Algebraic representation of language hierarchies

By T. GERGELY

1. Introduction

The investigation of the connections between completely different languages or between theories formulated within these languages is a problem of growing importance in System Science, in Theoretical Linguistics and in many branches of Computer Science. E.g. this problem has arisen in high level program specification (see e.g. BURSTALL—GOGUEN [6, 7] and DÖMÖLKI [9]) in abstract data type research (see e.g. HUPBACH [13]) and in computer system modelling (see e.g. RATTRAY—RUS [17]).

In order to establish a connection between two languages first a connection i.e. a method of translation between their syntax might be looked for. Another possibility is connected with the interpretation of one syntax into another by introducing appropriate mathematical tools (see e.g. MONK [15] and BLUM—ESTES [5]). However usually there are a lot of possibilities of interpretation. As to handle them together, i.e. to investigate the possible connections in a complex way, the so called theory morphisms have been introduced (see e.g. AGN [3], BURSTALL—GOGUEN [6] and WINKOWSKI [19]). It turned out that category theory provides an adequate frame for the required complex analysis. However it would be quite useful to characterize the category corresponding to language hierarchy by the use of a well developed "culture" like universal algebra. Here we show that this characterization is possible by the use of the culture of cylindric algebras.

Throughout the paper it is supposed that the reader is familiar with basic notions of universal algebra and category theory.

2. Locally finite dimensional cylindric algebras

Cylindric algebras provide a tool to handle classical first order logic properly in algebraical way. They are in the same relationship to first order logic as Boolean algebras are to propositional logic. Here we present the basic notions and properties of the theory of these algebras relevant to our aim.

Definition 2.1. A similarity type t is a pair of functions $\langle t_F, t_R\rangle$ such that $\mathrm{Rg}\, t_F \subseteq \omega$ and $\mathrm{Rg}\, t_R \subseteq \omega \setminus \{0\}$, $\mathrm{Do}\, t_F \cap \mathrm{Do}\, t_R = \emptyset$. The elements of $\mathrm{Do}\, t_F$ and $\mathrm{Do}\, t_R$

are called function and relation symbols, respectively. Here Dof and Rgf stand for the domain and range of the function f respectively. □

Note that a similarity type could be defined in such a way that it contains only relation symbols because functions are but special relations (cf. AGN [4]).

Let t be an arbitrary similarity type with $t_R=\emptyset$. The class of all t-type algebras will be denoted by Alg (t). The class of all t-type algebras forms a category denoted by **Alg** (t) in the usual way i.e. the class of objects is Ob (**Alg** (t))=Alg (t) and the class of morphisms consist of all the homomorphisms. Further on, the boldface version of a notion corresponding to a class of algebras refers to the corresponding category.

Let us fix and ordinal α and the following similarity type $l_\alpha=\{\langle +, 2\rangle, \langle \cdot, 2\rangle, \langle -, 1\rangle, \langle 0, 0\rangle, \langle 1, 0\rangle\}\cup\{\langle c_i, 0\rangle\colon i<\alpha\}\cup\{\langle d_{ij}, 0\rangle\colon i, j<\alpha\}$, which for the sake of convenience is denoted by

$$l_\alpha = \{\langle +, 2\rangle, \langle \cdot, 2\rangle, \langle -, 1\rangle, \langle 0, 0\rangle, \langle 1, 0\rangle, \langle c_i, 0\rangle, \langle d_{ij}, 0\rangle \colon i, j<\alpha\}.$$

Now we define a special subclass of Alg (l_α) as follows.

Definition 2.2. An l_α-type algebra $\mathfrak{A}=\langle A, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, -^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}}, c_i^{\mathfrak{A}}, d_{ij}^{\mathfrak{A}}\rangle_{i,j<\alpha}$ is said to be a cylindric algebra of dimension α iff it satisfies the conditions below. (For the sake of convenience we omit the supercript $\mathfrak{A}$ speaking about the concret operations of a model $\mathfrak{A}$, i.e. where it does not lead to ambiguity we simply write $\mathfrak{A}=\langle A, +, \cdot, -, 0, 1, c_i, d_{ij}\rangle_{i,j<\alpha}$.)

(i) $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra,
(ii) $c_i 0=0$,
(iii) $c_i x\cdot x=x$,
(iv) $c_i(x\cdot c_i y) = c_i x\cdot c_i y$
(v) $c_i c_j x=c_j c_i x$,
(vi) $d_{ii}=1$,
(vii) if $i\neq j$, n then $d_{jn}=c_i(d_{ji}\cdot d_{in})$,
(viii) if $i\neq j$ then $c_i(d_{ij}\cdot x)\cdot c_i(d_{ij}\cdot -x)=0$ for any $i, j<\alpha$. □

Further on the Gothic capital letters refer to algebras while the corresponding Roman capital letters do to their universe.

Let CA_α denote the class of all cylindric algebras of dimension α. The homomorphisms on CA_α are defined as usually, i.e. such that they preserve all operations of the cylindric algebras. The intuition for CA_α theory comes from cylindric set algebras a systematic exposition of which is HMTAN [12].

NOTATION. Sb $K\overset{d}{=}\{X\colon X\subseteq K\}$ for any class K.

Definition 2.3. Let $\mathfrak{A}\in\mathrm{Alg}\,(l_\alpha)$. The function $\Delta^{\mathfrak{A}}\colon A\to\mathrm{Sb}\,\alpha$, which renders to any $a\in A$ the following set $\Delta^{\mathfrak{A}}a\overset{d}{=}\{i\in\alpha\colon c_i^{\mathfrak{A}}\, a\neq a\}$ is said to be the *dimension-sensitivity function.* □

Definition 2.4. The following class of l_α-type algebras $\mathrm{LF}_\alpha=\{\mathfrak{A}\in\mathrm{Alg}\,(l_\alpha)\colon$ for any $a\in A$, $|\Delta^{\mathfrak{A}}a|<\omega\}$ is said to be the class of *locally finite dimensional algebras.* □

Proposition 2.1. Let $\mathfrak{A}, \mathfrak{B} \in \mathrm{Alg}(l_\alpha)$, $a \in A$ and let $f: \mathfrak{A} \to \mathfrak{B}$ be a homomorphism. Then $\Delta^{\mathfrak{B}} f(a) \subseteq \Delta^{\mathfrak{A}} a$.

Proof. Let $i \in \Delta f(a)$, i.e. $c_i f(a) \neq f(a)$. Since f is a homomorphism this is possible only in the case $c_i a \neq a$, i.e. when $i \in \Delta a$. □

Now let us define the locally finite dimensional cylindric algebras as follows.

Definition 2.5. $\mathrm{Lf}_\alpha \overset{\mathrm{d}}{=} \mathrm{CA}_\alpha \cap \mathrm{LF}_\alpha$. □

Now we turn to the relationships between first order logic and cylindric algebras.

First we recall some well-known notions of first order logic.

Let t be an arbitrary similarity type and α be an arbitrary ordinal. A t-type first order language of α variables with equality is a triple $\langle F_t^\alpha, M_t, \models \rangle$ where F_t^α is the set of all t-type formulas containing variable symbols belonging to the set $\{x_i: i \in \alpha\}$ of variables of cardinality $|\alpha|$, M_t denotes the class of all t-type models; $\models \subseteq M_t \times F_t^\alpha$ is the validity relation. It is supposed that the symbol $=$ of equality relation is interpreted in each model as identity.

If $\mathrm{Ax} \subseteq F_t^\alpha$ and $\varphi \in F_t^\alpha$ then $\mathrm{Ax} \models \varphi$ means that φ is a semantical consequence of Ax.

To each F_t^α there corresponds an l_α-type algebra the so called formula algebra $\mathfrak{F}_t^\alpha = \langle F_t^\alpha, +, \cdot, -, 0, 1, c_i, d_{ij}: i, j < \alpha \rangle$ where for any $\varphi, \psi \in F_t^\alpha$, $i, j < \alpha$

$\varphi + \psi$	stands for	$\varphi \vee \psi$,
$\varphi \cdot \psi$	stands for	$\varphi \wedge \psi$,
$-\varphi$	stands for	$\neg \varphi$,
0	stands for	$\neg x = x$,
1	stands for	$x = x$,
$c_i \varphi$	stands for	$\exists x_i \varphi$ and
d_{ij}	stands for	$x_i = x_j$.

Definition 2.6. A pair $T = \langle \mathrm{Ax}, F_t^\alpha \rangle$, where $\mathrm{Ax} \subseteq F_t^\alpha$ is said to be a *theory* in α variables. □

Note that a theory provides a sublanguage of $\langle F_t^\alpha, M_t, \models \rangle$, namely, the triple $\langle F_t^\alpha, \mathrm{Mod}(\mathrm{Ax}), \models \rangle$, where $\mathrm{Mod}(\mathrm{Ax}) \overset{\mathrm{d}}{=} \{\mathfrak{A} \in M_t: \mathfrak{A} \models \mathrm{Ax}\}$.

Let $T = \langle \mathrm{Ax}, F_t^\alpha \rangle$ be a theory and let $\equiv_T \subseteq F_t^\alpha \times F_t^\alpha$ be the semantic equivalence w.r.t. T defined as follows: For any $\varphi, \psi \in F_t^\alpha$, $\varphi \equiv_T \psi$ iff $\mathrm{Ax} \models \varphi \leftrightarrow \psi$. Further on for any $\varphi \in F_t^\alpha$ let $\varphi/\equiv_T$ denote the corresponding equivalence class, i.e. $\varphi/\equiv_T \overset{\mathrm{d}}{=} \{\psi \in F_t^\alpha: \varphi \equiv_T \psi\}$.

