An algebraic definition of attributed transformations

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1. Magmoids and rational theories

The concept of magmoid was introduced in [1]. A magmoid $M=(\{M_s|s\in S\}, \cdot, \otimes, e, e_0)$, is a many sorted algebra with sorting set S, the set of all pairs of non-negative integers. Further on we shall write M_q^p instead of $M_{\langle p,q\rangle}$. Binary operations $\cdot$ and $\otimes$ are called composition and tensor product, respectively. The following axioms must be valid in M:

(i) $\cdot : M_q^p \times M_r^q \to M_r^p$ is associative.

(ii) $\otimes : M_{q_1}^{p_1} \times M_{q_2}^{p_2} \to M_{q_1+q_2}^{p_1+p_2}$ is associative.

(iii) $(a_1 \cdot b_1) \otimes (a_2 \cdot b_2) = (a_1 \otimes a_2) \cdot (b_1 \otimes b_2)$ for all composable pairs $\langle a_1, b_1\rangle$, $\langle a_2, b_2$.

(iv) $e \in M_1^1$, $e_0 \in M_0^0$, and if e_n denotes $\underbrace{e \otimes \ldots \otimes e}_{n \text{ times}}$ $(n \geqq 1)$, then for each $p \geqq 0$, $q \geqq 0$, $a \in M_q^p$: $e_p \cdot a = a \cdot e_q = a \otimes e_0 = e_0 \otimes a = a$.

An element $a \in M_q^p$ will often be denoted by $a: p \to q$ if M is understood.

Let $\Sigma = \bigcup_{n \geqq 0} \Sigma_n$ be a finite ranked alphabet, and define the structure $T(\Sigma) = (\{T(\Sigma)_q^p\ p, q \geqq 0\}, \cdot, \otimes, e, e_0)$ as follows:

For arbitrary $p \geqq 0$ and $q \geqq 0$, $T(\Sigma)_q^p = \{\langle q; t_1, \ldots, t_p\rangle |$ for each $1 \leqq i \leqq p$, t_i is a finite Σ-tree over the variables $x_1, \ldots, x_q\}$. $\langle q;\rangle \in T(\Sigma)_q^0$ will be denoted by 0_q.

$$\langle q; t_1, \ldots, t_p\rangle \cdot \langle r; u_1, \ldots, u_q\rangle = \langle r; t_1[u_1, \ldots, u_q], \ldots, t_p[u_1, \ldots, u_q]\rangle,$$

where $[\ldots]$ denotes the composition of trees;

$$\langle q_1; t_1, \ldots, t_{p_1}\rangle \otimes \langle q_2; u_1, \ldots, u_{p_2}\rangle = \langle q_1+q_2; t_1, \ldots, t_{p_1}, u_1', \ldots, u_{p_2}'\rangle,$$

where $u_i' = u_i[x_{q_1+1}, \ldots, x_{q_1+q_2}]$; $e = \langle 1; x_1\rangle$, $e_0 = 0_0$.

We shall omit the component q of $\langle q; t_1, \ldots, t_p\rangle$ if it is understood. Moreover, we leave $\langle \ldots \rangle$ if $p=1$. It is known that $T(\Sigma)$ is a magmoid. $\tilde{T}(\Sigma)$ is a submagmoid of $T(\Sigma)$ such that $t = \langle q; t_1, \ldots, t_p\rangle \in \tilde{T}(\Sigma)_q^p$ if and only if the sequence of variables labeling the leaves of $t_1, \ldots, t_p$, read from left to the right, is exactly $x_1, \ldots, x_q$. $\tilde{T}(\Sigma)$ is the free magmoid generated by Σ, that is, every ranked alphabet map $h: \Sigma \to M^1$ into a magmoid M has a unique homomorphic extension $\bar{h}: \tilde{T}(\Sigma) \to M$. (Viewing $\sigma \in \Sigma_n$ as $\langle n; \sigma(x_1, \ldots, x_n)\rangle \in \tilde{T}(\Sigma)_n^1$).

Another important magmoid is θ, in which θ_q^p is the set of all mappings of $[p]=\{1, \ldots, p\}$ into $[q]$. Composition is that of mappings, and for $\vartheta_i \in \theta_{q_i}^{p_i}$, $i=1, 2$

$$\vartheta_1 \otimes \vartheta_2(j) = \begin{cases} \vartheta_1(j) & \text{if} \quad j \in [p_1], \\ \vartheta_2(j-p_1)+q_1 & \text{if} \quad p_1 < j \leqq p_2. \end{cases}$$

e and e_0 are the unique elements of θ_1^1 and θ_0^0, respectively. e_n will be denoted by id_n if $n \geqq 1$. The elements of θ are usually called torsions or base moprhisms.

