tree transformations induced by (deterministic) linear AT transducers by $(\mathcal{TDLA})\ \mathcal{TLA}$.

Theorem 4.2. The classes $\mathcal{TDA}$ and $\mathcal{TDLA} \circ \mathcal{TDLA}$ are incomparable.

Proof. It is obvious that there are tree transformations which are in $\mathcal{TDA}$ but not in $\mathcal{TDLA} \circ \mathcal{TDLA}$.

As we have seen the tree transformation $T(\mathbf{A})$ defined in Theorem 3.2 is not in $\mathcal{TDA}$. On the other hand $T(\mathbf{A})$ can be decomposed in the following way. Let $\mathbf{B}(=(T_F(X_2), B, T_F(X_2), P', b, rt'))$ be the AT transducer where

(i) $F = F_1 = \{f, g\}$,

(ii) $B = B_s \cup B_i, \quad B_s = \{b\}, \quad B_i = \{b_1, b_2\}$,

(iii) $P' = P'_f \cup P'_g \cup \left(\bigcup_{j=1}^{2} P'_{x_j}\right)$ with

$$P'_f = \{bf(z_1) \leftarrow bz_1,\ b_1 z_1 \leftarrow f(b_1 z_0),\ b_2 z_1 \leftarrow f(b_2 z_0)\},$$

$$P'_g = \{bg(z_1) \leftarrow bz_1,\ b_1 z_1 \leftarrow g(b_1 z_0),\ b_2 z_1 \leftarrow g(b_2 z_0)\},$$

$$P'_{x_j} = \{bx_j \leftarrow b_j z_0\} \quad (j = 1, 2),$$

(iv) $rt'(b_j) = x_j, \quad (j = 1, 2)$;

and $\mathbf{C}(=(T_F(X_2), C, T_{F'}(X_2), P'', c, rt''))$ be the AT transducer where

(i) $F' = F'_1 = \{f_1, f_2, g\}$,

(ii) $C = C_s \cup C_i, \quad C_s = \{c\}, \quad C_i = \{c_1, c_2\}$,

(iii) $P'' = P''_f \cup P''_g \cup \left(\bigcup_{j=1}^{2} P''_{x_j}\right)$ with

$$P''_f = \{cf(z_1) \leftarrow cz_1,\ c_1 z_1 \leftarrow f_1(c_1 z_0),\ c_2 z_1 \leftarrow f_2(c_2 z_0)\},$$

$$P''_g = \{cg(z_1) \leftarrow cz_1,\ c_1 z_1 \leftarrow g(c_1 z_0),\ c_2 z_1 \leftarrow g(c_2 z_0)\},$$

$$P''_{x_j} = \{cx_j \leftarrow c_j z_0\} \quad (j = 1, 2),$$

(iv) $rt''(c_j) = x_j \quad (j = 1, 2)$.

It is easy to see that both $\mathbf{B}$ and $\mathbf{C}$ is deterministic and linear, moreover, that $T(\mathbf{A}) = T(\mathbf{B}) \circ T(\mathbf{C})$ holds. This ends the proof. □

In case of $n=1$ Theorem 4.1 says that $\mathcal{TDA} \subset \mathcal{TDA}_s \circ \mathcal{TDA}$. If we exchange the factors of the composition then we have

Theorem 4.3. $\mathcal{TDA}\circ\mathcal{TDA}_s=\mathcal{TDA}$.

Proof. First we prove the inclusion $\mathcal{TDA}\circ\mathcal{TDA}_s\subseteq\mathcal{TDA}$.

For this purpose let $\mathbf{A}(=(T_F(X_n), A, T_G(Y_m), a_0, P, rt))$ and $\mathbf{B}(=(T_G(Y_m), B, T_H(U_r), b_0, P'))$ be deterministic AT transducers and suppose that $\mathbf{B}$ has only synthesized attributes.

