On finite nilpotent automata

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In this paper we consider the isomorphic and homomorphic realizations of finite nilpotent automata. First we characterize all finite subdirectly irreducible nilpotent automata. Secondly we give necessary and sufficient conditions for a system of automata to be isomorphically complete for the class of all finite nilpotent automata with respect to the α_i-products (see [2]). Finally, we characterize the homomorphically complete systems for the class of all finite nilpotent automata with respect to the α_i-products.

The terminology and notations will be used in accordance with [3]. By an automaton we always mean a finite automaton. It can be seen from the definition of nilpotent automata that if $\mathbf{A}=(X, A, \delta)$ is a nilpotent automaton with absorbent state a_0 then

(i) $\mathbf{A}$ is connected in the sense that for any $a, b \in A$ there exist $p, q \in X^*$ with $ap=bq$,

(ii) the binary relation $a \leqq b \Leftrightarrow (\exists p)\ (p \in X^*\ \&\ ap=b)$ is a partial ordering in A and a_0 is the greatest element in $(A, \leqq)$.

Theorem 1. A nilpotent automaton $\mathbf{A}=(X, A, \delta)$ $(|A| \geqq 2)$ is subdirectly irreducible if and only if

(1) there exists an $a_1 \in A \setminus \{a_0\}$ such that a_1 is the greatest element in $(A \setminus \{a_0\}, \leqq)$,

(2) for any $a, b \in A$ if $a \neq b$ and $\{a, b\} \subsetneqq \{a_0, a_1\}$ then there exists a $p \in X^+$ with $ap \neq bp$.

Proof. Theorem 1 will be proved in a similar way as the corresponding statement for commutative automata in [1].

In order to prove the necessity assume that $\mathbf{A}$ is subdirectly irreducible and (1) does not hold. Then $(A \setminus \{a_0\}, \leqq)$ has at least two maximal elements. Denote them by a_1 and a_2. Consider the following relations: for any $a, b \in A$

$a\sigma_1 b$ if and onyl if $\{a, b\} \subseteq \{a_0, a_1\}$ or $a=b$,

$a\sigma_2 b$ if and only if $\{a, b\} \subseteq \{a_0, a_2\}$ or $a=b$.

It is not difficult to see that σ_1 and σ_2 are nontrivial congruence relations of $\mathbf{A}$ and $\sigma_1 \cap \sigma_2 = \Delta_A$, where Δ_A denotes the equality relation of A. This is a contradiction. Now assume that (1) holds and (2) does not hold. Then there exist $u, v \in A$ such that $u \neq v$, $\{u, v\} \nsubseteq \{a_0, a_1\}$ and $up = vp$ for any $p \in X^+$. Consider the following relations: for any $a, b \in A$

$a\sigma_1 b$ if and only if $\{a, b\} \subseteq \{a_0, a_1\}$ or $a = b$,

$c\sigma_2 b$ if and only if $\{a, b\} \subseteq \{u, v\}$ or $a = b$.

It is clear that σ_1 and σ_2 are nontrivial congruence relations of **A** and $\sigma_1 \cap \sigma_2 = \Delta_A$, which is a contradiction.

To prove the sufficiency assume that (1) and (2) are satisfied by **A**, and **A** is subdirectly reducible. Then there exists a congruence relation ϱ of **A** such that $\varrho \neq \Delta_A$ and $a_0 \not\equiv a_1(\varrho)$. By $\varrho \neq \Delta_A$, there exist $u \neq v \in A$ with $u \equiv v(\varrho)$. Consider the nonvoid set

$$B = \big\{\{a, b\}\colon a, b \in A,\ a \neq b,\ (\exists p)\ (p \in X^* \text{ and } \{u, v\}p = \{a, b\})\big\}.$$

Define the binary relation $\leqq$ on B as follows: $\{a, b\} \leqq \{a', b'\}$ if and only if there is a word $p \in X^*$ satisfying $\{a, b\}p = \{a', b'\}$. It is obvious that $\leqq$ is a partial ordering in B. Let $\{\bar{a}, \bar{b}\}$ denote a maximal element of B. Then, by the definition of B, $\bar{a} \neq \bar{b}$ and $\bar{a} \equiv \bar{b}(\varrho)$. Therefore, $\{\bar{a}, \bar{b}\} \nsubseteq \{a_0, a_1\}$. On the other hand, $\{\bar{a}, \bar{b}\}$ is a maximal element in $(B, \leqq)$, thus, $\bar{a}p = \bar{b}p$ for any $p \in X^+$, contradicting condition (2). This ends the proof of Theorem 1.

By Theorem 1, we can give all subdirectly irreducible nilpotent automata directly. Indeed, let $m \geqq 1$ be a fixed natural number and consider the input set $X_m = \{x_1, \dots, x_m\}$. Take the sets $A_1^{(m)} = \{0\}$, $A_2^{(m)} = \{0, 1\}$,

$$A_{n+1}^{(m)} = A_n^{(m)} \cup \{(u_1, \dots, u_m)\colon u_1, \dots, u_m \in A_n^{(m)} \text{ and } \{u_1, \dots, u_m\} \cap (A_n^{(m)} \setminus A_{n-1}^{(m)}) \neq \emptyset\}$$

for all $n \geqq 2$. Now, define the automata $\mathbf{A}_n^{(m)}$ $n = 1, 2, \dots$ in the following way:

$$\mathbf{A}_1^{(m)} = (X_m, A_1^{(m)}, \delta_1), \text{ where } \delta_1(0, x_t) = 0 \quad (t = 1, \dots, m),$$

$$\mathbf{A}_2^{(m)} = (X_m, A_2^{(m)}, \delta_2), \text{ where } \delta_2(0, x_t) = \delta_2(1, x_t) = 0 \quad (t = 1, \dots, m),$$

and in case of $n > 2$

$$\mathbf{A}_n^{(m)} = (X_m, A_n^{(m)}, \delta_n) \text{ with } \delta_{n|A_{n-1}^{(m)} \times X_m} = \delta_{n-1} \text{ and } \delta_n((u_1, \dots, u_m), x_t) = u_t$$

$(t = 1, \dots, m)$ for any $(u_1, \dots, u_m) \in A_n^{(m)} \setminus A_{n-1}^{(m)}$, where the restriction of δ_n to $A_{n-1}^{(m)} \times X_m$ is denoted by $\delta_{n|A_{n-1}^{(m)} \times X_m}$.

Using Theorem 1 it is not difficult to prove that a nilpotent automaton **A** with the input set X_m is subdirectly irreducible if and only if there exists a natural number n such that **A** can be embedded isomorphically into a quasi-direct product $\mathbf{A}_n^{(m)}(X_m, \varphi)$ of $\mathbf{A}_n^{(m)}$ with a single factor. From this we get the following

Corollary. A system Σ of automata is isomorphically complete for the class of all nilpotent automata with respect to the quasi-direct product if and only if for any pair (n, m) of natural numbers $n, m \geqq 1$ there is an automaton $\mathbf{A} \in \Sigma$ such that $\mathbf{A}_n^{(m)}$ can be embedded isomorphically into a quasi-direct product of **A** with a single factor.

This Corollary shows that there exists no system of automata which is isomorphically complete for the class of all nilpotent automata with respect to the quasi-direct product and minimal.