A magmoid is called projective if it contains a submagmoid isomorphic to θ and every $a: p \to q$ is uniquely determined by its "projections", i.e. by the sequence $\langle \pi_p^i \cdot a | 1 \leqq i \leqq p \rangle$. π_p^i denotes the isomorphic image of the map $\pi_p^i: [1] \to [p]$ that picks out the integer i of $[p]$. $T(\Sigma)$ is projective, and it is the free projective magmoid generated by Σ. $P_F T(\Sigma)$ will denote the magmoid in which $(P_F T(\Sigma))_q^p = \{q; A_1, \ldots, \ldots, A_p\rangle |$ for each $i \in [p]$, A_i is a finite set of Σ-trees over the variables $x_1, \ldots, x_q\}$. (For the interpritation of the operations see [2].) $P_F T(\Sigma)$ is also projective. Let M be a projective magmoid, $a_1, \ldots, a_p \in M_q^1$. $\prec a_1, \ldots, a_p \succ$ will denote the unique element of M_q^p whose sequence of projections is $\langle a_1, \ldots, a_p \rangle$. This source-tupling can be viewed as a derived operation in M, and it can be extended as follows. Let $a_1: p_1 \to q$, $a_2: p_2 \to q$. Then $\prec a_1, a_2 \succ = \prec \pi_{p_1}^1 \cdot a_1, \ldots, \pi_{p_1}^{p_1} \cdot a_1, \pi_{p_2}^1 \cdot a_2, \ldots, \pi_{p_2}^{p_2} \cdot a_2 \succ$.

Rational theories were introduced in [3], based on the concept of algebraic theory. However, the only difference between nondegenerate algebraic theories and projective magmoids is that in algebraic theories source-tupling is a basic operation (and tensor product is a derived one). So, if we introduce rational theories by means of projective magmoids, we get a definition equivalent to the original one excluding the trivial degenerate rational theory.

A rational theory is also a many sorted algebra $R = (\{R_q^p | p, q \geqq 0\}, \cdot, \otimes, e, e_0, {}^+)$, where, apart from ${}^+$, R is a projective magmoid, the sets R_q^p are partially ordered, and ${}^+: R_{p+q}^p \to R_q^p$ is a new operation. For $f: p \to p+q$, f^+ is the least fixpoint of f, and some further conditions must hold concerning the ordering and the operations, that we do not list here.

Add a new symbol $\perp$ with rank 0 to Σ, to get the ranked alphabet $\Sigma_\perp$. There exists a rational theory $T_\infty(\Sigma)$ for which $T_\infty(\Sigma)_q^p = \{\langle q; t_1, \ldots, t_p \rangle |$ for each $i \in [p]$, t is a possibly infinite $\Sigma_\perp$-tree over the variables $x_1, \ldots, x_q\}$. For the interpretation of the operations, see [3]. It is known that $R(\Sigma)$, the free rational theory generated by Σ, is the smallest subtheory of $T_\infty(\Sigma)$ that contains $T(\Sigma)$ as a submagmoid.

Let $q \geqq 0$, $X_q = \{x_1, \ldots, x_q\}$, $\chi_q: \Sigma \to (\Sigma \cup X_q)^*$ such that for each $\sigma \in \Sigma_n$, length $(\chi_q(\sigma)) = n$. An infinite tree $t \in R(\Sigma)_q^p$ is called local of type χ_q if the following holds. If an interior node of t is labeled by $\sigma \in \Sigma_n$, then its direct descendants are labeled by $\chi_q(\sigma)$. If so, we will denote t by (ω, χ_q), where $\omega = \mathrm{root}(t) \in (\Sigma \cup X_q)^p$. $\mathrm{Rec}(\Sigma)$ will denote the smallest rational theory in $PT(\Sigma)$ that contains $P_F T(\Sigma)$ as a submagmoid.