Consider the AT transducer $\mathbf{C}(=(T_F(X_n), C, T_H(U_r), c_0, P'', rt''))$ defined as follows:

(a) $C=C_s\cup C_i$ where $C_s=B\times A_s$, $C_i=B\times A_i$ $(A=A_s\cup A_i)$;

(b) $c_0=(b_0, a_0)$;

(c) P'' is built in the way:

(i) for each $a(\in A_s)$, $b(\in B)$, $k(\geqq 0)$ and $f(\in F_k)$, if $af\leftarrow\bar{q}(a_1z_{i_1}, \ldots, a_lz_{i_l})\in P_f$ $(l\geqq 0;\ 0\leqq i_1, \ldots, i_l\leqq k;\ \bar{q}\in T_G(Y_m\cup Z_l))$ and $(b, \lambda)\overset{*}{\underset{\bar{q},\mathbf{B}}{\Longleftarrow}}q((b_1, z_{j_1}), \ldots, (b_t, z_{j_t}))$ $(t\geqq 0;\ 1\leqq j_1, \ldots, j_t\leqq l;\ q\in T_H(U_r\cup Z_t))$ then take into P''_f the rule $(b, a)f\leftarrow q((b_1, a_{j_1})z_{i_{j_1}}, \ldots, (b_t, a_{j_t})z_{i_{j_t}})$;

(ii) for each $a(\in A_s)$, $b(\in B)$, $x_j(\in X_n)$ if $ax_j\leftarrow\bar{q}(a_1z_0, \ldots, a_lz_0)\in P_{x_j}$ $(l\geqq 0,$ $\bar{q}\in T_G(Y_m\cup Z_l))$ and $(b, \lambda)\overset{*}{\underset{\bar{q},\mathbf{A}}{\Longleftarrow}}q((b_1, z_{j_1}), \ldots, (b_t, z_{j_t}))$ $(t\geqq 0;\ 1\leqq j_1, \ldots, j_t\leqq l;\ q\in T_H(U_r\cup Z_t))$ then take into P''_{x_j} the rule $(b, a)x_j\to q((b_1, a_{j_1})z_0, \ldots, (b_t, a_{j_t})z_0)$;

(iii) for each $i(\in A_i)$, $b(\in B)$, $k(\geqq 1)$, $f(\in F_k)$ and $1\leqq j\leqq k$ if $iz_j\leftarrow\bar{q}(a_1z_{i_1}, \ldots, a_lz_{i_l})\in P_f$ $(l\geqq 0;\ 0\leqq i_1, \ldots, i_l\leqq k;\ \bar{q}\in T_G(Y_m\cup Z_l))$ and $(b, \lambda)\overset{*}{\underset{\bar{q},\mathbf{B}}{\Longleftarrow}}q((b_1, z_{j_1}), \ldots, (b_t, z_{j_t}))$ $(t\geqq 0;\ 1\leqq j_1, \ldots, j_t\leqq l;\ q\in T_H(U_r\cup Z_t))$ then take into P''_f the rule $(b, i)z_j\leftarrow q((b_1, a_{j_1})z_{i_{j_1}}, \ldots, (b_t, a_{j_t})z_{i_{j_t}})$;

(d) for each $(b, i)(\in C_i)$ let $rt''((b, i))=q(\in T_H(U_r))$ if $rt(i)=\bar{q}(\in T_G(Y_m))$ and $(b, \lambda)\overset{*}{\underset{\bar{q},\mathbf{B}}{\Longleftarrow}}q$ hold.

We can prove the following: for each $p(\in T_F(X_n))$, $q(\in T_H(U_r))$, $a(\in A)$, $b(\in B)$ and $w(\in \mathrm{path}\,(p))$: $(\exists q'(\in T_G(Y_m)))\ ((a, w)\overset{*}{\underset{p,\mathbf{A}}{\Longleftarrow}}q'$ and $(b, \lambda)\overset{*}{\underset{q',\mathbf{B}}{\Longleftarrow}}q)$ if and only if $((b, a), w)\overset{*}{\underset{p,\mathbf{C}}{\Longleftarrow}}q$.

The proof of the only if part is performed by an induction on the length of the derivation $(a, w)\overset{*}{\underset{p,\mathbf{A}}{\Longleftarrow}}q'$.