Definition 2.7. The equivalence classes $\varphi/\equiv_T$ ($\varphi \in F_t^\alpha$) are said to be *concepts* of the corresponding theory T. The set of concepts of a theory T is $C_T \overset{\mathrm{d}}{=} F_t^\alpha/\equiv_T$, where $F_t/\equiv_T$ means the factorization of the set of formulas into such classes any two elements of which are semantically equivalent w.r.t. T. □

Note that the classes of C_T contain both open and closed formulas. (A formula is closed if each variable symbol occurs bound in it.) With respect to the open formulas it is important to remark that interpreting them in a model the variable symbols occurring free should be handled as constants. (See Examples below.)

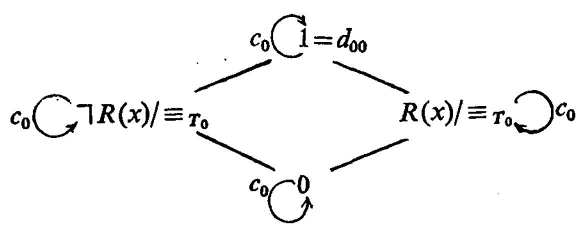
On the base of the set of concepts of a theory T we define another l_α-type algebra.

Definition 2.8. The concept algebra of a theory T is defined as follows. $\mathfrak{C}_T = \mathfrak{F}_t^\alpha / \equiv_T$, hence $\mathfrak{C}_T = \langle C_T, +, \cdot, -, 0, 1, c_i, d_{ij}: i, j < \alpha \rangle$. □

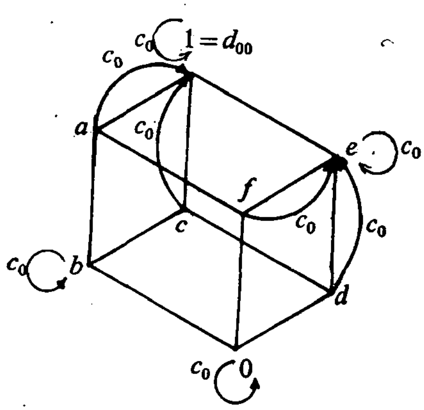
To see that this definition is correct one has to check that $\equiv_T$ is a congruence relation on the algebra $\mathfrak{F}_t^\alpha$.

Let us illustrate the notion of concept algebra by the following

EXAMPLES. a) Let $T_0 = \langle \mathrm{Ax}_0, F_{t_0}^1 \rangle$ be a theory, where $t_0 = \langle \emptyset, \{\langle R, 1 \rangle\} \rangle$ and $\mathrm{Ax}_0 = \{(\exists x R(x) \to \forall x R(x))\}$. Then the corresponding concept algebra is as follows. (About the graphical representation of algebras see AGN [4].)



b) Let $T_1 = \langle \mathrm{Ax}_1, F_{t_1}^1 \rangle$ be a theory where $t_1 = \langle \emptyset, \{\langle A, 1 \rangle\} \rangle$ and $\mathrm{Ax}_1 = \{\exists x \neg A(x)\}$. Then the corresponding concept algebra is as follows, where



$a = \neg A(x)/\equiv_{T_1}$,
$b = \neg \exists x A(x)/\equiv_{T_1}$,
$c = (\neg \exists x A(x) \vee A(x))/\equiv_{T_1}$,
$d = A(x)/\equiv_{T_1}$,
$e = \exists x A(x)/\equiv_{T_1}$ and
$f = (\neg A(x) \vee \exists x A(x))/\equiv_{T_1}$. □

Let C_α be the class of concept algebras with α variables, i.e. $C_\alpha \stackrel{\mathrm{d}}{=} \{\mathfrak{C}_T: T = \langle \mathrm{Ax}, F_t^\alpha \rangle, \mathrm{Ax} \subseteq F_t^\alpha, t \text{ is an arbitrary similarity type}\}$.

Note that concept algebras $\mathfrak{C}_T$ are denoted in Definition 12.22 of MONK [15] by $\mathfrak{M}_\Gamma^L$ (where L is a first order language and Γ is a set of sentences in L).

No we turn to the investigation of the connection of the classes C_α and Lf_α.

Proposition 2.2. Let $\mathfrak{C}_{\langle \mathrm{Ax}, F_t^\alpha \rangle} \in C_\alpha$. Then $\mathfrak{C}_{\langle \mathrm{Ax}, F_t^\alpha \rangle} \in \mathrm{Lf}_\alpha$.

Proof. Any formula $\varphi \in F_t^\alpha$ contains finitely many variables, the set of which, say, is $\mathrm{Var}\ \varphi$. Let $x_k \in \mathrm{Var}\ \varphi$ for some $k < \alpha$, then $\varphi \equiv_T \exists x_k \varphi$. Thus $\Delta\varphi \subseteq$

$\subseteq \{i: x_i \in \text{Var}\, \varphi\}$ so it is finite. It is easy to verify that $\mathfrak{C}_{\langle Ax, F_t \rangle}$ satisfies conditions (i)—(viii) of Definition 2.2. □

Let $\mathbf{C}_\alpha$ be defined to be the full subcategory of $\mathbf{Alg}(l_\alpha)$ such that $\text{Ob}\, \mathbf{C}_\alpha = C_\alpha$.

Now we turn to the investigation of the role of the category $\mathbf{C}_\alpha$ w.r.t. other subcategories of $\mathbf{Alg}(l_\alpha)$. First we recall (see MAC LANE [14])

Definition 2.9. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two arbitrary categories. A functor F from $\mathbf{A}_1$ into $\mathbf{A}_2$ is defined to be a pair $F=(F_{\text{Ob}}, F_{\text{Mor}})$ of functions $F_{\text{Ob}}\colon \text{Ob}\, \mathbf{A}_1 \to \text{Ob}\, \mathbf{A}_2$ and $F_{\text{Mor}}\colon \text{Mor}\, \mathbf{A}_1 \to \text{Mor}\, \mathbf{A}_2$ such that (i)—(iii) below hold:

(i) If $f \in \text{Hom}(A, B)$ in $\mathbf{A}_1$ then $F_{\text{Mor}}(f) \in \text{Hom}\big(F_{\text{Ob}}(A), F_{\text{Ob}}(B)\big)$ in $\mathbf{A}_2$;

(ii) $F_{\text{Mor}}(f \circ g) = F_{\text{Mor}}(f) \circ F_{\text{Mor}}(g)$ for all $f, g \in \text{Mor}\, \mathbf{A}_1$;

(iii) $F_{\text{Mor}}(\text{Id}_A) = \text{Id}_{F_{\text{Ob}}(A)}$ for any $A \in \text{Ob}\, \mathbf{A}_1$.

Here $\text{Id}_A\colon A \to A$ is the identity morphism corresponding to A. Note that instead of F_{Ob} and F_{Mor} we often write only F.

For a category $\mathbf{A}$ the identity functor $\text{Id}_{\mathbf{A}}$ sends A to A and f to f for all $A \in \text{Ob}\, \mathbf{A}$ and $f \in \text{Mor}\, \mathbf{A}$.

The categories $\mathbf{A}_1$ and $\mathbf{A}_2$ are *equivalent* iff there is a functor $F\colon \mathbf{A}_1 \to \mathbf{A}_2$, to which there is a backward functor $G\colon \mathbf{A}_2 \to \mathbf{A}_1$ and there are two natural isomorphisms $\theta\colon F \circ G \dot{\to} \text{Id}_{\mathbf{A}_2}$ and $\nu\colon G \circ F \dot{\to} \text{Id}_{\mathbf{A}_1}$.

The categories $\mathbf{A}_1$ and $\mathbf{A}_2$ are *isomorphic* iff there are functors $F\colon \mathbf{A}_1 \to \mathbf{A}_2$ and $G = \mathbf{A}_2 \to \mathbf{A}_1$ such that $G \circ F = \text{Id}_{\mathbf{A}_1}$ and $F \circ G = \text{Id}_{\mathbf{A}_2}$. □

Theorem 2.3. Let $\alpha \geqq \omega$ be an arbitrary infinite ordinal. The categories $\mathbf{Lf}_\alpha$ and $\mathbf{C}_\alpha$ are equivalent.

This theorem immediately follows from the following

Theorem 2.4. Let $\alpha \geqq \omega$. There are two full and faithful one-one functors $F\colon \mathbf{C}_\alpha \to \mathbf{Lf}_\alpha$ and $G\colon \mathbf{Lf}_\alpha \to \mathbf{C}_\alpha$ and two natural isomorphisms $\theta\colon F \circ G \to \text{Id}_{\mathbf{Lf}_\alpha}$ and $\nu\colon G \circ F \dot{\to} \text{Id}_{\mathbf{C}_\alpha}$ such that the functions F, G, θ and ν are definable (in a parameter free way) in ZFC set theory by formulas which are absolute (in set theoretical sense) and moreover these functions are primitive recursive (in the sense of DEVLIN [8] p. 29).

Proof. I. First we define the functors.