2. The magmoid $R(k, l)$

Definition 2.1. Let R be a rational theory, $k \geqq 1, l \geqq 0$ integers. Define $R(k, l) = (\{R(k, l)_q^p | p, q \geqq 0\}, \cdot, \otimes, e, e_0)$ to be the following structure:

(i) $R(k, l)_q^p = R_{k \cdot q + l \cdot p}^{k \cdot p + l \cdot q}$;

(ii) if $a \in R(k, l)_q^p$, $b \in R(k, l)_r^q$, then

$$a \cdot b = \langle\!\langle \mu^{k \cdot p}, \nu_{l \cdot r} \rangle\!\rangle \cdot \langle\!\langle a \cdot \vartheta_{p,q,r}, b \cdot \psi_{p,q,r} \rangle\!\rangle^{+},$$

where

$$\mu_m^n (= \mu^n \text{ if } m \text{ is understood}) = \mathrm{id}_n \otimes 0_m \in \theta_{n+m}^n,$$

$$\nu_m^n (= \nu_m \text{ if } n \text{ is understood}) = 0_n \otimes \mathrm{id}_m \in \theta_{n+m}^m,$$

$$\vartheta_{p,q,r} = \nu_{k \cdot q}^{k \cdot p + l \cdot q} \otimes \nu_{l \cdot p}^{(k+l) \cdot r},$$

$$\psi_{p,q,r} = 0_{k \cdot p} \otimes \langle\!\langle \nu_{k \cdot r}^{(k+l) \cdot q + l \cdot r}, \mu_{k \cdot q + (k+l) \cdot r}^{l \cdot q} \rangle\!\rangle \otimes 0_{l \cdot p}.$$

See also Fig. 1.

(iii) if $a \in R(k, l)_{q_1}^{p_1}$, $b \in R(k, l)_{q_2}^{p_2}$, then

$$a \otimes b = \langle\!\langle \mu_{l \cdot q_1}^{k \cdot p_1} \otimes \mu_{l \cdot q_2}^{k \cdot p_2}, \nu_{l \cdot q_1}^{k \cdot p_1} \otimes \nu_{l \cdot q_2}^{k \cdot p_2} \rangle\!\rangle \cdot (a \otimes b) \cdot \langle\!\langle \mu_{l \cdot p_1}^{k \cdot q_1} \otimes \mu_{l \cdot p_2}^{k \cdot q_2}, \nu_{l \cdot p_1}^{k \cdot q_1} \otimes \nu_{l \cdot p_2}^{k \cdot q_2} \rangle\!\rangle^{-1}.$$

(iv) $e = \mathrm{id}_{k+l}$, $e_0 = 0_0$.

(We shall never add any distinctive mark to the sign of the operations when working in different magmoids in the same time, because only one interpretation is reasonable anywhere in the context.)

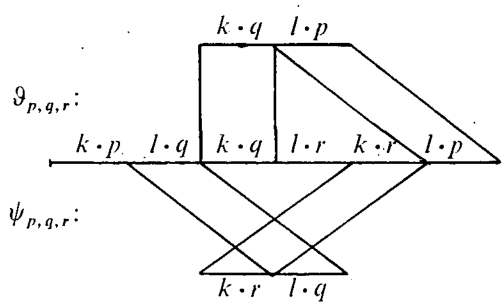

Fig. 1

Theorem 2.2. $R(k, l)$ is a magmoid.

Proof. All the requirements can be proved by the same method, so we only show the associativity of composition. Let

$$a = \langle\!\langle \underline{a}_1, \ldots, \underline{a}_{k \cdot p}, \bar{a}_1, \ldots, \bar{a}_{l \cdot q} \rangle\!\rangle \in R(k, l)_q^p,$$

$$b = \langle\!\langle \underline{b}_1, \ldots, \underline{b}_{k \cdot q}, \bar{b}_1, \ldots, \bar{b}_{l \cdot r} \rangle\!\rangle \in R(k, l)_r^q, \tag{1}$$

$$c = \langle\!\langle \underline{c}_1, \ldots, \underline{c}_{k \cdot r}, \bar{c}_1, \ldots, \bar{c}_{l \cdot s} \rangle\!\rangle \in R(k, l)_s^r.$$

We must prove that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. Both sides of this equation can be considered as a polynomial in R over the variables $\underline{a}_i, \bar{a}_j, \ldots, \underline{c}_i, \bar{c}_j$. Since R is arbitrary, we have to show that these polynomials are identical. Let Σ be the smallest finite ranked alphabet satisfying the following conditions:

(i) for arbitrary $i \in [k \cdot p]$ and $j \in [l \cdot q]$, $\underline{A}_i, \bar{A}_j \in \Sigma_{k \cdot q + l \cdot p}$,

(ii) for arbitrary $i \in [k \cdot q]$ and $j \in [l \cdot r]$, $\underline{B}_i, \bar{B}_j \in \Sigma_{k \cdot r + l \cdot q}$,

(iii) for arbitrary $i \in [k \cdot r]$ and $j \in [l \cdot s]$, $\underline{C}_i, \bar{C}_j \in \Sigma_{k \cdot s + l \cdot r}$.