If $\mathrm{lt}\left((a, w)\overset{*}{\underset{p,\mathbf{A}}{\Longleftarrow}}q'\right)=1$ then one of the following cases is valid:

$$a\in A_s,\quad \mathrm{lb}_p(w)=f,\quad af\leftarrow q'\in P_f; \tag{13}$$

$$a\in A_s,\quad \mathrm{lb}_p(w)=x_j,\quad ax_j\leftarrow q'\in P_{x_j}; \tag{14}$$

$$\begin{aligned} a\in A_i,\quad & w=vj\quad (v\in\mathbf{N}^*, j\in\mathbf{N}),\quad \mathrm{lb}_p(v)=f(\in F_k,\ k\geqq 1),\\ & 1\leqq j\leqq k \text{ and } az_j\leftarrow q'\in P_f; \end{aligned} \tag{15}$$

$$a\in A_i,\quad w=\lambda,\quad q'=rt(a). \tag{16}$$

Thus, what we wanted to prove it holds by definition in all of the four cases.

Now let $\mathrm{lt}\big((a, w) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} q'\big) > 1$. Then the derivation $(a, w) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} q'$ can be written as

$$(a, w) \underset{p,\mathbf{A}}{\Leftarrow} q_0'\big((a_1, w_1), \dots, (a_l, w_l)\big) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} q_0'(q_1', \dots, q_l') = q' \qquad (17)$$
$$(l \geqq 1;\ a_1, \dots, a_l \in A;\ q_0' \in T_G(Y_m \cup Z_l)).$$

Let $q_0(\in T_H(Z_t))$ be the tree for which

$$(b, \lambda) \overset{*}{\underset{q_0'\mathbf{B}}{\Leftarrow}} q_0\big((b_1, z_{j_1}), \dots, (b_t, z_{j_t})\big) \quad (t \geqq 0;\ 1 \leqq j_1, \dots, j_t \leqq l) \qquad (18)$$

is valid. Then the derivation $(b, \lambda) \overset{*}{\underset{q',\mathbf{B}}{\Leftarrow}} q$ can be specified in the following form:

$$(b, \lambda) \overset{*}{\underset{q',\mathbf{B}}{\Leftarrow}} q_0\big((b_1, v_1), \dots, (b_t, v_t)\big) \overset{*}{\underset{q',\mathbf{B}}{\Leftarrow}} q_0(q_1, \dots, q_t) = q \qquad (19)$$

where $\mathrm{str}_{q'}(v_s) = q_{j_s}'$ for all $s(=1, \dots, t)$. Taking into account this latter fact as well as the derivation $(b_s, v_s) \overset{*}{\underset{q',\mathbf{B}}{\Leftarrow}} q_s$ $(s=1, \dots, t)$ by Lemma 3.1 we get

$$(b_s, \lambda) \overset{*}{\underset{q_{j_s}',\mathbf{B}}{\Leftarrow}} q_s \quad (s = 1, \dots, t). \qquad (20)$$

Studying (17) we can say that three cases are possible, namely

$$a \in A_s,\quad \mathrm{lb}_p(w) = f(\in F_k \text{ for some } k \geqq 0); \qquad (21)$$

$$a \in A_s,\quad \mathrm{lb}_p(w) = x_j(\in X_n); \qquad (22)$$

$$a \in A_i,\quad w = vj,\quad \mathrm{lb}_p(v) = f\ (\in F_k \text{ for some } k(\geqq 1) \text{ and } 1 \leqq j \leqq k). \qquad (23)$$

We only detail case (21) because the others can be done similarly. Then, from (21) and (17) we obtain

$$af(z_1, \dots, z_k) \leftarrow q_0'(a_1 z_{i_1}, \dots, a_l z_{i_l}) \in P_f \quad (0 \leqq i_1, \dots, i_l \leqq k) \qquad (24)$$

and

$$(a_s, w_s) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} q_s' \quad \text{for all} \quad s(= 1, \dots, l). \qquad (25)$$