1. Let $\mathfrak{A} \in \text{Ob}\, \mathbf{Lf}_\alpha$. From 12.18, 12.25 and 12.28 of MONK [15], see also Theorem 5.2 of AGN [1] and Proposition 1 in [16], it follows that there is a theory $T_{\mathfrak{A}}$, i.e. a similarity type $t_{\mathfrak{A}}$ together with the corresponding set of formulas $F^\alpha_{t_{\mathfrak{A}}}$ and a set $Ax_{\mathfrak{A}}$ of axioms such that $\mathfrak{A} \cong \mathfrak{C}_{T_{\mathfrak{A}}}$. Moreover from the proof of 12.28 of MONK [15] it follows that there is a function $F_{\text{Ob}}\colon \text{Ob}\, \mathbf{Lf}_\alpha \to \text{Ob}\, \mathbf{C}_\alpha$ such that

(i) for any $\mathfrak{A} \in \text{Ob}\, \mathbf{Lf}_\alpha$ $F_{\text{Ob}}(\mathfrak{A}) = \mathfrak{C}_{T_{\mathfrak{A}}}$;

(ii) there exists a function $\theta\colon \text{Ob}\, \mathbf{Lf}_\alpha \to \text{Mor}\, \mathbf{Lf}_\alpha$ such that $\theta(\mathfrak{A}) = \text{Is}\big(F_{\text{Ob}}(\mathfrak{A}), \mathfrak{A}\big)$ for any $\mathfrak{A} \in \text{Ob}\, \mathbf{Lf}_\alpha$. Here $\text{Is}(\mathfrak{A}, \mathfrak{B})$ denotes the set of isomorphisms from $\mathfrak{A}$ onto $\mathfrak{B}$.

(iii) the functions F_{Ob} and θ are definable in ZFC, i.e. there are set theoretic formulas $\varphi(x, y)$ and $\psi(x, y)$ such that

$$\text{ZFC} \vdash (\forall x \in \text{Ob}\, \mathbf{Lf}_\alpha)(\exists !\, y \varphi(x, y) \wedge \exists !\, y \psi(x, y))$$

and

$$\text{ZFC} \vdash (\forall x \in \text{Ob}\, \mathbf{Lf}_\alpha)\, \forall y, z\big((\varphi(x, y) \wedge \psi(x, z)) \to (y \in \text{Ob}\, \mathbf{C}_\alpha \wedge z \in \text{Is}\,(x, y))\big).$$

Above we assumed that Ob $\mathbf{Lf}_\alpha$ and Ob $\mathbf{C}_\alpha$ are also definable in ZFC, i.e. the expression "$y \in \text{Ob}\,\mathbf{C}_\alpha$" and "$y \in \text{Ob}\,\mathbf{Lf}_\alpha$" are formulas of one free variable y in ZFC. We omit the proof that this assumption is justified. Similarly "$z \in \text{Is}\,(x, y)$" is also a formula of ZFC of free variables x, y and z.

Moreover the formulas $\varphi(x, y)$ and $\psi(x, y)$ are absolute (in set theoretical sense).

(iv) The functions F_{Ob} and θ are primitive recursive in the sense of DEVLIN [8], i.e. they can be generated by the schemata (i)—(vii) of [8], p. 29. (And, even more we believe that these functions are rudimentary.)

Let $f \in \text{Mor}\,\mathbf{Lf}_\alpha$, namely let $f \in \text{Hom}\,(\mathfrak{A}, \mathfrak{B})$ for some $\mathfrak{A}, \mathfrak{B} \in \text{Ob}\,\mathbf{Lf}_\alpha$. We define $F_{\text{Mor}}(f) \stackrel{\text{d}}{=} [\theta(\mathfrak{B})]^{-1} \circ f \circ \theta(\mathfrak{A})$. Then clearly $F_{\text{Mor}}(f) \in \text{Hom}\,\big(F_{\text{Ob}}(\mathfrak{A}), F_{\text{Ob}}(\mathfrak{B})\big) \subseteq$ $\subseteq \text{Mor}\,\mathbf{C}_\alpha$.

It is not difficult to verify that this function preserves composition and identity. Thus the pair $F = \langle F_{\text{Ob}}, F_{\text{Mor}} \rangle$ is a functor. Since the function θ is definable by an absolute formula of ZFC so is F_{Mor} and thus so is the functor F as well.

Now we show that the functor F is one-one.

a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be two different elements of Ob $\mathbf{Lf}_\alpha$. Recall that at the beginning of the proof to every $\mathfrak{A} \in \text{Ob}\,\mathbf{Lf}_\alpha$ a theory $T_\mathfrak{A}$ was associated in a fixed way such that $T_\mathfrak{A}$ should be the theory constructed from $\mathfrak{A}$ in the proof of 12.28 [15]. We also recall that for any $\mathfrak{A} \in \text{Ob}\,\mathbf{Lf}_\alpha$ $F(\mathfrak{A}) = C_{T_\mathfrak{A}}$.

(i) First we suppose that $\mathfrak{A} \neq \mathfrak{B}$ because $A \neq B$. In this case using the construction provided by MONK in the proof of 12.28 [15] we get different $F_{t_\mathfrak{A}}, F_{t_\mathfrak{B}}$, i.e. $F_{t_\mathfrak{A}} \neq F_{t_\mathfrak{B}}$. Hence $\mathbf{C}_{T_\mathfrak{A}} \neq \mathbf{C}_{T_\mathfrak{B}}$.

(ii) Let $A = B$. Since $\mathfrak{A} \neq \mathfrak{B}$ there is at least one operation symbol h say of n arguments and there are $a_1, \ldots, a_n \in A$ such that $h^\mathfrak{A}(a_1, \ldots, a_n) = a_0$ but $h^\mathfrak{B}(a_1, \ldots, a_n) \neq a_0$. Therefore $\text{Ax}_\mathfrak{A} \neq \text{Ax}_\mathfrak{B}$.

Hence $\mathbf{C}_{T_\mathfrak{A}} \neq \mathbf{C}_{T_\mathfrak{B}}$. Thus F_{Ob} is one-one.

b) Since F_{Ob} is one-one it is sufficient to prove that F_{Mor} is one-one on Hom $(\mathfrak{A}, \mathfrak{B})$ for each $\mathfrak{A}, \mathfrak{B} \in \text{Ob}\,\mathbf{Lf}_\alpha$.

Let $f \circ g \in \text{Hom}\,(\mathfrak{A}, \mathfrak{B})$ be two elements of Mor $\mathbf{Lf}_\alpha$ such that $f \neq g$. By the definition of F_{Mor} obviously $F_{\text{Mor}}(f) \neq F_{\text{Mor}}(g)$. Thus F_{Ob} and F_{Mor} are one-one functions and F is so as well.

2. Now let us define the functor $G: \mathbf{C}_\alpha \to \mathbf{Lf}_\alpha$. From Proposition 2.2 it follows that for G we can choose the identical embedding, i.e. let $G = \langle G_{\text{Ob}}, G_{\text{Mor}} \rangle$ be such that for any $\mathfrak{A} \in \text{Ob}\,\mathbf{C}_\alpha$ and $f \in \text{Mor}\,\mathbf{C}_\alpha$, $G_{\text{Ob}}(\mathfrak{A}) = \mathfrak{A}$ and $G_{\text{Mor}}(f) = f$. Clearly the functor G is definable by an absolute set theoretic formula and it is one-one, full and faithful.

From the above observations we have the following

Lemma 2.4.1. For any $\mathfrak{A} \in \text{Ob}\,\mathbf{Lf}_\alpha$ and $f \in \text{Mor}\,\mathbf{Lf}_\alpha$

$$G \circ F(\mathfrak{A}) = F(\mathfrak{A}),\ G \circ F(f) = F(f)$$

and for any $\mathfrak{A} \in \text{Ob}\,\mathbf{C}_\alpha$ and $f \in \text{Mor}\,\mathbf{C}_\alpha$

$$F \circ G(\mathfrak{A}) = F(\mathfrak{A}),\ F \circ G(f) = F(f).$$

II. Now we turn to the construction of the appropriate natural isomorphisms.

1. First we show that the function θ: Ob $\mathbf{Lf}_\alpha \to$ Mor $\mathbf{Lf}_\alpha$ defined in I.1 (ii) of this proof is a natural transformation from $G \circ F$ to $\mathrm{Id}_{\mathbf{Lf}_\alpha}$ which we denote following MAC LANE [14] by θ: $G \circ F \dot{\to} \mathrm{Id}_{\mathbf{Lf}_\alpha}$.

We would need a diagram of type

$$\begin{array}{ccc} G\circ F(\mathfrak{A}) & \xrightarrow{G\circ F(f)} & G\circ F(\mathfrak{B}) \\ \Big\downarrow \theta(\mathfrak{A}) & & \Big\downarrow \theta(\mathfrak{B}) \\ \mathrm{Id}_{\mathbf{Lf}_\alpha}(\mathfrak{A}) & \xrightarrow{\mathrm{Id}_{\mathbf{Lf}_\alpha}(f)} & \mathrm{Id}_{\mathbf{Lf}_\alpha}(\mathfrak{B}) \end{array} \qquad (*)$$

By Lemma 2.4.1 instead of the above diagram it is enough to consider the following one:

$$\begin{array}{ccc} F(\mathfrak{A}) & \xrightarrow{F(f)} & F(\mathfrak{B}) \\ \Big\downarrow \theta(\mathfrak{A}) & & \Big\downarrow \theta(\mathfrak{B}) \\ \mathfrak{A} & \xrightarrow{f} & \mathfrak{B} \end{array}$$

This diagram exists, so by Lemma 2.4.1 the diagram $(*)$ does exist as well.

By the definition of F_{Mor} we have: $F_{\mathrm{Mor}}(f)=[\theta(\mathfrak{B})]^{-1}\circ f\circ\theta(\mathfrak{A})$. Now it is easy to establish that the diagram commutes.