Change the small letters to capital ones in (1), to obtain the elements A, B, C of $R(\Sigma)$. Clearly, it is enough to show that $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ holds in $R(\Sigma)(k, l)$. However, it is easy to check that $(A \cdot B) \cdot C = A \cdot (B \cdot C) = (\omega, \chi_n)$, where $n = k \cdot s + l \cdot p$ and

$$\omega = \langle \underline{A}_1, \ldots, \underline{A}_{k \cdot p}, \bar{C}_1, \ldots, \bar{C}_{l \cdot s} \rangle,$$

$$\chi_n(\underline{A}_i) = \chi_n(\bar{A}_j) = \langle \underline{B}_1, \ldots, \underline{B}_{k \cdot q}, x_{k \cdot s + 1}, \ldots, x_{k \cdot s + l \cdot p} \rangle,$$

$$\chi_n(\underline{B}_i) = \chi_n(\bar{B}_j) = \langle \underline{C}_1, \ldots, \underline{C}_{k \cdot r}, \bar{A}_1, \ldots, \bar{A}_{l \cdot q} \rangle,$$

$$\chi_n(\underline{C}_i) = \chi_n(\bar{C}_j) = \langle x_1, \ldots, x_{k \cdot s}, \bar{B}_1, \ldots, \bar{B}_{l \cdot r} \rangle$$

for any appropriate choise of the integers i and j.

Let $\xi \colon R \to R'$ be a homomorphism between rational theories. Clearly, ξ defines a homomorphism $\xi(k, l) \colon R(k, l) \to R'(k, l)$, and so the operator (k, l) becomes a functor.

3. Attributed transformations

Definition 3.1. An attributed transducer is a 6-tuple $\mathfrak{A} = (\Sigma, R, k, l, h, S)$, where

(i) Σ is a finite ranked alphabet, $S \notin \Sigma$;

(ii) R is a rational theory, $k \geqq 1, l \geqq 0$ are integers;

(iii) $h \colon \Sigma_S \to R(k, l)$ is a ranked alphabet map, where $\Sigma_S = \Sigma \cup \{S\}$ with S having rank 1, and $h(S) = a \otimes 0_l$ for some $a \in R_k^{k+l}$. We say that $h(S)$ is a synthesizer.

$\tau_{\mathfrak{A}} \colon \tilde{T}(\Sigma)_0^1 \to R_0^1$, the transformation induced by $\mathfrak{A}$, is the following function: $\tau_{\mathfrak{A}}(t) = a$, where $\pi_k^1 \cdot h(S(t)) = a \otimes 0_l$. It is clear that $\tau_{\mathfrak{A}}(t)$ is uniquely determined by this imlicit form. (As it is usual, we denoted the unique homomorphic extension of h also by h.)

Definition 3.2. An attributed tree transducer is a 6-tuple $\mathfrak{A} = (\Sigma, \Delta, k, l, h, S)$, where Σ, k, l and S are as in the previous definition, Δ is a finite ranked alphabet, $h \colon \Sigma_S \to P_F T(\Delta)$ is such that $h((\Sigma_S)_n) \subseteqq P_F T(\Delta)_{k \cdot n + l}^{k + l \cdot n}$ and $h(S) \in P_F T(\Delta)_k^{k+l}$. To define the transformation $\tau_{\mathfrak{A}}$, consider the attributed transducer $\mathfrak{B} = (\Sigma, \mathrm{Rec}(\Delta), k, l, h, S)$. $\mathfrak{B}$ is correct, since $P_F T(\Delta) \subseteqq \mathrm{Rec}(\Delta)$ and $h(S)$ is a synthesizer. Now

$$\tau_{\mathfrak{A}} = \{\langle t, u \rangle \mid t \in \tilde{T}(\Sigma)_0^1, \quad u \in \tau_{\mathfrak{B}}(t)\}.$$

$\mathfrak{A}$ is called deterministic if for arbitrary $n \geqq 0$ and $\sigma \in (\Sigma_S)_n$ all the components of $h(\sigma)$ contain at most one element.