We say that an automaton **A** can be *realized homomorphically* by an α_i-product of automata $\mathbf{A}_t$ $(t=1, \ldots, k)$ if there exists a subautomaton **B** of an α_i-product of automata $\mathbf{A}_t$ $(t=1, \ldots, k)$ such that **A** is a homomorphic image of **B**.

We are going to use the following obvious statements.

Lemma 1. If an automaton **A** can be embedded isomorphically into an α_0-product of automata $\mathbf{A}_t$ $(t=1, \ldots, k)$ and for some $1 \leqq i \leqq k$ the automaton $\mathbf{A}_i$ can be embedded isomorphically into an α_0-product of automata $\mathbf{B}_j$ $(j=1, \ldots, s)$ then the automaton **A** can be embedded isomorphically into an α_0-product of automata $\mathbf{A}_1, \ldots, \mathbf{A}_{i-1}, \mathbf{B}_1, \ldots, \mathbf{B}_s, \mathbf{A}_{i+1}, \ldots, \mathbf{A}_k$.

Lemma 2. If an automaton **A** can be realized homomorphically by an α_0-product of automata $\mathbf{A}_t$ $(t=1, \ldots, k)$ and for some $1 \leqq i \leqq k$ the automaton $\mathbf{A}_i$ can be realized homomorphically by an α_0-product of automata $\mathbf{B}_j$ $(j=1, \ldots, s)$ then the automaton **A** can be realized homomorphically by an α_0-product of automata $\mathbf{A}_1, \ldots, \mathbf{A}_{i-1}, \mathbf{B}_1, \ldots, \mathbf{B}_s, \mathbf{A}_{i+1}, \ldots, \mathbf{A}_k$.

Let us denote by $\mathbf{R}_n = (\{x_1, \ldots, x_{n-1}\}, \{1, \ldots, n\}, \delta_n)$ the automaton, where $\delta_n(t, x_s) = \min\,(t+s, n)$ for any $1 \leqq t \leqq n$, $x_s \in \{x_1, \ldots, x_{n-1}\}$ and $n \geqq 2$.

Concerning the isomorphic realizations of nilpotent automata with respect to the α_0-product we have the following result.

Theorem 2. A system Σ of automata is isomorphically complete for the class of all nilpotent automata with respect to the α_0-product if and only if one of the following four conditions is satisfied by Σ:

(1) there exists an automaton in Σ which has three different states b, c, d and four input signs y, z, v, w (need not be different) such that $by=b$, $bz=c$, $cv=dv=bw=d$ hold,

(2) Σ contains an automaton which has two different states b, c and two input signs y, z such that $b=cy=by$ and $bz=c$ hold,

(3) Σ contains an automaton which has two different states b, c and two input signs y, z with $b=by$, $bz=cz=c$,

(4) for any natural number $n \geqq 3$ there exists an automaton in Σ which has n different states a_t $(t=1, \ldots, n)$ and input signs $x_k^{(t)}$ $(t=1, \ldots, n-1)$ $(k=1, \ldots, n-t)$ such that $a_t x_k^{(t)} = a_{t+k}$ if $1 \leqq t \leqq n-1$, $1 \leqq k \leqq n-t$ furthermore, $a_n x_1^{(n-1)} = a_n$ hold.

Proof. In order to prove the necessity assume that Σ is isomorphically complete for the class of all nilpotent automata with respect to the α_0-product. Let $n \geqq 3$ be arbitrary and consider the automaton $\mathbf{R}_n$. Since $\mathbf{R}_n$ is nilpotent, by our assumption, $\mathbf{R}_n$ can be embedded isomorphically into an α_0-product $\prod_{t=1}^{s} \mathbf{A}_t(\{x_1, \ldots, x_{n-1}\}, \varphi)$ of automata from Σ. Let μ denote a suitable isomorphism, and for any $i \in \{1, \ldots, n\}$ let $(a_{i1}, \ldots, a_{is})$ be the image of i under μ. Denote by m the least index for which $a_{nm} \neq a_{n-1m}$ holds. Observe that if $a_{im} = a_{nm}$ for some $1 \leqq i < n-1$ then (2) holds, while (3) holds if $a_{im} = a_{n-1m}$. Furthermore, if $a_{im} \notin \{a_{n-1m}, a_{nm}\}$ $(i=1, \ldots, n-2)$ and $a_{im} = a_{jm}$ for some indices $1 \leqq i < j < n-1$ then Σ satisfies condition (1) by $\mathbf{A}_m$. In the remaining case the elements a_{im} $(i=1, \ldots, n)$ are pairwise different and this implies that $\mathbf{A}_m$ has the property required in (4). Therefore, since n was arbitrary, if none of conditions (1), (2) and (3) is satisfied by Σ then (4) holds.

We have already shown the necessity of our statement. Conversely, assume that (1) holds by $\mathbf{B}\in\Sigma$. We shall prove that every nilpotent automaton can be embedded isomorphically into an α_0-power of $\mathbf{B}$. We proceed by induction on the number of states of the automaton. The case $n\leqq 2$ is trivial. Now let $n>2$ and assume that for any $m<n$ the statement is valid. Denote by $\mathbf{A}=(X, A, \delta)$ an arbitrary nilpotent automaton with n states. If $\mathbf{A}$ is subdirectly reducible then $\mathbf{A}$ can be embedded isomorphically into a direct product of nilpotent automata with fewer states than n. Therefore, by our induction hypothesis and Lemma 1, the statement is valid. Now assume that $\mathbf{A}$ is subdirectly irreducible. Then $\mathbf{A}$ has elements a_0 and a_1 satisfying (1) in Theorem 1. Define the congruence relation σ of $\mathbf{A}$ in the following manner: for any $a, b\in A$ $a\sigma b$ if and only if $\{a, b\}\subseteq\{a_0, a_1\}$ or $a=b$. The quotient automaton $\mathbf{A}_1=\mathbf{A}/\sigma$ is nilpotent with $n-1$ states. Consider the α_0-product $\mathbf{A}_1\times\mathbf{B}(X, \varphi)$, where $\varphi_1(x)=x$ and

$$\varphi_2(\sigma(a), x)=\begin{cases} y & \text{if } \sigma(a)\neq\sigma(a_0) \text{ and } \delta(a, x)\in A\setminus\sigma(a_0),\\ z & \text{if } \sigma(a)\neq\sigma(a_0) \text{ and } \delta(a, x)=a_1,\\ w & \text{if } \sigma(a)\neq\sigma(a_0) \text{ and } \delta(a, x)=a_0,\\ v & \text{if } \sigma(a)=\sigma(a_0),\end{cases}$$

for any $x\in X, \sigma(a)\in A/\sigma$. It can be easily proved that the correspondence

$$\nu(a)=\begin{cases}(\sigma(a), b) & \text{if } a\in A\setminus\sigma(a_0),\\ (\sigma(a), c) & \text{if } a=a_1,\\ (\sigma(a), d) & \text{if } a=a_0,\end{cases}$$

is an isomorphism of $\mathbf{A}$ into the α_0-product $\mathbf{A}_1\times\mathbf{B}(X, \varphi)$. Therefore, by our induction assumption and Lemma 1, $\mathbf{A}$ can be decomposed in the required form.

The sufficiencies of conditions (2) and (3) can be proved in a similar way as the sufficiency of (1).