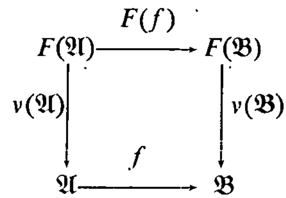
$$\theta(\mathfrak{B})\circ F(f) = \theta(\mathfrak{B})\circ[\theta(\mathfrak{B})]^{-1}\circ f\circ\theta(\mathfrak{A}) = f\circ\theta(\mathfrak{A}).$$

So θ: $G\circ F \to \mathrm{Id}_{\mathbf{Lf}_\alpha}$ is a natural transformation. Since for each $\mathfrak{A}\in$ Ob $\mathbf{Lf}_\alpha$, $\theta(\mathfrak{A})\in \mathrm{Is}\,(G\circ F(\mathfrak{A}), \mathrm{Id}_{\mathbf{Lf}_\alpha}(\mathfrak{A}))$ we have that θ is a natural isomorphism.

2. Now we define ν: $F\circ G \dot{\to} \mathrm{Id}_{\mathbf{C}_\alpha}$. Let $\nu \overset{\mathrm{d}}{=} \theta \restriction \mathbf{C}_\alpha$. That is ν: Ob $\mathbf{C}_\alpha \to$ Mor $\mathbf{Lf}_\alpha$ such that for any $\mathfrak{A}\in$ Ob $\mathbf{C}_\alpha$, $\nu(\mathfrak{A})=\theta(\mathfrak{A})$. Then for any $\mathfrak{A}\in$ Ob $\mathbf{C}_\alpha$, $\nu(\mathfrak{A})\in$ Is $(F\circ G(\mathfrak{A}), \mathrm{Id}_{\mathbf{C}_\alpha}(\mathfrak{A}))$. Let $\mathfrak{A}, \mathfrak{B}\in$ Ob $\mathbf{C}$ and $f\in$ Hom $(\mathfrak{A}, \mathfrak{B})$. Consider the following diagram

$$\begin{array}{ccc} F\circ G(\mathfrak{A}) & \xrightarrow{F\circ G(f)} & F\circ G(\mathfrak{B}) \\ \Big\downarrow \nu(\mathfrak{A}) & & \Big\downarrow \nu(\mathfrak{B}) \\ \mathrm{Id}_{\mathbf{C}_\alpha}(\mathfrak{A}) & \xrightarrow{\mathrm{Id}_{\mathbf{C}_\alpha}(f)} & \mathrm{Id}_{\mathbf{C}_\alpha}(\mathfrak{B}) \end{array}$$

By Lemma 2.4.1 instead of the above diagram it is enough to consider the following one



In II.1 we have already seen that this diagram commutes. Thus $\nu\colon F \circ G \dot{\to} \mathrm{Id}_{\mathbf{C}_\alpha}$ is a natural isomorphism.

III. The definiability of F, G, θ, ν by absolute set theoretic parameter free formulas follows from this property of F_{Ob} and θ established in I.1 (iii) and from the construction of F, G, θ, ν by using F_{Ob} and θ.

The primitive recursiveness of the functions $F_{\mathrm{Ob}}, F_{\mathrm{Mor}}, G_{\mathrm{Ob}}, G_{\mathrm{Mor}}, \theta, \nu$ can be established analogously. □

The above theorem raises the question about the isomorphism of the categories under consideration. We show that isomorphism does occur, indeed.

Theorem 2.5. Let $\alpha \geqq \omega$. The categories $\mathbf{Lf}_\alpha$ and $\mathbf{C}_\alpha$ are isomorphic, i.e. $\mathbf{Lf}_\alpha \cong \mathbf{C}_\alpha$.

Proof. To prove the statement we construct an isomorphism $H\colon \mathbf{Lf}_\alpha \to \mathbf{C}_\alpha$, which is a one-one and onto functor, both on objects and on morphisms. For the construction of H first we define a covering of the category $\mathbf{Lf}_\alpha$ and then we define H on this covering such that the image of H covers the category $\mathbf{C}_\alpha$.

By Theorem 2.4 we have a one-one endofunctor $F\colon \mathbf{C}_\alpha \to \mathbf{Lf}_\alpha$ and a natural isomorphism $\theta\colon F \dot{\to} \mathrm{Id}_{\mathbf{Lf}_\alpha}$, which sends F into $\mathrm{Id}_{\mathbf{Lf}_\alpha}$.

(Note that here we use the fact provided by Lemma 2.4.1 that $G\colon \mathbf{Lf}_\alpha \to \mathbf{C}_\alpha$ is an identity functor.)

First we construct the covering of $\mathrm{Ob}\,\mathbf{Lf}_\alpha$ by induction as follows.

Take $L_0 \overset{\mathrm{d}}{=} \mathrm{Ob}\,\mathbf{Lf}_\alpha$.

We need the following notation. Let $\mathbf{A}$ be an arbitrary category and R be a functor on $\mathbf{A}$. Then for any subclass $S \subseteqq \mathrm{Ob}\,\mathbf{A}$ the R image of S is defined as follows

$$R^* S \overset{\mathrm{d}}{=} \{R_{\mathrm{Ob}}(\mathfrak{A}) : \mathfrak{A} \in S\}.$$

Take $K_0 \overset{\mathrm{d}}{=} \mathrm{Ob}\,\mathbf{C}_\alpha$. (It is evident that $K_0 \subseteqq L_0$.)

Furthermore let

$$L_1 \overset{\mathrm{d}}{=} F^* L_0 \quad \text{(Clearly } L_1 \subseteqq K_0\text{.)}$$

$$K_1 \overset{\mathrm{d}}{=} F^* K_0 \quad \text{(Since } K_0 \subseteqq L_0 \text{ we have } K_1 \subseteqq L_1\text{.)}$$

Let us suppose that the classes L_n and K_n have already been defined up to some n.

Then let

$$L_{n+1} \overset{\mathrm{d}}{=} F^* L_n \quad \text{and} \quad K_{n+1} \overset{\mathrm{d}}{=} F^* K_n.$$

Thus the classes L_n and K_n have been defined for any $n \in \omega$ by induction. They are illustrated by Fig 1.

For any $n \in \omega$ let $W_n \overset{d}{=} K_n \setminus L_{n+1}$ and let $W \overset{d}{=} \bigcup_{n \in \omega} W_n$.

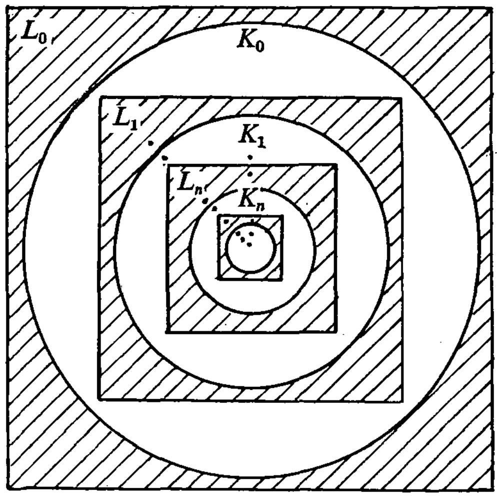


Fig. 1

Moreover let $D \overset{d}{=} L_0 \setminus W$ (note that $D = \bigcup_{n \in \omega} (L_n \setminus K_n)$).

On Fig. 1. the white area corresponds to W and the dark one to D.

It follows from the cónstruction that Ob $\mathbf{Lf}_\alpha$ is covered by the disjoint union of D and W, i.e. Ob $\mathbf{Lf}_\alpha = D \cup W$.

Now we construct a covering to $\mathbf{C}_\alpha$ by giving a function $H_{\mathrm{Ob}} = \mathrm{Ob}\, \mathbf{Lf}_\alpha \rightarrowtail\!\!\!\!\twoheadrightarrow \mathrm{Ob}\, \mathbf{C}_\alpha$ as follows.

For any $\mathfrak{A} \in D$ let $H_{\mathrm{Ob}}(\mathfrak{A}) \overset{d}{=} F(\mathfrak{A})$ and for any $\mathfrak{B} \in W$ let $H_{\mathrm{ob}}(\mathfrak{B}) = \mathfrak{B}$, i.e. $H_{\mathrm{Ob}} = (F_{\mathrm{Ob}} \restriction D) \cup \mathrm{Id} \restriction W$. Clearly H_{Ob}: Ob $\mathbf{Lf}_\alpha \rightarrowtail\!\!\!\!\twoheadrightarrow$ Ob $\mathbf{C}_\alpha$ is one-one and onto Ob $\mathbf{C}_\alpha$ since Ob $\mathbf{Lf}_\alpha = L_0$ and Ob $\mathbf{C}_\alpha = K_0$ that is H_{Ob}: $L_0 \rightarrowtail\!\!\!\!\twoheadrightarrow K_0$. Note that $H_{\mathrm{Ob}} = F_{\mathrm{Ob}} \restriction D \cup G_{\mathrm{Ob}}^{-1} \restriction W$.

Now we define the mapping H_{Mor}: Mor $\mathbf{Lf}_\alpha \rightarrowtail\!\!\!\!\twoheadrightarrow$ Mor $\mathbf{C}_\alpha$. We distinguish four cases:

1. Let $\mathfrak{A}, \mathfrak{B} \in W$ and $f \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$. Then we define $H_{\mathrm{Mor}}(f) \overset{d}{=} f$.

2. Let $\mathfrak{A}, \mathfrak{B} \in D$ and $f \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$. Then we define $H_{\mathrm{Mor}}(f) \overset{d}{=} F(f)$.