Taking into account the relations (18) and (24), by the definition of P'' it follows that

$$(b, a)f(z_1, \dots, z_k) \leftarrow q_0\big((b_1, a_{j_1}) z_{i_{j_1}}, \dots, (b_t, a_{j_t}) z_{i_{j_t}}\big) \in P_F''. \qquad (26)$$

Since $1 \leqq j_1, \dots, j_t \leqq l$ thus, from (25),

$$(a_{j_s}, w_{j_s}) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} q_{j_s}' \qquad (27)$$

follows for all $s(=1, \dots, t)$, moreover, from this and by (20) and the induction hypothesis, we obtain $\big((b_s, a_{j_s}), w_{j_s}\big) \overset{*}{\underset{p,\mathbf{C}}{\Leftarrow}} q_{j_s}'$ $(s=1, \dots, t)$. Finally, from these latters and the derivation $\big((b, a), w\big) \overset{*}{\underset{p,\mathbf{C}}{\Leftarrow}} q_0\big((b_1, a_{j_1}) w_{j_1}, \dots, (b_t, a_{j_t}) w_{j_t}\big)$ (flowing from (26)) we have what we wanted to prove.

In order to prove the if part of our Theorem let us suppose that the derivation $d=((b,a),w)\overset{*}{\underset{p,\mathbf{C}}{\Leftarrow}}q$ holds and let $\operatorname{lt}(d)=1$. There are four possible cases. Three of them can be specified as (21), (22) and (23) and the fourth is the case $a\in A_i$, $w=\lambda$. Because of similarity, we deal with case (21) only. Since $\operatorname{lt}(d)=1$ thus $(b,a)f\leftarrow q\in P''_f$ follows by the definition of the length of a derivation. From this we get $af(z_1,\dots,z_k)\leftarrow q'_0(a_1z_{i_1},\dots,a_lz_{i_l})\in P_f(l\geqq 0;\ 0\leqq i_1,\dots,i_l\leqq k;\ q'_0\in T_G(Y_m\cup Z_l))$ and $(b,\lambda)\overset{*}{\underset{q'_0,\mathbf{B}}{\Leftarrow}}q$. Consider the derivation $(a,w)\overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}}q'_0((a_1,w_1),\dots,(a_l,w_l))$ $\overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}}q'_0(q'_1,\dots,q'_l)$, which exists because of $\mathbf{A}$ is completely defiend. If we take the tree $q'=q'_0(q'_1,\dots,q'_l)$ then $(b,\lambda)\overset{*}{\underset{q',\mathbf{B}}{\Leftarrow}}q$ holds obviously.

Suppose now that $\operatorname{lt}(d)>1$. Then d can be written in the following form:

$$((b,a),w)\underset{p,\mathbf{C}}{\Leftarrow}q_0((b_1,a_1)w_1,\dots,(b_t,a_t)w_t)\overset{*}{\underset{p,\mathbf{C}}{\Leftarrow}}q_0(q_1,\dots,q_t)=q. \tag{28}$$

Then three different cases exist, namely (21), (22) and (23). Again, since these cases are similar we deal with case (21) only. In this case, (28) means that

$$(b,a)f(z_1,\dots,z_k)\leftarrow q_0((b_1,a_1)z_{i_1},\dots,(b_t,a_t)z_{i_t})\in P''_f, \tag{29}$$

moreover,

$$((b_s,a_s),w_s)\overset{*}{\underset{p,\mathbf{C}}{\Leftarrow}}q_s\quad\text{for all}\quad s(=1,\dots,t). \tag{30}$$