Now assume that condition (4) holds. We proceed by induction on the number of states of the automaton. The case $n\leqq 2$ is trivial. Let $n>2$ and assume that the statement is valid for any $v<n$. Denote by $\mathbf{A}=(X, A, \delta)$ an arbitrary nilpotent automaton with n states. If $\mathbf{A}$ is subdirectly reducible then, by our induction assumption and Lemma 1, the statement is valid. Now assume that $\mathbf{A}$ is subdirectly irreducible and let $X=\{x_1, \ldots, x_m\}$. Then, by the observation connecting with Theorem 1, there exists an automaton $\mathbf{A}_s^{(m)}$ such that $\mathbf{A}$ can be embedded isomorphically into $\mathbf{A}_s^{(m)}(X_m, \psi)$. Denote by $\bar{s}$ the least natural number for which $\mathbf{A}$ can be embedded isomoriphcally into $\mathbf{A}_{\bar{s}}^{(m)}(X_m, \psi)$. Let μ denote a suitable isomorphism. Since Σ satisfies (4) there exists an automaton $\mathbf{B}\in\Sigma$ which has $\bar{s}$ different states a_j $(j=1, \ldots, \bar{s})$ and input signs $x_k^{(t)}$ $(t=1, \ldots, \bar{s}-1)$ $(k=1, \ldots, \bar{s}-t)$ such that $a_t x_k^{(t)}= =a_{t+k}$ $(t=1, \ldots, \bar{s}-1)$ $(k=1, \ldots, \bar{s}-t)$ and $a_{\bar{s}} x_1^{(\bar{s}-1)}=a_{\bar{s}}$ hold. Now consider the α_0-product $\mathbf{A}_1\times\mathbf{B}(X, \varphi)$ where $\mathbf{A}_1$ is defined in the same way as above and $\varphi_1(x)=x$,

$$\varphi_2(\sigma(a), x)=\begin{cases} x_{i-j}^{(\bar{s}-i+1)} & \text{if } \mu(a)\in A_i\setminus A_{i-1} \text{ for some } 3\leqq i\leqq\bar{s} \text{ and}\\ & \mu(\delta(a, x))\in A_j\setminus A_{j-1} \text{ for some } 1<j<i \text{ or}\\ & \mu(\delta(a, x))\in A_j \text{ with } j=1,\\ x_1^{(\bar{s}-1)} & \text{if } \sigma(a)=\sigma(a_0),\end{cases}$$

for any $x \in X$, $\sigma(a) \in A/\sigma$. It is not difficult to prove that the correspondence

$$\nu(a) = \begin{cases} (\sigma(a), a_{\bar{s}-i+1}) & \text{if } \mu(a) \in A_i \setminus A_{i-1} \text{ for some } 3 \leqq i \leqq \bar{s}, \\ (\sigma(a_0), a'_{\bar{s}-1}) & \text{if } \mu(a) \in A_2 \setminus A_1, \\ (\sigma(a_0), a_{\bar{s}}) & \text{if } \mu(a) \in A_1, \end{cases}$$

is an isomorphism of **A** into the α_0-product $\mathbf{A}_1 \times \mathbf{B}(X, \varphi)$. Thus, by our induction assumption and Lemma 1, we have a required decomposition of **A**. This completes the proof of Theorem 2.

The following theorem holds for α_i-products with $i \geqq 1$.

Theorem 3. A system Σ of automata is isomorphically complete for the class of all nilpotent automata with respect to the α_i-product $(i \geqq 1)$ if and only if one of the following three conditions is satisfied by Σ:

(1) there exists an automaton in Σ which has two different states b, c and three input signs y, z, v (need not be different) such that $by=b$ and $bz=cv=c$ hold,

(2) Σ contains an automaton which has two different states b, c and three input signs y, z, v (need not be different) such that $by=cv=b$ and $bz=c$ hold,

(3) for any natural number $n \geqq 3$ there exists an automaton in Σ which has n different states a_j $(j=1, \ldots, n)$ and input signs $x_k^{(t)}$ $(t=1, \ldots, n-1)$ $(k=1, \ldots, n-t)$, y such that $a_t x_k^{(t)} = a_{t+k}$ $(t=1, \ldots, n-1)$ $(k=1, \ldots, n-t)$ and $a_n y = a_n$.

Proof. The necessity of these conditions can be proved in a similar way as in the proof of Theorem 2. To prove the sufficiency, again, by Theorem 2, it is enough to show that an α_0-product of α_1-products with single factors is an α_1-product. But this is immediate from the definition of the α_i-products.

For any natural, number $n \geqq 1$ denote by $\mathbf{I}_n = (\{x\}, \{1, \ldots, n\}, \delta_n)$ the automaton satisfying $\delta_n(i, x) = \min(i+1, n)$ for all $i \in \{1, \ldots, n\}$. Furthermore, for any natural number $n \geqq 3$ denote by $\mathbf{Q}_n = (\{x, y\}, \{1, \ldots, n\}, \delta_n)$ the automaton for which $\delta_n(i, x) = \delta_n(i, y) = \min(i+1, n)$ for all $i \neq n-2$, $i \in \{1, \ldots, n\}$ and $\delta_n(n-2, x) = n-1$, $\delta_n(n-2, y) = n$.

In the sequel we shall need a more general concept of a subautomaton. The automaton $\mathbf{B} = (Y, B, \delta')$ is an *X-subautomaton* of $\mathbf{A} = (X, A, \delta)$ if $Y \subseteq X$, $B \subseteq A$ and $\delta_{|B \times Y} = \delta'$.

Take an automaton $\mathbf{A} = (X, A, \delta)$. Let $a \in A$ and $x \in X$ be arbitrary. The X-subautomaton generated by a and x is called a *cycle* and it will be denoted by (a, x). (Also, this notation (a, x) will be used to denote the set of states of this X-subautomaton.) For a cycle (a, x) there exist natural numbers $n \geqq 1$ and $m \geqq 1$ such that

(i) $n-1$ is the least exponent for which there exists a $t > n-1$ with $ax^{n-1} = ax^t$,

(ii) m is the least nonzero natural number for which $ax^{n-1} = ax^{n+m-1}$ holds,

(iii) the states $a, ax, \ldots, ax^{n+m-2}$ are pairwise different.

In this case we say that (a, x) is a *cycle of type* (n, m).

Observe an important property of cycles which we are going to use in the proofs of Theorems 4 and 5. Let $\mathbf{A} = (a, x)$ be a cycle of type (n, m) and let $\mathbf{B} = (b, x)$ be a cycle of type $(\bar{n}, \bar{m})$, where **A** and **B** have the same input sign x. Then the automaton **B** is a homomorphic image of **A** if and only if $\bar{n} \leqq n$ and $\bar{m} | m$ hold.