3. Let $\mathfrak{A} \in D$, $\mathfrak{B} \in W$ and $f \in \mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$.

Since θ: $F \dot{\to} \mathrm{Id}_{\mathbf{Lf}_\alpha}$ is a natural isomorphism we have $F(\mathfrak{B}) \xrightarrow{\theta(\mathfrak{B})} \mathfrak{B} = H(\mathfrak{B})$. Then $H(\mathfrak{A}) = F(\mathfrak{A}) \xrightarrow{F(f)} F(\mathfrak{B}) \xrightarrow{\theta(\mathfrak{B})} H(\mathfrak{B})$. We define $H_{\mathrm{Mor}}(f) \overset{d}{=} \theta(\mathfrak{B}) \circ F(f)$. It is evident that $H(f) \in \mathrm{Hom}(H(\mathfrak{A}), H(\mathfrak{B}))$.

4. Let $\mathfrak{A} \in D$, $\mathfrak{B} \in W$ and $f \in \mathrm{Hom}(\mathfrak{B}, \mathfrak{A})$. For this case we define $H_{\mathrm{Mor}}(f) \overset{d}{=} F(f) \circ [\theta(\mathfrak{B})]^{-1}$. By the above cases 1—4 the mapping H_{Mor}: Mor $\mathbf{Lf}_\alpha \to$ Mor $\mathbf{C}_\alpha$ is defined. Since by Theorem 2.4 the functor F is full, faithful and one-one, it is

easy to verify that the mapping H_{Mor} is onto and one-one such that for any $\mathfrak{A}, \mathfrak{B} \in \text{Ob } \mathbf{Lf}_\alpha$ and $f \in \text{Hom}(\mathfrak{A}, \mathfrak{B})$ we have $H_{\text{Mor}}(f) \in \text{Hom}(H_{\text{Ob}}(\mathfrak{A}), H_{\text{Ob}}(\mathfrak{B}))$. For illustration to H_{Mor} see Fig. 2.

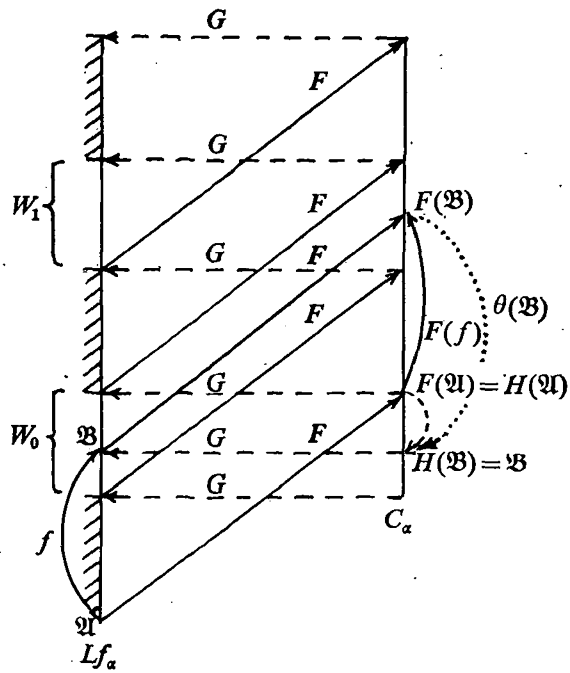


Fig. 2

Let $H \stackrel{\text{d}}{=} \langle H_{\text{Ob}}, H_{\text{Mor}} \rangle$. For the verification that H is a functor, properties (i)—(iii) displayed in Definition 2.9 should be established. The properties (i) and (iii) are satisfied by definition. Let $f \in \text{Hom}(\mathfrak{A}, \mathfrak{B})$ and $g \in \text{Hom}(\mathfrak{B}, \mathfrak{C})$. To verify property (ii) the following cases should be checked.

a) $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in D$,

b) $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in W$,

c) $\mathfrak{A}, \mathfrak{B} \in D$, $\mathfrak{C} \in W$,

d) $\mathfrak{A}, \mathfrak{B} \in W$, $\mathfrak{C} \in D$,

e) $\mathfrak{A} \in D$, $\mathfrak{B}, \mathfrak{C} \in W$,

f) $\mathfrak{A} \in W$, $\mathfrak{B}, \mathfrak{C} \in D$,

g) $\mathfrak{A}, \mathfrak{C} \in D$, $\mathfrak{B} \in W$,

h) $\mathfrak{A}, \mathfrak{C} \in W$, $\mathfrak{B} \in D$,

i) $\mathfrak{B}, \mathfrak{C} \in D$, $\mathfrak{A} \in W$

and

j) $\mathfrak{B}, \mathfrak{C} \in W, \quad \mathfrak{A} \in D.$

From the above cases we check the most difficult ones, namely g) and j)

g) $\mathfrak{A}, \mathfrak{C} \in D, \quad \mathfrak{B} \in W.$

By using the corresponding definitions we have the following diagram

$$\begin{array}{ccccc}
\mathfrak{C} & \qquad & H(\mathfrak{C}) = F(\mathfrak{C}) & = \!= \!= & F(\mathfrak{C}) \\
g \Big\uparrow & & H(g) = F(g) \circ [\theta(\mathfrak{B})]^{-1} \Big\uparrow & & \Big\uparrow F(g) \\
\mathfrak{B} & & H(\mathfrak{B}) = \mathfrak{B} & \xleftarrow{\;\theta(\mathfrak{B})\;} & F(\mathfrak{B}) \\
f \Big\uparrow & & H(f) = \theta(\mathfrak{B}) \circ F(f) \Big\uparrow & & \Big\uparrow F(f) \\
\mathfrak{A} & & H(\mathfrak{A}) = F(\mathfrak{A}) & = \!= \!= & F(\mathfrak{A})
\end{array}$$

By using the fact that F is a functor, from the above diagram we have

$$H(g) \circ H(f) = F(g) \circ [\theta(\mathfrak{B})]^{-1} \circ \theta(\mathfrak{B}) \circ F(f) =$$
$$= F(g) \circ \mathrm{Id}_{F(\mathfrak{B})} \circ F(f) = F(g) \circ F(f) = F(g \circ f).$$

Hence, by definition, we get

$$F(g \circ f) = H(g \circ f) \quad \text{since} \quad \mathfrak{A}, \mathfrak{C} \in D.$$

j) $\mathfrak{A} \in D, \quad \mathfrak{B}, \mathfrak{C} \in W.$

By using the corresponding definitions we have the following diagram

$$\begin{array}{ccccc}
\mathfrak{C} & \qquad & H(\mathfrak{C}) = \mathfrak{C} & \xleftarrow{\;\theta(\mathfrak{C})\;} & F(\mathfrak{C}) \\
g \Big\uparrow & & H(g) = g \Big\uparrow & & \Big\uparrow F(g) \\
\mathfrak{B} & & H(\mathfrak{B}) = \mathfrak{B} & \xleftarrow{\;\theta(\mathfrak{B})\;} & F(\mathfrak{B}) \\
f \Big\uparrow & & H(f) = \theta(\mathfrak{B}) \circ F(f) \Big\uparrow & & \Big\uparrow F(f) \\
\mathfrak{A} & & H(\mathfrak{A}) = F(\mathfrak{A}) & = \!= \!= & F(\mathfrak{A})
\end{array}$$

By using the fact that θ is a natural transformation and that F is a functor we get from the above diagram

$$H(g)\circ H(f) = g\circ\theta(\mathfrak{B})\circ F(f) =$$
$$= \theta(\mathfrak{C})\circ F(g)\circ F(f) = \theta(\mathfrak{C})\circ F(g\circ f)$$

which, by definition, is $H(g\circ f)$, since $\mathfrak{A}\in D$ and $\mathfrak{C}\in W$.

Thus H is a functor and by its construction, H is one-one and onto and thus H establishes an isomorphic connection between the categories $\mathbf{Lf}_\alpha$ and $\mathbf{C}_\alpha$. □

Some questions w.r.t. the functor H arise. Namely, we have the following

OPEN PROBLEMS:

— Is there an absolute isomorphism $M\colon \mathbf{Lf}_\alpha \rightarrowtail\!\!\twoheadrightarrow \mathbf{C}_\alpha$?
— Is the functor H constructed in the above proof definable by a quantifier free formula in ZFC?
— Is the functor H primitive recursive in the sense of DEVLIN [8]?
— Is there any isomorphism $I\colon \mathbf{Lf}_\alpha \rightarrowtail\!\!\twoheadrightarrow \mathbf{C}_\alpha$ which is rudimentary in the sense of DEVLIN [8]?

3. Category of theories

Let α be an ordinal. Definition 2.6 provides the notion of theories of α variables. However without supposing further conditions two theories T_1 and T_2 can have e.g. different sets Ax_1 and Ax_2 but one of these sets might be derivated from the other one by the use of an appropriate calculus, i.e. by the use of pure syntactical transformations. I.e. despite of their differences in their presentations the theories are equivalent. To avoid such cases we slightly modify Definition 2.6.

Definition 3.1. Let α be a fixed ordinal. Let t be an arbitrary similarity type and $\mathrm{Ax}\subseteq F_t^\alpha$. Take $\mathrm{Ax}^*\stackrel{d}{=}\{\varphi\colon \mathrm{Ax}\models\varphi\}$.

The pair $\langle \mathrm{Ax}^*, F_t^\alpha\rangle$ is said to be a *saturated theory* of α variables. □

Further on when speaking about a theory we have a saturated one in mind.

In the case of saturated theories we often identify a theory $T=\langle \mathrm{Ax}, F_t^\alpha\rangle$ with the set Ax of axioms.

Now we define how a theory can be interpreted in an other one.

Definition 3.2. Let $T_1=\langle \mathrm{Ax}_1, F_{t_1}^\alpha\rangle$ and $T_2=\langle \mathrm{Ax}_2, F_{t_2}^\alpha\rangle$ be theories in α variables. Let $m\colon F_{t_1}^\alpha\to F_{t_2}^\alpha$.