By the definition of P''_f, this implies that

$$af(z_1,\dots,z_k)\leftarrow q'_0(a'_1z'_1,\dots,a'_l\,z'_l)\in P_f \tag{31}$$

for some $l(\geqq 0)$ and $q'_0(\in T_G(Z_l))$, furthermore,

$$(b,\lambda)\overset{*}{\underset{q'_0,\mathbf{B}}{\Leftarrow}}q_0(b_1z_{i_1},\dots,b_tz_{i_t}) \tag{32}$$

and $a_s=a'_{i_s}$, $z_{i_s}=z'_{i_s}$ $(s=1,\dots,t)$. Then from (31), it follows that

$$(a,w)\overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}}q'_0((a'_1,w'_1),\dots,(a'_l,w'_l)), \tag{33}$$

furthermore, $w_s=w'_{i_s}$ for all $s(=1,\dots,t)$. Then, by the induction hypotesis and (30), we obtain

$$(\exists q''_s)\big((a_s,w_s)\overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}}q''_s\ \text{and}\ (b,\lambda)\overset{*}{\underset{q''_s,\mathbf{B}}{\Leftarrow}}q_s\big)\quad\text{for all}\ s(=1,\dots,t). \tag{34}$$

Define the trees $q'_r(\in T_G(Y_m))$ $(r=1,\dots,l)$ as follows

$$q'_r=\begin{cases}q''_s & \text{if}\quad r=i_s\quad\text{for some}\quad s(=1,\dots,t)\\ \text{the tree which can be derived from}\\ (a'_r,w'_r)\ \text{in}\ p\ \text{with}\ \mathbf{A},\ \text{otherwise.}\end{cases}$$

(Note that if in the above definition both $r=i_s$ and $r=i_{s'}$ hold for some $r(=1,\dots,l)$ and $s\neq s'$ then $w_s=w_{s'}$ holds because of $w'_{i_s}=w'_{i_{s'}}$, and $a_s=a_{s'}$

holds because of $a'_{i_s}=a'_{i_{s'}}$. Thus, also $q''_s=q''_{s'}$. On the other hand if $r=i_s$ does not hold for any s then q'_r exists because **A** is completely defined.)

Finally, we are going to show that the tree $q'=q'_0(q'_1,\dots,q'_t)$ $(\in T_G(Y_m))$ is suitable. Indeed, it follows from (33) and the definition of q'_r $(r=1,\dots,t)$ that $(a,w)\overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}}q'$. Moreover, it follows from (32) that $(b,\lambda)\overset{*}{\underset{q',\mathbf{B}}{\Leftarrow}}q_0(b_1v_1,\dots,b_tv_t)$ and $\mathrm{str}_{q'}(v_s)=q'_{i_s}$ for all $s(=1,\dots,t)$. Taking into account that $q'_{i_s}=q''_s$, from (33) and by lemma 3.1, we have $(b,\lambda)\overset{*}{\underset{q',\mathbf{B}}{\Leftarrow}}q$. This ends the proof of the if part.

If we choose $a=a_0$, $w=\lambda$ then we have $\mathcal{TDA}\circ\mathcal{TDA}_s\subseteq\mathcal{TDA}$.

The equality follows from the fact that every tree transformation being in $\mathcal{TDA}$ can be decomposed by itself and the identity and this latter is in $\mathcal{TDA}_s$. □

If in the former theorem $A_i=\emptyset$ then $C_i=\emptyset$, hence, we obtain

Corollary 4.2. $\mathcal{TDA}_s\circ\mathcal{TDA}_s=\mathcal{TDA}_s$. □

Finally, we want to show that if we apart from the condition that **A** is deterministic in Theorem 4.3 then this equality does not remain valid. Namely, we have the stronger

Theorem 4.4. The classes $\mathcal{TA}_s\circ\mathcal{TDA}_s$ and $\mathcal{TA}$ are incomparable.

Proof. It is easy to show that the tree transformation given in Example 2.1 is not in $\mathcal{TA}_s\circ\mathcal{TDA}_s$.