Theorem 4. A system Σ of automata is homomorphically complete for the class of all nilpotent automata with respect to the α_0-product if and only if one of the following three conditions is satisfied by Σ:

(1) there exists an automaton in Σ which has states b, c, d, input sign z and input words p, r, q such that $|p|\geqq 1$, $b\neq c$, $bz=cz$, $czq=c$, $dp=d$ and $dr=b$ hold,

(2) (i) Σ contains an automaton which has a state b and input signs $x_1, \ldots, x_k, y$ such that $bx_1\ldots x_k=b$ and $bx_1\neq by$,

(ii) for any natural number $n\geqq 3$ there exist a nonzero natural number m and an automaton in Σ having $n+m-1$ different states a_t $(t=1, \ldots, n+m-1)$ and input signs x_t $(1\leqq t<n+m-1)$ for which $a_t x_t=a_{t+1}$ $(1\leqq t<n+m-1)$ and $a_{n+m-1}x_{n-1}=a_n$ hold,

(3) (i) for any natural number $n\geqq 3$ there exists an automaton in Σ which has n different states b_t $(t=1, \ldots, n)$ and input signs x_t $(1\leqq t<n)$ such that $b_t x_t=b_{t+1}$ if $1\leqq t\leqq n-2$ and $b_{n-2}x_{n-1}=b_n$,

(ii) for any natural number $n\geqq 3$ there exist $m\geqq 1$ and an automaton in Σ having $n+m-1$ different states a_t $(t=1, \ldots, n+m-1)$ and input signs x_t $(1\leqq t<n+m-1)$ for which $a_t x_t=a_{t+1}$ $(1\leqq t<n+m-1)$ and $a_{n+m-1}x_{n-1}=a_n$ hold.

Proof. In order to prove the necessity assume that Σ is homomorphically complete for the class of all nilpotent automata with respect to the α_0-product. If Σ satisfies condition (1) then we are ready. Consider the case when Σ does not satisfy condition (1). We shall show that in this case (2) (ii) and, henceforth, (3) (ii) also hold. Indeed, let $n\geqq 3$ and consider the automaton $\mathbf{I}_n$. As Σ is homomorphically complete $\mathbf{I}_n$ can be realized homomorphically by an α_0-product of automata from Σ, i.e. there exists a subautomaton $\mathbf{A}$ of an α_0-product of automata from Σ such that $\mathbf{I}_n$ is a homomorphic image of $\mathbf{A}$. Let us denote by $\prod_{t=1}^{s} \mathbf{A}_t(\{x\}, \varphi)$ such an α_0-product and let μ be a suitable homomorphism. Let a be a counter image of the state 1 under μ, i.e. $\mu(a)=1$. Consider the cycle (a, x) in $\mathbf{A}$. It is obvious that (a, x) is a cycle of type $(\bar{n}, m)$ for some $m\geqq 1$ and $\bar{n}\geqq n$. From this we get that a cycle of type (n, m) can be embedded isomorphically into the α_0-product $\prod_{t=1}^{s} \mathbf{A}_t(\{x\}, \varphi)$. Let us denote by $\mathbf{B}=(b, x)$ the cycle of type (n, m) and by ν the isomorphism in question. Further on, we write $b_1=b$, $b_{t+1}=bx^t(1\leqq t<n+m-1)$. For any t $(1\leqq t\leqq n+m-1)$ let $(a_{t1}, \ldots, a_{ts})$ be the image of b_t under ν. Now consider the congruence relations $\pi_1\geqq\pi_2\geqq\ldots\geqq\pi_s$ on $\mathbf{B}$ which are defined in the following way: for any $1\leqq r\leqq s$ $b_i\equiv b_j(\pi_r)$ $b_i, b_j\in(b, x)$ if and only if $a_{it}=a_{jt}$ $(t=1, \ldots, r)$. Since the quotient automaton $\mathbf{B}/\pi_r$ is a homomorphic image of $\mathbf{B}$ we obtain that $\mathbf{B}/\pi_r$ is a cycle of type (n_r, m_r) for some natural numbers n_r, m_r, where $n_r\leqq n$ and $m_r|m$. On the other hand, by $\pi_1\geqq\pi_2\geqq\ldots\geqq\pi_s$ we get $n_1\leqq\leqq n_2\leqq\ldots\leqq n_s=n$. Now, if $n_1=n$ then the automaton $\mathbf{A}_1$ has the property required in (2) (ii). In the opposite case there exists a natural number r $(1\leqq r<s)$ such that $n_r<n$ and $n_{r+1}=n$. It is not difficult to see that in this case a cycle of type (n, m_{r+1}) can be embedded isomorphically into the α_0-product $\mathbf{B}/\pi_r\times\mathbf{A}_{r+1}(\{x\}, \psi)$, where $\psi_1(x)=x$, $\psi_2(\pi_r(b_t), x)=\varphi_{r+1}(a_{t1}, \ldots, a_{tr}, x)$ for any $\pi_r(b_t)\in B/\pi_r$. For the sake of simplicity let $m_{r+1}=m$ and denote by $\mathbf{D}=(d, x)$ and $\mathbf{C}=(c, x)$ a cycle of type (n, m) and (n_r, m_r), respectively. Therefore, we obtain that $\mathbf{D}$ can be embedded isomorphically into an α_0-product $\mathbf{C}\times\mathbf{A}_{r+1}(\{x\}, \varphi')$ under a suitable isomorphism τ.

We write $d_1=d$, $d_{t+1}=dx^t$ $(1\leqq t<n+m-1)$, and for any t $(1\leqq t\leqq n+m-1)$ let (c_t, a_t) be the image of d_t under τ. Since $n_r<n$ and $m_r|m$, $c_{n-1}=c_{n+m-1}$. From this it follows that $a_{n-1}\neq a_{n+m-1}$ and $\delta_{r+1}(a_{n-1}, z)=\delta_{r+1}(a_{n+m-1}, z)$ for some input sign $z\in X_{r+1}$. Now observe that the states $a_1, \ldots, a_n$ are pairwise different and $\{a_1, \ldots, a_{n-1}\}\cap\{a_n, \ldots, a_{n+m-1}\}=\emptyset$. Indeed, in the opposite case it can easily be seen that the automaton $\mathbf{A}_{r+1}$ has the property required in (1) and this is a contradiction. On the other hand, if $a_1, \ldots, a_n$ are pairwise different and $\{a_1, \ldots, a_{n-1}\}\cap\cap\{a_n, \ldots, a_{n+m-1}\}=\emptyset$ then it is not difficult to prove that $\mathbf{A}_{r+1}$ satisfies the conditions required in (2) (ii). Since n was arbitrary we get that Σ satisfies condition (2) (ii).

Now assume that Σ does not satisfy condition (2) (i). We shall show that in this case (3) holds. Indeed, let $n\geqq 3$ be arbitrary and consider the automaton $\mathbf{Q}_n$. By our assumption, $\mathbf{Q}_n$ can be realized homomorphically by an α_0-product $\prod_{t=1}^{s}\mathbf{A}_t(\{x, y\}), \varphi)$ of automata from Σ. Denote by μ a suitable homomorphism. Let b be a counter image of the state 1 under μ. Consider the states $b_1=b$, $b_{t+1}=bx^t$ $(1\leqq t<n-1)$, $b_n=b_{n-2}y$ in the α_0-product. They are pairwise different since their images under μ are pairwise different. Let $b_t=(a_{t1}, \ldots, a_{ts})$ for any t $(1\leqq t\leqq n)$. Denote by k the least index for which $a_{n-1k}\neq a_{nk}$. It can be easily seen that if there exist indices i, j $(1\leqq i<j\leqq n)$ with $a_{ik}=a_{jk}$ then Σ satisfies (2) (i) by $\mathbf{A}_k$, which is a contradiction. Therefore, the states a_{tk} $(1\leqq t\leqq n)$ are pairwise different. Then $\mathbf{A}_k$ has the property required in (3) (i). Since n is arbitrary we obtain that Σ satisfies (3). This ends the proof of the necessity.