The triple $\langle T_1, m, T_2\rangle$ is said to be an *interpretation* going from T_1 into T_2 iff the following conditions hold:

a) $m(x_i=x_j)=x_i=x_j$ for every $i, j<\alpha$;
b) $m(\varphi\wedge\psi)=m(\varphi)\wedge m(\psi)$, $m(\neg\varphi)=\neg m(\varphi)$;
$m(\exists x_i\varphi)=\exists x_i m(\varphi)$ for all $\varphi, \psi\in F_{t_1}^\alpha$, $i<\alpha$;
c) $\mathrm{Ax}_2\models m(\varphi)$ for all $\varphi\in F_{t_1}^\alpha$ such that $\mathrm{Ax}_1\models\varphi$.

We shall often say that m is an interpretation but in these cases we actually mean $\langle T_1, m, T_2\rangle$. By saying that $\langle T_1, m, T_2\rangle$ is an interpretation we mean that $\langle T_1, m, T_2\rangle$ is an interpretation of the theory T_1 in the theory T_2. □

Let m, n be two interpretations of T_1 in T_2.

The interpretations $\langle T_1, m, T_2\rangle$, $\langle T_1, n, T_2\rangle$ are defined to be *semantically equivalent,* in symbols $m \equiv n$, iff the following condition holds:

$$\mathrm{Ax}_2 \models (m(\varphi) \leftrightarrow n(\varphi)) \quad \text{for all } \varphi \in F^\alpha_{t_1}.$$

Let $\langle T_1, m, T_2\rangle$ be an interpretation. We define the equivalence class $m/\equiv$ of m or more precisely $\langle T_1, m, T_2\rangle\ /\equiv$ to be: $m/\equiv \stackrel{d}{=} \{\langle T_1, n, T_2\rangle : n \equiv m$ and n is an interpretation of T_1 in $T_2\}$.

Now we are ready to define the connection between two theories T_1 and T_2.

Definition 3.3. Let T_1 and T_2 be two theories of α variables.

By a *theory morphism* $\mu\colon T_1 \to T_2$ going from T_1 into T_2 we understand an equivalence class of interpretations of T_1 in T_2, i.e. μ is a theory morphism $\mu\colon T_1 \to T_2$ iff $\mu = m/\equiv$ for some interpretation $\langle T_1, m, T_2\rangle$. □

Definition 3.4. (i) $\mathbf{TH}_\alpha$ is defined to be the quadruple $\mathbf{TH}_\alpha \stackrel{d}{=} \langle \mathrm{Ob}\,\mathbf{TH}_\alpha, \mathrm{Mor}\,\mathbf{TH}_\alpha, \circ, \mathrm{Id}\rangle$, where the mappings $\circ\colon \mathrm{Mor}\,\mathbf{TH}_\alpha \times \mathrm{Mor}\,\mathbf{TH}_\alpha \to \mathrm{Mor}\,\mathbf{TH}_\alpha$ and $\mathrm{Id}\colon \mathrm{Ob}\,\mathbf{TH}_\alpha \to \mathrm{Mor}\,\mathbf{TH}_\alpha$ are defined in (ii)—(iii) below and $\mathrm{Ob}\,\mathbf{TH}_\alpha \stackrel{d}{=} \{T\colon T$ is a saturated theory in α variables$\}$, $\mathrm{Mor}\,\mathbf{TH}_\alpha \stackrel{d}{=} \{\langle T_1, \mu, T_2\rangle : \mu$ is a theory morphism $\mu\colon T_1 \to T_2$ and $T_1, T_2 \in \mathrm{Ob}\,\mathbf{TH}_\alpha\}$.

(ii) Let $\mu\colon T_1 \to T_2$ and $\nu\colon T_2 \to T_3$ be two theory morphisms. We define the *composition* $\nu \circ \mu\colon T_1 \to T_3$ to be the unique theory morphism for which there exists $m \in \mu$ and $n \in \nu$ such that $\nu \circ \mu = (n \circ m)\ /\equiv$, where the function $(n \circ m)\colon F^\alpha_{t_1} \to F^\alpha_{t_3}$ is defined by $(n \circ m)(\varphi) = n(m(\varphi))$ for all $\varphi \in F^\alpha_{t_1}$.

(iii) Let $T = \langle \mathrm{Ax}, F^\alpha_t\rangle$ be a theory. The identity function $\mathrm{Id}_{F^\alpha_t}$ is defined to be $\mathrm{Id}_{F^\alpha_t} \stackrel{d}{=} \{\langle \varphi, \varphi\rangle : \varphi \in F^\alpha_t\}$.

The *identity morphism* Id_T on T is defined to be $\mathrm{Id}_T \stackrel{d}{=} (\mathrm{Id}_{F^\alpha_t})/\equiv$. □

Proposition 3.1. $\mathbf{TH}_\alpha$ is a category.

Proof. The statement follows from the two properties bellow:

a) the composition defined in (ii) of Definition 3.4 is associative, i.e. let $\mu_1\colon T_1 \to T_2$, $\mu_2\colon T_2 \to T_3$ and $\mu_3\colon T_3 \to T_4$ be theory morphisms and let $m_i \in \mu_i$ for $i \in \{1, 2, 3\}$. By associativity of composition of ordinary mappings $m_3 \circ m_2 \circ m_1 \in \mu_3 \circ (\mu_2 \circ \mu_1)$ and $m_3 \circ m_2 \circ m_1 \in (\mu_3 \circ \mu_2) \circ \mu_1$ proving $\mu_3 \circ \mu_2 \circ \mu_1 = (m_3 \circ m_2 \circ m_1)/\equiv\, = (\mu_3 \circ \mu_2) \circ \mu_1$;

b) the identity morphism is Id_T defined by (iii) of Definition 3.4. Let $\mu\colon T_1 \to T_2$, then for some $m \in \mu$, $m \circ \mathrm{Id}_{T_1}(\varphi) = m(\mathrm{Id}_{T_1}(\varphi)) = m(\varphi) = \mathrm{Id}_{T_2}\, m(\varphi)$, for any $\varphi \in F^\alpha_t$, i.e. $\mu \circ \mathrm{Id}_{T_1} = \mathrm{Id}_{T_2} \circ \mu = \mu$. □

The main properties of the category $\mathbf{TH}_\alpha$ are investigated in AGN [4]. Here we show how the category of theories can be characterized algebraically.

Theorem 3.2. The categories $\mathbf{C}_\alpha$ and $\mathbf{TH}_\alpha$ are isomorphic.

Proof. First we define a functor $F\colon \mathbf{TH}_\alpha \to \mathbf{C}_\alpha$.

a) We define the object part $F_{\mathrm{Ob}}\colon \mathrm{Ob}\,\mathbf{TH}_\alpha \to \mathrm{Ob}\,\mathbf{C}_\alpha$ of F as follows. Let $T = \langle \mathrm{Ax}, F^\alpha_t\rangle \in \mathrm{Ob}\,\mathbf{TH}_\alpha$ be arbitrary. Recall that in Definition 2.8 the concept algebra $\mathfrak{C}_T$ of the theory T was defined to be $\mathfrak{F}_t/\equiv_T$ that is $\mathfrak{C}_T=\mathfrak{F}_T^\alpha/\{\langle\varphi,\psi\rangle\colon \mathrm{Ax}\models(\varphi\leftrightarrow\psi)\}$. We define $F(T)\overset{d}{=}F_{\mathrm{Ob}}(T)\overset{d}{=}\mathfrak{C}_T$ for every $T\in\mathrm{Ob}\,\mathbf{TH}_\alpha$. By this the function $F_{\mathrm{Ob}}\colon \mathrm{Ob}\,\mathbf{TH}_\alpha\to\mathrm{Ob}\,\mathbf{C}_\alpha$ is defined.

b) Let $\mu\colon T_1\to T_2\in\mathrm{Mor}\,\mathbf{TH}_\alpha$.

We define $F_{\mathrm{Mor}}(\mu)\overset{d}{=}\{\langle x,y\rangle\in C_{T_1}\times C_{T_2}\colon$ there exist a $\varphi\in x$ and an $m\in\mu$ such that $m(\varphi)\in y\}$.

It is not hard to check that $F_{\mathrm{Mor}}(\mu)\colon \mathfrak{C}_{T_1}\to\mathfrak{C}_{T_2}$ is a function, and, by Definition 2.9, it follows that $F_{\mathrm{Mor}}(\mu)\in\mathrm{Hom}\,(\mathfrak{C}_{T_1},\mathfrak{C}_{T_2})=\mathrm{Hom}\,(F(T_1),F(T_2))\subseteq\mathrm{Mor}\,\mathbf{C}_\alpha$, i.e. $F_{\mathrm{Mor}}(\mu)$ is a homomorphism.

c) We have defined a function $F_{\mathrm{Mor}}\colon \mathrm{Mor}\,\mathbf{TH}_\alpha\to\mathrm{Mor}\,\mathbf{C}_\alpha$. Let $F\overset{d}{=}\langle F_{\mathrm{Ob}},F_{\mathrm{Mor}}\rangle$. Now we prove that F is a functor. F_{Mor} satisfies the following properties:

(i) for any $T\in\mathrm{Ob}\,\mathbf{TH}_\alpha$, $F_{\mathrm{Mor}}(\mathrm{Id}_T)=\mathrm{Id}_{\mathfrak{C}_T}$,

(ii) let $\mu_1\colon T_1\to T_2$ and $\mu_2\colon T_2\to T_3$. Then $F_{\mathrm{Mor}}(\mu_2\circ\mu_1)(\varphi)=F_{\mathrm{Mor}}(m\circ n)/\equiv_{T_3}(\varphi)=n(m(\varphi))/\equiv_{T_2}=F_{\mathrm{Mor}}(\mu_2)(m(\varphi)/\equiv_{T_2})=F_{\mathrm{Mor}}(\mu_2)\circ F_{\mathrm{Mor}}(\mu_1)$ for any $\varphi\in F_t^\alpha$. Here $m\in\mu_1$ and $n\in\mu_2$.