On the other hand consider the AT transducer $\mathbf{A}(=(T_F(X_1), A, T_G(X_1), a, P))$ where

(i) $F=F_1=\{f\},\quad G=G_1=\{g_1,g_2\};$

(ii) $A=A_s=\{a\};$

(iii) $P\ =P_f\cup P_{x_1},$ where

$P_f\ =\{af(z_1)\leftarrow g_1(az_1),\ af(z_1)\leftarrow g_2(az_1)\},$

$P_{x_1}=\{ax_1\leftarrow x_1\};$

and the AT transducer $\mathbf{B}(=(T_G(X_1), B, T_H(X_1), b, P'))$ where

(i) $H=H_1\cup H_2,\quad H_1=\{h_1\},\quad H_2=\{h_2\};$

(ii) $B=B_s=\{b\};$

(iii) $P'\ =P'_{g_1}\cup P'_{g_2}\cup P'_{x_1}$ where

$P'_{g_1}=\{bg_1(z_1)\leftarrow h_1(bz_1)\},\ P'_{g_2}=\{bg_2(z_1)\leftarrow h_2(bz_1,bz_1)\},$

$P'_{x_1}=\{bx_1\rightarrow x_1\}.$

Let $T_1=T(\mathbf{A}), T_2=T(\mathbf{B})$ and $T=T_1\circ T_2$. Obviously, $T_1\in\mathcal{TA}_s, T_2\in\mathcal{TDA}_s$.

Since both **A** and **B** contain only one synthesized attribute it is obvious to show by induction on n that

$$T|_{f^n(x_1)}=\{(f^n(x_1),h_1(q))\mid q\in T|_{f^{n-1}(x_1)}\}\cup\{(f^n(x_1),h_2(q,q))\mid q\in T|_{f^{n-1}(x_1)}\}$$

for each $n(\geqq 1)$. Taking into account that $T|_{x_1}=(x_1, x_1)$ it is easy to show that the images of tree $f^n(x_1)$ are "simmetrical" for all $n(\geqq 1)$. Moreover, it can be seen that the tree $f^n(x_1)$ has 2^n images, and 2^{n-1} of them are of form $h_2(q, q)$.

Assume that $T=T(\mathbf{C})$ for some AT transducer $\mathbf{C}(=(T_F(X_1), C, T_H(X_1), C_s', P'', rt))$. Let

$$K = |C_s|, \quad L = |C_i| \quad (\text{where } C = C_s \cup C_i),$$

$$M = |\{q \mid q \text{ is the right side of some rule of } P'' \text{ and has the form } h_2(q_1, q_2)\}|.$$

Let us fix an arbitrary integer $n(\geqq 1)$. Consider the derivation of some image $h_2(q, q)$ of the tree $p=f^n(x_1)$. This derivation can be written in the following way:

$$\begin{aligned}(a, \lambda) \underset{p,\mathbf{C}}{\overset{*}{\Leftarrow}} (b, w) \underset{p,\mathbf{C}}{\Leftarrow} h_2\big(q_1((a_1, v_1), \ldots, (a_n, v_n)), q_2((b_1, w_1), \ldots, (b_m, w_m))\big) \\ \underset{p,\mathbf{C}}{\overset{*}{\Leftarrow}} h_2\big(q_1(r_1, \ldots, r_n), q_2(s_1, \ldots, s_m)\big) = h_2(q, q)\end{aligned} \tag{35}$$

for some $a(\in C_s')$. Observe that it holds: if $(a_j, v_j) \underset{p,\mathbf{C}}{\overset{*}{\Leftarrow}} r_j'$ then $r_j=r_j'$ $(j=1, \ldots, n)$ and if $(b_k, w_k) \underset{p,\mathbf{C}}{\overset{*}{\Leftarrow}} s_k'$ then $s_k=s_k'$ $(k=1, \ldots, m)$. Indeed, in the opposite case if $r_j \neq r_j'$ would hold for some $j(=1, \ldots, n)$ then — since the images of p are simmetrical — we should have $q_1(r_1, \ldots, r_j, \ldots, r_n)=q$ and $q_1(r_1, \ldots, r_j', \ldots, r_n)=q$ and it is a contradiction.

Thus, each derivation (35) is determined by the attribute occurence (b, w) and the alternative of the rule applied there. As the number of attribute occurences is $(K+L)(n+1)$ and the number of alternatives of a rule is at most M we obtain that the number of images of p, which has the form $h_2(q, q)$, is at most $(K+L)M(n+1)$. This is again a contradiction provided n is sufficiently large. □

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(Received March 2, 1981)