The proof of sufficiency consists of two steps. First we shall show that if one of the conditions (1), (2), (3) is satisfied by Σ then the automaton $\mathbf{Q}_n$ can be realized homomorphically by an α_0-product of automata from Σ for any $n\geqq 3$. Secondly, it is proved that every nilpotent automaton can be realized homomorphically by an α_0-product of automata from $\{\mathbf{Q}_n\colon n\geqq 3\}$. By Lemma 2, this will complete the proof of sufficiency.

Indeed, suppose that Σ satisfies (1) by the automaton $\mathbf{A}(\in\Sigma)$. We show that the automaton $\mathbf{I}_n$ can be realized homomorphically by an α_0-power of $\mathbf{A}$ for any $n\geqq 2$. This statement is proved by induction on n. Let $n=2$ and take the states $b, c(\in A)$ and the input sign z of $\mathbf{A}$ for which $b\neq c$ and $bz=cz$. Consider the cycle (b, z). Let (k, l) be the type of (b, z). If $k>1$, then $\mathbf{I}_2$ can be realized homomorphically by an α_0-product of (b, z) with a single factor. If $k=1$ then, by $b\neq c$ and $bz=cz$, it can be easily seen that $c\notin(b, z)$. In this case $\mathbf{I}_2$ can be realized homomorphically by an α_0-product of (c, z) with a single factor. Now let $n>2$ and assume that our statement is valid for any $m<n$. We distinguish two cases depending on the value of k.

First suppose that $k>1$ in the type (k, l) of (b, z). Since Σ satisfies (1) by $\mathbf{A}$, there exist a state $d(\in A)$ and input words p, r with $|p|\geqq 1$, $dp=d$, $dr=b$. Let $p=x_1\ldots x_i$ and let $r=y_1\ldots y_j$ if r is nonempty. Consider the α_0-product $\mathbf{I}_{n-1}\times\times\mathbf{A}(\{x\}, \varphi)$, where $\varphi_1(x)=x$,

$$\varphi_2(t, x)=\begin{cases} z & \text{if } t=n-1,\\ y_{j-v} & \text{if } |r|\geqq 1 \text{ and } t=n-2-v \text{ for some } 0\leqq v<j,\\ x_{i-v} & \text{if } t=n-2-|r|-ui-v \quad \text{[for some } 0\leqq v<i \text{ and } u=0, 1, \ldots\end{cases}$$

for all $1 \leqq t < n$. Define the state a of $\mathbf{A}$ in the following manner: $a = dy_1 \dots y_v$ if $|r| \geqq 1$ and $n = j+2-v$ for some $1 \leqq v < j$; $a = d$ if $|r| \geqq 1$, $n = j+2$; $a = dx_1 \dots x_{i-v}$ if $|r| \geqq 1$ and $n = j+2+ui+v$ for some $0 \leqq v < i$ and $u = 0, 1, \dots$; $a = bx_1 \dots x_{i-v}$ if $|r| = 0$ and $n = 2+ui+v$ for some $0 \leqq v < i$ and $u = 0, 1, \dots$. It can be easily seen that $\mathbf{I}_n$ is a homomorphic image of the subautomaton generated by $(1, a)$ in the α_0-product $\mathbf{I}_{n-1} \times \mathbf{A}(\{x\}, \varphi)$. From this, by our induction assumption and Lemma 2, we obtain a required decomposition of $\mathbf{I}_n$.

Now assume that $k=1$. In this case $c \notin (b, z)$ and thus, by $cz = bz$, we have $c \neq cz$. On the other hand $czq = c$, thus $q = z_1 \dots z_i$ where $i \geqq 1$. Consider the α_0-product $\mathbf{I}_{n-1} \times \mathbf{A}(\{x\}, \varphi)$, where $\varphi_1(x) = x$ and

$$\varphi_2(t, x) = \begin{cases} z & \text{if } \ t = n-1-u(i+1) \quad \text{for some } u = 0, 1, \dots, \\ z_{i-v+1} & \text{if } \ t = n-1-u(i+1)-v \text{ for some } 1 \leqq v \leqq i \text{ and } u = 0, 1, \dots \end{cases}$$

for all $1 \leqq t < n$. Define the state $a (\in A)$ in the following way: $a = cz$ if $n = 1 + u(i+1)$ for some $u = 0, 1, \dots$; $a = czz_1 \dots z_{i-v+1}$ if $n = 1 + u(i+1) + v$ for some $1 \leqq v \leqq i$ and $u = 0, 1, \dots$. It is not difficult to see that $\mathbf{I}_n$ is a homomorphic image of the subautomaton generated by $(1, a)$ in $\mathbf{I}_{n-1} \times \mathbf{A}(\{x\}, \varphi)$. This yields a required decomposition of $\mathbf{I}_n$.

Now let $n \geqq 3$ be arbitrary and consider the automaton $\mathbf{Q}_n$. We know that $dx_1 \dots x_i = d$. We write $d = d_1$ and $d_{t+1} = d_t x_t$ $(1 \leqq t < i)$. Without loss of generality we may assume that the states $d_1, \dots, d_i$ are pairwise different. We show that there exist an index j $(1 \leqq j \leqq i)$ and an input sign w of $\mathbf{A}$ such that $d_j x_j \neq d_j w$. Indeed, in the opposite case $d_t x_t = d_t x$ holds for any input sign x and d_t $(t = 1, \dots, i)$. Since $d_1 r = b$ and $d_1 rzq = bzq = czq = c$, there exist $1 \leqq t_1, t_2 \leqq i$ with $b = d_{t_1}$ and $c = d_{t_2}$. On the other hand, $bz = cz$ from which $t_1 = t_2$ and, henceforth $b = c$ follows, yielding a contradiction. (Observe that we have proved that $\mathbf{A}$ has the property required in (2) (i).) Now let j $(1 \leqq j \leqq i)$ denote an index such that $d_j x_j \neq d_j w$ for some input sign w of $\mathbf{A}$. Take the following α_0-product $\mathbf{I}_n \times \mathbf{A}(\{x, y\}, \varphi)$, where $\varphi_1(x) = x$,

$$\varphi_2(t, x) = \begin{cases} x_{j+v} & \text{if } i > 1 \text{ and } t = n-2+v \quad \text{for some } 0 \leqq v \leqq i-j, \\ x_{j+v-i} & \text{if } i > 1 \text{ and } t = n-2+v \quad \text{for some } i-j < v \leqq 2, \\ x_{j-v} & \text{if } i > 1 \text{ and } t = n-2-v \quad \text{for some } 1 \leqq v < j, \\ x_{i-v} & \text{if } i > 1 \text{ and } t = n-2-j-ui-v \text{ for some } 0 \leqq v < i \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{and } u = 0, 1, \dots, \\ x_1 & \text{if } \ i = 1, \end{cases}$$

and

$$\varphi_2(t, y) = \begin{cases} w & \text{if } \ t = n-2, \\ \varphi_2(t, x) & \text{otherwise,} \end{cases}$$

for all $1 \leqq t \leqq n$. Define the state $a (\in A)$ in the following manner: $a = d_{v+1}$ if $i > 1$ and $n = j+2-v$ for some $0 \leqq v < j$; $a = d_{i-v}$ if $i > 1$ and $n = j+3+ui+v$ for some $0 \leqq v < i$ and $u = 0, 1, \dots$; $a = d_1$ if $i = 1$. It can be easily seen that the automaton $\mathbf{Q}_n$ is a homomorphic image of the subautomaton generated by $(1, a)$ in $\mathbf{I}_n \times \mathbf{A}(\{x, y\}, \varphi)$. By Lemma 2, we got a required decomposition of $\mathbf{Q}_n$, and thus we have proved the homomorphic realizations of automata $\mathbf{Q}_n$ by Σ if Σ satisfies condition (1).