Thus the pair of functions $F=\langle F_{\mathrm{Ob}},F_{\mathrm{Mor}}\rangle$ is a functor $F\colon \mathbf{TH}_\alpha\to\mathbf{C}_\alpha$.

Next we prove that $F_{\mathrm{Ob}}\colon \mathrm{Ob}\,\mathbf{TH}_\alpha\rightarrowtail\!\!\twoheadrightarrow\mathrm{Ob}\,\mathbf{C}_\alpha$ is a set theoretic isomorphism, that is F_{Ob} is one-one and onto.

(i) Let $T_i=\langle\mathrm{Ax}_i,F_{t_i}^\alpha\rangle\in\mathrm{Ob}\,\mathbf{TH}_\alpha$ for $i\in\{1,2\}$. Assume $T_1\neq T_2$.

Case 1. $t_1\neq t_2$. Then $F(T_1)\neq F(T_2)$ since $\bigcup C_{T_1}=F_{t_1}^\alpha\neq F_{t_2}^\alpha=\bigcup C_{T_2}$.

Case 2. $t_1=t_2$. Then $\mathrm{Ax}_1\neq\mathrm{Ax}_2$. Recall that by the definition of $\mathbf{TH}_\alpha$ we have $\mathrm{Ax}_i=\mathrm{Ax}_i^*$ for $i\in\{1,2\}$. Thus $1^{F(T_1)}=\mathrm{Ax}_1^*=\mathrm{Ax}_1\neq\mathrm{Ax}_2=\mathrm{Ax}_2^*=1^{F(T_2)}$.

Cases 1—2 prove $F(T_1)\neq F(T_2)$ and hence $F_{\mathrm{Ob}}\colon \mathrm{Ob}\,\mathbf{TH}_\alpha\rightarrowtail\mathrm{Ob}\,\mathbf{C}_\alpha$ is proved to be one-one.

(ii) Let $\mathfrak{A}\in\mathrm{Ob}\,\mathbf{C}_\alpha$ be arbitrary. By the definition of $\mathbf{C}_\alpha$ then there exists a theory $T=\langle\mathrm{Ax},F_t^\alpha\rangle$ such that $\mathfrak{A}\cong\mathfrak{C}_T$. Let $T^*=\langle\mathrm{Ax}^*,F_t^\alpha\rangle$. Clearly $T^*\in\mathrm{Ob}\,\mathbf{TH}_\alpha$ and $F(T^*)=\mathfrak{C}_{T^*}=\mathfrak{C}_T=\mathfrak{A}$.

We proved that $\mathrm{Rg}\,F_{\mathrm{Ob}}=\mathrm{Ob}\,\mathbf{C}_\alpha$ and hence $F_{\mathrm{Ob}}\colon \mathrm{Ob}\,\mathbf{TH}_\alpha\rightarrowtail\!\!\twoheadrightarrow\mathrm{Ob}\,\mathbf{C}_\alpha$ is proved to be a set theoretic isomorphism.

Next we prove that F_{Mor} is a set theoretic isomorphism on the Hom-sets.

Let $T_i=\langle\mathrm{Ax}_i,F_{t_i}^\alpha\rangle\in\mathrm{Ob}\,\mathbf{TH}_\alpha$ for $i\in\{1,2\}$.

(i) Let $\mu\colon T_1\to T_2$ and $\nu\colon T_1\to T_2$ be different, i.e. $\mu\neq\nu$. Then $(\exists m\in\mu)(\exists n\in\nu)(\exists\varphi\in F_{t_1}^\alpha)\ \mathrm{Ax}_2\not\models(m(\varphi)\leftrightarrow n(\varphi))$. Let these m,n,φ be fixed. Then

$$F(\mu)(\varphi/\equiv_{T_1})=m(\varphi)/\equiv_{T_2}\neq n(\varphi)/\equiv_{T_2}=F(\mu)(\varphi/\equiv_{T_2}).$$

Thus F_{Mor} is *one-one.*

(ii) Let $f\in\mathrm{Hom}\,(F(T_1),F(T_2))$ be an arbitrary homomorphism from the algebra $\mathfrak{C}_{T_1}$ into the algebra $\mathfrak{C}_{T_2}$. Let $\mathrm{At}\subseteq F_{t_1}^\alpha$ be the set of all atomic formulas in $F_{t_1}^\alpha$ not involving equality, i.e. $\mathrm{At}\overset{d}{=}\{R(x_{i_1},\dots,x_{i_n})\colon R\in\mathrm{Do}\,t_1$ and $t_1(R)=n$ and $i_1,\dots,i_n\in\alpha\}$. Note that $(x_i=x_j)\notin\mathrm{At}$ for any $i,j\in\alpha$.

For every $i\in\{1,2\}$ we define the homomorphism $\mathrm{nat}_i\colon \mathfrak{F}_{t_i}\twoheadrightarrow\mathfrak{F}_{t_i/\equiv_{T_i}}$ as follows $\mathrm{nat}_i\,(\varphi)\overset{d}{=}\varphi/\equiv_{T_i}$ for each $\varphi\in F_{t_i}^\alpha$.

Let $c\colon F_{t_2}^\alpha/\equiv_{T_2}\to F_{t_2}^\alpha$ be a choice function that is $\mathrm{nat}_2\circ c=\mathrm{Id}_{C_{T_2}}$. Let $n\overset{d}{=}(c\circ f\circ\mathrm{nat}_1)\restriction\mathrm{At}$. Then $n\colon \mathrm{At}\to F_{t_2}$ is such that $\mathrm{nat}_2\circ n=(f\circ\mathrm{nat}_1)\restriction\mathrm{At}$.

Since At freely generates the algebra $\mathfrak{F}_{t_1}^{\alpha}$ there is a unique homomorphic extension $m\colon \mathfrak{F}_{t_1}^{\alpha}\to\mathfrak{F}_{t_2}^{\alpha}$ of n to the algebra $\mathfrak{F}_{t_1}$, i.e. $m\restriction\mathrm{At}=n$. The diagram

$$\begin{array}{ccc} \mathfrak{F}_{t_1} & \xrightarrow{\;m\;} & \mathfrak{F}_{t_2} \\ {\scriptstyle \mathrm{nat}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{nat}_2} \\ \mathfrak{C}_{t_1} & \xrightarrow{\;f\;} & \mathfrak{C}_{T_2} \end{array}$$

commutes since $f\circ\mathrm{nat}_1\restriction\mathrm{At}=\mathrm{nat}_2\circ n=\mathrm{nat}_2\circ m\restriction\mathrm{At}$ and At generates $\mathfrak{F}_{t_1}$.

Assume $\mathrm{Ax}_1\models\varphi$. Then $f(\mathrm{nat}_1(\varphi))=1^{F(T_2)}=\mathrm{nat}_2(m(\varphi))$ and hence $m(\varphi)\in\mathrm{Ax}_2^*=$ $=\mathrm{nat}_2(1^{\mathfrak{F}_{t_2}^{\alpha}})=\mathrm{nat}_2\,(x_0=x_0)$.

This proves that $\langle T_1,\ m,\ T_2\rangle$ is an interpretation and hence $m/{\equiv}\colon T_1\to$ $\to T_2\in\mathrm{Mor}\,\mathbf{TH}_\alpha$.

By the definition of F_{Mor} we have $F(m/{\equiv})=f$. We have proved that $\mathrm{Rg}\,F_{\mathrm{Mor}}=\mathrm{Mor}\,\mathbf{C}_\alpha$. Then by the above considerations $F\colon \mathbf{TH}_\alpha\rightarrowtail\!\!\!\to\mathbf{C}_\alpha$ is an isomorphism proving $\mathbf{TH}_\alpha\cong\mathbf{C}_\alpha$. □

From Theorems 2.5 and 3.2 we have the following representation theorem.

Theorem 3.3. The categories $\mathbf{Lf}_\alpha$ and $\mathbf{TH}_\alpha$ are isomorphic. □

By the representation theorem (Theorem 3.3) we can investigate the category $\mathbf{TH}_\alpha$ through the investigation of the properties of the category $\mathbf{Lf}_\alpha$, since $\mathbf{TH}_\alpha\cong\mathbf{Lf}_\alpha$.

Before using this possibility we recall some well known notions.

By a small category we understand a category $\mathbf{C}=\langle\mathrm{Ob}\,\mathbf{C},\ \mathrm{Mor}\,\mathbf{C}\rangle$ such that Mor **C** is a set.

Definition 3.5. Let **K** be an arbitrary category. By a diagram in the category **K** we undestand a functor $D\colon \mathbf{C}\to\mathbf{K}$, where **C** is a small category.

The category **C** is called the *index* category of the diagram D.

Definition 3.6. Let **K** be an arbitrary category and let $D\colon \mathbf{I}\to\mathbf{K}$ be a diagram. Let $\mathbf{I}=\langle I, M\rangle$.

(i) A *cone* over D is a system $\langle H, \langle h_i\colon i\in I\rangle\rangle$ such that $H\in\mathrm{Ob}\,\mathbf{K}$ and for each $i\in I$, $h_i\colon H\to D(i)\in\mathrm{Mor}\,\mathbf{K}$ and for every $f\in M$ if $f\colon i\to j$ in **I** then $D(f)\circ h_i=h_j$ in **K**.