Now assume that Σ satisfies condition (2). First we show that the automaton $\mathbf{I}_n$ can be realized homomorphically by an α_0-product of automata from Σ for any $n \geqq 2$. We prove this by induction on n. Let $n=2$. Since Σ satisfies (2) (ii), there exists an automaton $\mathbf{A}$ in Σ which has $m+2$ different states $a_1, \ldots, a_{m+2}$ and input signs x_t $(1 \leqq t < m+2)$ such that $a_t x_t = a_{t+1}$ if $1 \leqq t < m+2$, and $a_{m+2} x_2 = a_3$. Take the cycle (a_2, x_2) in $\mathbf{A}$. If the type of (a_2, x_2) is (k, l) with $k > 1$ then $\mathbf{I}_2$ is a homomorphic image of an α_0-product of (a_2, x_2) with a single factor, and thus $\mathbf{I}_2$ can be realized homomorphically by an α_0-product of $\mathbf{A}$ with a single factor. In the opposite case, it is not difficult to see that $a_{m+2} \notin (a_2, x_2)$, and thus $\mathbf{I}_2$ is a homomorphic image of an α_0-product of the cycle (a_{m+2}, x_2) with a single factor. Therefore, $\mathbf{I}_2$ can be realized homomorphically by an α_0-product of $\mathbf{A}$ with a single factor. Now let $n > 2$ and assume that our statement is valid for any $j < n$. Since Σ satisfies (2) (ii) there exists an automaton $\mathbf{A}$ in Σ having different states $a_t (t=1, \ldots, n+m-1)$ and input signs x_t $(1 \leqq t < n+m-1)$ such that $a_t x_t = a_{t+1}$ if $1 \leqq t < n+m-1$ and $a_{n+m-1} x_{n-1} = a_n$. We distinguish two cases.

First assume that $k > 1$ in the type (k, l) of the cycle (a_{n-1}, x_{n-1}). Consider the α_0-product $\mathbf{I}_{n-1} \times \mathbf{A}(\{x\}, \varphi)$, where $\varphi_1(x) = x$ and $\varphi_2(t, x) = x_t$ for all $1 \leqq t < n$. It is clear that $\mathbf{I}_n$ is a homomorphic image of the subautomaton generated by $(1, a_1)$ in $\mathbf{I}_{n-1} \times \mathbf{A}(\{x\}, \varphi)$. From this, similarly as above, we get a required decomposition of $\mathbf{I}_n$.

Now suppose that $k=1$. Then one can prove that $a_{n+m-1} \notin (a_{n-1}, x_{n-1})$ and thus $m > 1$. Consider the α_0-product $\mathbf{I}_{n-1} \times \mathbf{A}(\{x\}, \varphi)$, where $\varphi_1(x) = x$ and

$$\varphi_2(t, x) = \begin{cases} x_{n-1} & \text{if } t = n-1-um \text{ for some } u = 0, 1, \ldots, \\ x_{n+m-v} & \text{if } t = n-um-v \text{ for some } 2 \leqq v \leqq m \text{ and } u = 0, 1, \ldots \end{cases}$$

for all $1 \leqq t < n$. Let $a = a_{n+m-1}$ if $n = um+2$ for some $u = 0, 1, \ldots$ and $a = a_{n+m-v}$ if $n = 1+um+v$ for some $2 \leqq v \leqq m, u = 0, 1, \ldots$. It is not difficult to see that $\mathbf{I}_n$ is a homomorphic image of the subautomaton generated by $(1, a)$ in $\mathbf{I}_{n-1} \times \mathbf{A}(\{x\}, \varphi)$. This yields a required decomposition of $\mathbf{I}_n$.

It remained to decompose the automata $\mathbf{Q}_n$. Since condition (2) (i) is satisfied by Σ and only this condition was used in the previous decomposition of $\mathbf{Q}_n$ (see the observation made in the proof) the automaton $\mathbf{Q}_n$ can be realized homomorphically by an α_0-product of automata from Σ for any $n \geqq 3$.

Now let us suppose that Σ satisfies condition (3). Since conditions (3) (ii) and (2) (ii) coincide, by the proof of the decomposition of automata $\mathbf{I}_n$ in the case Σ satisfies (2), we have that the automaton $\mathbf{I}_n$ can be realized homomorphically by an α_0-product of automata from Σ for any $n \geqq 2$. Let $n \geqq 3$ be arbitrary and consider the automaton $\mathbf{Q}_n$. Since Σ satisfies (3) (i) there exists an automaton $\mathbf{B}$ in Σ which has n different states b_t $(t=1, \ldots, n)$ and input signs x_t $(1 \leqq t < n)$ such that $b_t x_t = b_{t+1}$ if $1 \leqq t < n-1$ and $b_{n-2} x_{n-1} = b_n$. Take the α_0-product $\mathbf{I}_n \times \mathbf{B}(\{x, y\}, \varphi)$, where $\varphi_1(x) = x$ and

$$\varphi_2(t, x) = \begin{cases} x_t & \text{if } 1 \leqq t < n-2, \\ x_{n-2} & \text{otherwise}, \end{cases}$$

$$\varphi_2(t, y) = \begin{cases} x_t & \text{if } 1 \leqq t < n-2, \\ x_{n-1} & \text{otherwise}, \end{cases}$$

for all $1 \leqq t \leqq n$. It can be easily seen that $\mathbf{Q}_n$ is a homomorphic image of the subautomaton generated by $(1, b_1)$ in $\mathbf{I}_n \times \mathbf{B}(\{x, y\}, \varphi)$. Therefore, we have a required decomposition of $\mathbf{Q}_n$. This ends the first step of the proof of the sufficiency.

To prove that arbitrary nilpotent automaton can be realized homomorphically by an α_0-product of automata from $\{\mathbf{Q}_m: m \geqq 3\}$, by Theorem 2 and Lemma 2, it is enough to show that the automaton $\mathbf{R}_n$ can be realized homomorphically by an α_0-product of automata from $\{\mathbf{Q}_m: m \geqq 3\}$ for any $n \geqq 3$.

Let $n \geqq 3$ be arbitrary. In order to decompose $\mathbf{R}_n$ consider the automata $\mathbf{R}_n^{(j)}$ $(1 \leqq j < n)$ given by $\mathbf{R}_n^{(j)} = (\{x_1, \ldots, x_{n-1}\}, \{1, \ldots, n\}, \delta_n^{(j)})$, where

$$\delta_n^{(j)}(t, x_s) = \begin{cases} \min(t+1, n) & \text{if} \quad s \neq j, \\ \min(t+j, n) & \text{if} \quad s = j, \end{cases}$$

for any $1 \leqq t \leqq n$ and $x_s \in \{x_1, \ldots, x_{n-1}\}$. Take the direct product $\prod_{j=1}^{n-1} \mathbf{R}_n^{(j)}$ and let $\mathbf{W}$ denote its subautomaton generated by $(1, \ldots, 1)$. Observe that $a_i \geqq a_1$ holds if $1 \leqq i \leqq n-1$ for any state $(a_1, \ldots, a_{n-1})$ of the subautomaton $\mathbf{W}$. Therefore, if $a_s = k$ holds for some $1 \leqq s \leqq n-1$ and $1 \leqq k \leqq n$ then $a_1 + \sum_{i=2}^{n-1} (a_i - a_1) \geqq k$. Now define the mapping μ: $W \to \{1, \ldots, n\}$ in the following way:

$$\mu(a_1, \ldots, a_{n-1}) = \min\left(a_1 + \sum_{i=2}^{n-1} (a_i - a_1), n\right).$$

By the observation above, it is not difficult to prove that the mapping μ is a homomorphism of $\mathbf{W}$ onto $\mathbf{R}_n$.

Now let $1 \leqq j < n$ be arbitrary. For the decomposition of $\mathbf{R}_n^{(j)}$ consider the automaton $\mathbf{R}_{n,k}^{(j)} = (\{x_1, \ldots, x_{n-1}\}, \{1, \ldots, n\}, \delta_{n,k}^{(j)})$ for all k $(1 \leqq k \leqq n-1)$, where

$$\delta_{n,k}^{(j)}(t, x_s) = \begin{cases} \min(t+s, n) & \text{if} \quad t = k \quad \text{and} \quad j = s, \\ \min(t+1, n) & \quad \text{otherwise}, \end{cases}$$

for any $1 \leqq t \leqq n$ and $x_s \in \{x_1, \ldots, x_{n-1}\}$. Take the direct product $\prod_{k=1}^{n-1} \mathbf{R}_{n,k}^{(j)}$ and denote by $\mathbf{U}$ its subautomaton generated by $(1, \ldots, 1)$. Observe that for any state $(a_1, \ldots, a_{n-1}) \in U$ $0 \leqq a_i - a_{n-1} \leqq j-1$ holds provided $1 \leqq i \leqq n-1$ and $a_t = r$ $(t = r, \ldots, n-2)$ if $a_{n-1} = r$ for some r, where $1 \leqq r < n-1$. Now define the mapping μ: $U \to \{1, \ldots, n\}$ in the following way:

$$\mu(a_1, \ldots, a_{n-1}) = \min\left(a_{n-1} + \sum_{i=1}^{n-2} (a_i - a_{n-1}), n\right).$$

By the observation above, it can be seen that the mapping μ is a homomorphism of $\mathbf{U}$ onto $\mathbf{R}_n^{(j)}$.

Now let $1 \leqq k \leqq n-1$ be arbitrary. If $j=1$ or $n-2 \leqq k$ then $\mathbf{R}_{n,k}^{(j)}$ can be embedded isomorphically into an α_0-product of $\mathbf{Q}_n$ with a single factor. Let us suppose that $1<j$ and $1\leqq k<n-2$. For the decomposition of $\mathbf{R}_{n,k}^{(j)}$ consider the α_0-product $\mathbf{A}=\prod_{i=k+2}^{n}\mathbf{Q}_i(\{x_1,\dots,x_{n-1}\},\varphi)$, where

$$\varphi_1(x_s)=\begin{cases}x & \text{if } s\neq j,\\ y & \text{if } s=j,\end{cases}\quad \varphi_2(a_1,x_s)=\begin{cases}y & \text{if } a_1=k+2,\\ x & \text{otherwise},\end{cases}$$

$$\varphi_{i+1}(a_1,\dots,a_i,x_s)=\begin{cases}y & \text{if } a_1<a_2<\dots<a_i,\\ x & \text{otherwise},\end{cases}$$

for any $x_s\in\{x_1,\dots,x_{n-1}\}$, $2\leqq i\leqq n-k-2$, $a_t\in\{1,\dots,t+k+1\}$ $(1\leqq t\leqq n-k-2)$. Let $v=n-k-1$ and take the following sets of states of $\mathbf{A}$:

$A_1=\{(a_1,\dots,a_v)\colon a_1\leqq k+2$ and $a_i=a_{i+1}$ $(i=1,\dots,v-1)\}$,

$A_2=\{(a_1,\dots,a_v)\colon a_1=k+2$ and $(\exists s)$ $(2\leqq s\leqq v$ and $a_i<a_{i+1}$ if $i\leqq s-1$ and $a_i=a_{i+1}$ if $s\leqq i<v)\}$,

$A_3=\{(a_1,\dots,a_v)\colon a_1=k+2$ and $(\exists s)$ $(1\leqq s<v$ and $a_i<a_{i+1}$ if $1\leqq i\leqq s-1$ and $a_i=a_s-1$ if $s<i\leqq v)\}$.

It can be shown, by a sort computation, that $\mathbf{B}=(\{x_1,\dots,x_{n-1}\},\bigcup_{i=1}^{3}A_i,\delta|_{\left(\bigcup_{i=1}^{3}A_i\right)\times\{x_1,\dots,x_{n-1}\}})$ is a subautomaton of $\mathbf{A}$. Now define the mapping $\mu\colon\bigcup_{i=1}^{3}A_i\to\{1,\dots,n\}$ in the following way:

$$\mu(a_1,\dots,a_v)=\begin{cases}\max\limits_{1\leqq i\leqq v}a_i & \text{if } (a_1,\dots,a_v)\in A_1\cup A_2,\\ \min(a_v+j-1,n) & \text{if } (a_1,\dots,a_v)\in A_3.\end{cases}$$

It is not difficult to prove that the mapping μ is a homomorphism of $\mathbf{B}$ onto $\mathbf{R}_{n,k}^{(j)}$. This ends the proof of Theorem 4.

The following Theorem holds for α_i-products with $i\geqq 1$.

Theorem 5. A system Σ of automata is homomorphically complete for the class of all nilpotent automata with respect to the α_i-product $(i\geqq 1)$ if and only if one of the following two conditions is satisfied by Σ:

(1) Σ contains an automaton which has a state b and input signs $x_1,\dots,x_k,y$ such that $bx_1\dots x_k=b$ and $bx_1\neq by$,

(2) (i) for any natural number $n\geqq 3$ there exists an automaton in Σ which has n different states b_t $(t=1,\dots,n)$ and input signs x_t $(1\leqq t<n)$ such that $b_tx_t=b_{t+1}$ if $1\leqq t<n-1$ and $b_{n-2}x_{n-1}=b_n$,

(2) (ii) for any $n\geqq 3$ there exist $m\geqq 1$ and an automaton in Σ such that it has $n+m-1$ different states a_t $(t=1,\dots,n+m-1)$ and input signs x_t $(1\leqq t\leqq n+m-1)$ for which $a_tx_t=a_{t+1}$ $(1\leqq t<n+m-1)$ and $a_{n+m-1}x_{n+m-1}=a_n$ hold.

Proof. The necessity can be proved in a similar way as in the proof of Theorem 4. (One need consider the homomorphic realization of $\mathbf{Q}_n$.)