(ii) The *limit* of D in **K** is a cone $\langle G, \langle g_i\colon i\in I\rangle\rangle$ over D such that for every cone $\langle H, \langle h_i\colon i\in I\rangle\rangle$ over D there is a unique morphism $\mu\colon H\to G$ such that for any $i\in I$, $h_i\circ\mu=g_i$.

(iii) The *colimit* of D is defined exactly as above but all the arrows are reversed. Thus a colimit is a cocone $\langle\langle g_i\colon i\in I\rangle, G\rangle$ with $g_i\colon D(i)\to G$ etc.

Definition 3.7. A category **K** is said to be complete and cocomplete if for every diagram D in **K** both the limit and the colimit of D exist in **K**.

Theorem 3.4. The category $\mathbf{TH}_\alpha$ is complete and cocomplete if $\alpha\geqq\omega$.

Proof. Since $\mathbf{TH}_\alpha\cong\mathbf{Lf}_\alpha$ by Theorem 3.3 it is enough to prove that $\mathbf{Lf}_\alpha$ is complete and cocomplete. Let $\mathrm{Re}_\alpha\overset{\mathrm{d}}{=}\mathrm{HSP}\,\mathrm{Lf}_\alpha$, that is $\mathrm{Re}_\alpha\subseteq\mathrm{Alg}\,(l_\alpha)$ is the smallest variety containing Lf_α. Let $\mathbf{Re}_\alpha$ be the full subcategory of $\mathbf{Alg}\,(l_\alpha)$ with Ob $\mathbf{Re}_\alpha = Re_\alpha$. Then $\mathbf{Lf}_\alpha$ is a full subcategory of $\mathbf{Re}_\alpha$. It is well known that any variety is complete and cocomplete, see e.g. Proposition III.5.11 of TSALENKO—SHULGEIFER [18]. Let $D\colon \mathbf{I}\to\mathbf{Lf}_\alpha$ be a diagram in $\mathbf{Lf}_\alpha$. Let $\langle\mathfrak{A}, \langle h_i\colon i\in I\rangle\rangle$ be the limit of D in $\mathbf{Re}_\alpha$. It is easy to prove (see e.g. Corollary 2.1.6 of HMT [11]) that the greatest Lf_α-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ exists, that is $\mathfrak{A}\supseteq\mathfrak{B}\in Lf_\alpha$ and for every $\mathfrak{C}\in Lf_\alpha$ such that $\mathfrak{C}\subseteq\mathfrak{A}$ then $\mathfrak{C}\subseteq\mathfrak{B}$. In other words $\mathfrak{B}$ is the greatest member ot $Lf_\alpha\cap S\mathfrak{A}$, where $S\mathfrak{A}$ is the set of all subalgebras of $\mathfrak{A}$ and $\mathfrak{B}\subseteq\mathfrak{A}$ denotes that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. It is easy to check that $\langle\mathfrak{B}, \langle h_i{\restriction}B\colon i\in I\rangle\rangle$ is the limit of D in $\mathbf{Lf}_\alpha$.

Let $\langle h_i\colon i\in I, \mathfrak{A}\rangle$ be the colimit of D in $\mathbf{Re}_\alpha$. We prove that it is also the colimit of D in $\mathbf{Lf}_\alpha$. To this end it is enough to prove that $\mathfrak{A}\in Lf_\alpha$. Let $X= =\bigcup\{\mathrm{Rg}\, h_i\colon i\in I\}$. Then $X\subseteq A$, X generates $\mathfrak{A}$ and $(\forall y\in X)|\Delta y|<\omega$ since y is the homomorphic image of some $z\in D(i)\in Lf_\alpha$. Then $\mathfrak{A}\in Lf_\alpha$ by Theorem 2.1.5 in HMT [11]. □

We proved that $\mathbf{Lf}_\alpha$ is complete and cocomplete, moreover, we proved that $\mathbf{Lf}_\alpha$ is cocomplete in $\mathbf{Re}_\alpha$, that is the colimits of diagrams $D\colon \mathbf{I}\to\mathbf{Lf}_\alpha$ when computed in $\mathbf{Re}_\alpha$ coincide with those when computed in $\mathbf{Lf}_\alpha$. As a contrast we recall the following from GERGELY [10]. *$\mathbf{Lf}_\alpha$ is not cocomplete in $\mathbf{Alg}\,(l_\alpha)$, moreover, $\mathbf{Lf}_\alpha$ is not cocomplete in $\mathbf{Bo}_\alpha$* as $Bo_\alpha\subseteq Alg\,(l_\alpha)$ was defined in HMT [11], neither is it cocomplete in the variety I Crs_α as defined in HMTAN [12] as these are proved in GERGELY [10]. I Crs_α=HSP $Crs_\alpha\supseteq Lf_\alpha$ was proved in NÉMETI [16].

4. Conclusion

Here analysing the connection between the categories $\mathbf{TH}_\alpha$ and $\mathbf{CA}_\alpha$ only the theories were represented by cylindric algebras. However having a theory $T\subseteq F_t^\alpha$ not only the representation of T but that of the models $\mathfrak{A}\in \mathrm{Mod}\ T$ of the theory T, or that of the subclasses $K\subseteq\mathrm{Mod}\ T$ of the models can be done by the use of CA's. E.g. in NÉMETI [16], classes of models were represented by the use of the tools introduced in AGN [2] but from the point of view of the categories presently introduced only the objects were considered. Thus, for the entire investigation, morphisms should be considered as well. This investigation will be done elsewhere.

On the whole the present paper emphasizes the usefulness of certain universal algebraic tools to handle the category of all theories of α variables.

Thus all results concerning the subclass Lf_α of l_α-type algebras can be used directly to investigate language hierarchies.

This provides the possibility to represent and analyse formal semantics of language hierarchies by the use of a very important subclass of l_α-type cylindric algebras the so called locally independently-finite cylindric algebras, introduced in AGN [1]. These algebras were later called regular in HMTAN [12]. At the same time the established connection provides quite a concrete content to the notion of Lf_α which was introduced in HMT [11].

Theorem 3.3 provides an opposite possibility as well, namely, to establish some new results about Lf_α by using the tools of Category Theory.

RESEARCH INSTITUTE FOR
APPLIED COMPUTER SCIENCE
CSALOGÁNY U. 30—32.
BUDAPEST, HUNGARY
H—1536

References

[1] AGN, (ANDRÉKA, H., GERGELY, T., NÉMETI, I.): Toward a general theory of logics, Part 1, KFKI—73—67, Budapest, 1973.

[2] AGN: On universal algebraic construction of logics, *Studia Logica,* v. 36, 1977, pp. 9—47.

[3] AGN: Model theoretic semantics for many purpose languages and language hierarchies, Proceedings of COLING'80, Tokyo, 1980.

[4] AGN: Investigation in the language hierarchies, SZÁMKI-Working paper, Budapest, 1981.

[5] BLUM, E. K., D. R. ESTES, A generalization of the homomorphism concept, *Algebra Universalis,* v. 7, 1977, pp. 143—161.

[6] BURSTALL, R. M., J. A. GOGUEN, Putting theories together to make specifications, Proceedings of 5th IJCAI, Massachusets, 1977, pp. 1045—1058.

[7] BURSTALL, R. M., J. A. GOGUEN, The semantics of CLEAR, a specification language, D. BJØRNER, ed., *Abstract software specifications,* Lecture Notes in Computer Science, v. 86, Springer-Verlag, 1980, pp. 292—332.

[8] DEVLIN, K. J., *Aspects of constructibility,* Lecture Notes in Mathematics, v. 354, Springer-Verlag, Berlin, 1973.

[9] DÖMÖLKI, B., An example of hierarchical program specification, D. BJØRNER, ed., *Abstract software specifications,* Lecture Notes in Computer Science, v. 86, Springer-Verlag, 1980, pp. 333—353.

[10] GERGELY, T., Mono-coreflexivity of cylindric algebras, SZÁMKI-Working paper, Budapest, 1981.

[11] HMT (HENKIN, L., MONK, J. D., TARSKI, A.): *Cylindric algebras,* Part I, North Holland, Amsterdam, 1971.

[12] HMTAN (HENKIN, L., MONK, J. D., TARSKI, A., ANDRÉKA, H,, NÉMETI, I.): *Cylindric set algebras,* Lecture Notes in Mathematics, v. 883, Springer-Verlag, 1981.

[13] HUPBACH, U. L., Abstract implementation of abstract data type, P. DEMBINSK, ed., *Mathematical foundations of computer science,* Lecture Notes in Computer Science, v. 88, Springer-Verlag, 1980.

[14] MAC LANE, S., *Categories for the working mathematicians,* Springer-Verlag, N. Y., 1971.

[15] MONK, J. D., *Mathematical logic,* Springer-Verlag, New-York, 1976.

[16] NÉMETI, I., Connections between cylindric algebras and initial algebra semantics of CF languages, DÖMÖLKI, B., GERGELY, T. eds., *Mathematical logic in computer science,* North-Holland, 1981, pp. 561—606.

[17] RATTRAY, C. M. I., T. RUS, Task-hierarchy, a mathematical device for computer system modelling, Proceeding of First International Symposium on Mathematical Modelling, Missouri, 1977, pp. 1—15.

[18] TSALENKO, M. S., E. G. SHULGEĬFER, Osnovȳ teorii kategoriĭ, Nauka, Moskva, 1974.

[19] WINKOWSKI, J., Towards an understanding of computer simulation, *Fund. Inform.,* v. 1, 1978, pp. 277—289.

(Received April 27, 1981)