In order to verify the sufficiency assume that Σ satisfies (1) by $\mathbf{A}=(X, A, \delta)$. From the proof of Theorem 4 it follows that every nilpotent automaton can be realized homomorphically by an α_0-product of automata from $\{\mathbf{A}\}\cup\{\mathbf{I}_m\colon m\geqq 2\}$. Therefore, using the fact that the α_0-product of α_1-products is an α_1-product, it is enough to show that the automaton $\mathbf{I}_n$ can be realized homomorphically by an α_1-power of $\mathbf{A}$ for any $n\geqq 2$. Indeed, let $n\geqq 2$ be arbitrary. Write $b_1=b$ and $b_{t+1}=b_t x_t$ $(t=1, \ldots, k-1)$. Without loss of generality we may assume that the states $b_1, \ldots, b_k$ are pairwise different. We distinguish three cases.

First suppose that $\{b_1 y, b_1 y^2, \ldots\}\cap\{b_1, \ldots, b_k\}=\emptyset$. Then take the α_1-power $\mathbf{A}^{n-1}(\{x\}, \varphi)$, where $\varphi_1(u_1, x)=y$,

$$\varphi_t(u_1, \ldots, u_t, x)=\begin{cases} y & \text{if } \{u_1, \ldots, u_{t-1}\}\cap\{b_1, \ldots, b_k\}=\emptyset, \\ x_j & \text{if } \{u_1, \ldots, u_{t-1}\}\cap\{b_1, \ldots, b_k\}\neq\emptyset \text{ and } u_t=b_j, \\ \text{arbitrary input sign from } X \text{ otherwise,} \end{cases}$$

for any state $(u_1, \ldots, u_{n-1})\in A^{n-1}$ and $2\leqq t\leqq n-1$. Define the state $(a_1, \ldots, a_{n-1})$ of the α_1-product in the following way:

$$a_1=b_1,\ a_{t+1}=\begin{cases} b_{j-1} & \text{if } a_t=b_j \text{ for some } 1<j\leqq k, \\ b_k & \text{if } a_t=b_1 \end{cases}$$

where $t=1, \ldots, n-2$. Let $\mathbf{U}$ denote the subautomaton of $\mathbf{A}^{n-1}(\{x\}, \varphi)$ which is generated by $(a_1, \ldots, a_{n-1})$. It is not difficult to see that $\mathbf{I}_n$ is a homomorphic image of $\mathbf{U}$ and thus, we got a required decomposition of $\mathbf{I}_n$.

Now assume that $\{b_1 y, b_1 y^2, \ldots\}\cap\{b_1, \ldots, b_k\}\neq\emptyset$ and $b_1 y\notin\{b_1, \ldots, b_k\}$. Denote by $s>1$ the least natural number for which $b_1 y^s\in\{b_1, \ldots, b_k\}$. There exists such an s. Take the α_1-power $\mathbf{A}^{n-1}(\{x\}, \varphi)$, where

$$\varphi_1(u_1, x)=\begin{cases} y & \text{if } u_1\in\{b_1 y_1, \ldots, b_1 y^{s-1}\}, \\ x_j & \text{if } u_1=b_j \text{ for some } 1\leqq j\leqq k, \\ \text{arbitrary input sign from } X \text{ otherwise,} \end{cases}$$

$$\varphi_t(u_1, \ldots, u_t, x)=\begin{cases} y & \text{if } u_t\in\{b_1 y, \ldots, b_1 y^{s-1}\}, \\ y & \text{if } u_t=b_1 \text{ and } u_{t-1}\in\{b_1 y, b_1 y^2, \ldots\}, \\ x_j & \text{if } u_t=b_j \text{ for some } 1<j\leqq k, \\ x_1 & \text{if } u_t=b_1 \text{ and } u_{t-1}\notin\{b_1 y, b_1 y^2, \ldots\}, \\ \text{arbitrary input sign from } X \text{ otherwise,} \end{cases}$$

for any $(u_1, \ldots, u_{n-1})\in A^{n-1}$, $2\leqq t\leqq n-1$. Define the state $(a_1, \ldots, a_{n-1})\in A^{n-1}$ in the following way:

$$a_1=b_1 y^{s-1}, \ldots, a_{s-1}=b_1 y, \quad a_s=b_1 \text{ and}$$

$$a_{t+1}=\begin{cases} b_{j-1} & \text{if } a_t=b_j, \text{ for some } 1<j\leqq k, \\ b_k & \text{if } a_t=b_1, \end{cases}$$

where $s\leqq t<n-1$. Denote by $\mathbf{U}$ the subautomaton generated by $(a_1, \ldots, a_{n-1})$. It can be seen easily that $\mathbf{I}_n$ is a homomorphic image of $\mathbf{U}$, which yields a required decomposition of $\mathbf{I}_n$.

Finally, assume that $b_1 y \in \{b_1, \ldots, b_k\}$. Then $k \geqq 2$ and $b_1 y = b_i$ for some $i \neq 2$, $1 \leqq i \leqq k$. Let $D = \{b_2, \ldots, b_{i-1}\}$ if $i \neq 1$ and $D = \{b_2, \ldots, b_k\}$ if $i = 1$. Consider the α_1-power $\mathbf{A}^{n-1}(\{x\}, \varphi)$, where

$$\varphi_1(u_1, x) = \begin{cases} y & \text{if} \quad u_1 = b_1, \\ x_j & \text{if} \quad u_1 = b_j \quad \text{for some} \quad 2 \leqq j \leqq k, \\ \text{arbitrary input sign from } X \text{ otherwise,} \end{cases}$$

$$\varphi_t(u_1, \ldots, u_t, x) = \begin{cases} y & \text{if} \quad u_t = b_1 \quad \text{and} \quad \{u_1, \ldots, u_{t-1}\} \cap D = \emptyset, \\ x_j & \text{if} \quad u_t = b_j \quad \text{for some} \quad 2 \leqq j \leqq k, \\ x_1 & \text{if} \quad u_t = b_1 \quad \text{and} \quad \{u_1, \ldots, u_{t-1}\} \cap D \neq \emptyset, \\ \text{arbitrary input sign from } X \text{ otherwise,} \end{cases}$$

for any $(u_1, \ldots, u_{n-1}) \in A^{n-1}$, $2 \leqq t \leqq n-1$. Let b_r denote that element of D which has the greatest index. Define the state $(a_1, \ldots, a_{n-1}) \in A^{n-1}$ in the following way:

$$a_1 = b_r, \quad a_2 = b_{r-1}, \ldots, a_r = b_1 \text{ and}$$

$$a_{t+1} = \begin{cases} b_{j-1} & \text{if} \quad a_t = b_j \quad \text{for some} \quad 1 < j \leqq k, \\ b_k & \text{if} \quad a_t = b_1, \end{cases}$$

where $r \leqq t < n-1$. Denote by $\mathbf{U}$ the subautomaton generated by $(a_1, \ldots, a_{n-1})$. It is not difficult to prove that $\mathbf{I}_n$ is a homomorphic image of $\mathbf{U}$ and thus, we have a required decomposition of $\mathbf{I}_n$.

It remained to prove the sufficiency of condition (2). But this can be seen easily, using Theorem 4 and the fact that the α_0-product of α_1-products is an α_1-product. This ends the proof of Theorem 5.

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(Received Jan. 19, 1981)