Example 3.3. Let $k=l=2$,

$$\Sigma=\Sigma_0\cup\Sigma_1,\ \Sigma_0=\{\bar{a}\},\ \Sigma_1=\{f\},\ \Delta=\Delta_0\cup\Delta_1,\ \Delta_0=\{\bar{a}\},\ \Delta_1=\{f,g\},$$

$$h(f)=\langle 4; f(x_1), f(x_2), g(x_3), g(x_4)\rangle,\ h(\bar{a})=\langle 2; x_1, x_2\rangle,\ h(S)=\langle 4; x_1, \bar{a}, x_2, \bar{a}\rangle.$$

(Braces enclosing singletons are omitted.) Then $\mathfrak{A}=(\Sigma, \Delta, k, l, h, S)$ is a deterministic attributed tree transducer, and it is easy to see that for all $n\geqq 0$

$$h(f^n(x_1)) = \langle 4; f^n(x_1), f^n(x_2), g^n(x_3), g^n(x_4)\rangle.$$

Hence, $h(f^n(\bar{a}))=\langle 2; f^n g^n(x_1), f^n g^n(x_2)\rangle$, and

$$\tau_{\mathfrak{A}} = \{\langle f^n(\bar{a}), f^n g^n f^n g^n(\bar{a})\rangle | n\geqq 0\}.$$

Definition 3.2 might be interpreted as follows. Let $t\in\tilde{T}(\Sigma)^1$, α a node in t having some label $\sigma\in\Sigma_n$. A component of $h(\sigma)$ describes how to compute the value of a synthesized attribute of α (the first k components), or an inherited attribute of an immediate descendant of α (the last $l\cdot n$ components) as a function (polynomial) of the synthesized attributes of the immediate descendants (the variables $x_1, \ldots, x_{k\cdot q}$) and the inherited attributes of α itself (the variables $x_{k\cdot q+1}, \ldots, x_{k\cdot q+l}$). The role of the synthesizer $h(S)$ is to produce the final result of the computation.

It will be convenient to identify the nodes of a tree $t\in\tilde{T}(\Sigma)^1_q$ with the set $\mathrm{nds}(t)\subseteq \mathbf{N}^*\times(\Sigma\cup X_q)$, and the leaves of t with $\mathrm{lvs}(t)\subseteq\mathbf{N}^*\times X_q$ as follows:

(i) if $t=x_1$, then $\mathrm{nds}(t)=\mathrm{lvs}(t)=\{\langle\lambda, x_1\rangle\}$;

(ii) if $t=t_0\cdot(\mathrm{id}_{p-1}\otimes\sigma(x_1, \ldots, x_n)\otimes\mathrm{id}_{q-p})$ with $t_0\in\tilde{T}(\Sigma)^1_q$, $q\geqq 1$, $p\in[q]$, $n\geqq 0$, $\sigma\in\Sigma_n$, then $\mathrm{nds}(t)=\bigcup_{i=1}^{5} V_i$, where

$V_1=\{\langle w, x_j\rangle | j\in[p-1]$ and $\langle w, x_j\rangle\in\mathrm{lvs}(t_0)\}$,
$V_2=\{\langle w, x_j\rangle | j\geqq p+n$ and $\langle w, x_{j-n+1}\rangle\in\mathrm{lvs}(t_0)\}$,
$V_3=\{\langle wj, x_{p+j-1}\rangle | j\in[n]$ and $\langle w, x_p\rangle\in\mathrm{lvs}(t_0)\}$,
$V_4=\mathrm{nds}(t_0)\setminus\mathrm{lvs}(t_0)$,
$V_5=\{\langle w, \sigma\rangle\}$, where $\langle w, x_p\rangle\in\mathrm{lvs}(t_0)$.
$\mathrm{lvs}(t)=V_1\cup V_2\cup V_3$.

It is easy to verify that $\mathrm{nds}(t)$ and $\mathrm{lvs}(t)$ are uniquely defined by the above construction, and for each $w\in\mathbf{N}^*$ there exists at most one $\alpha\in\mathrm{nds}(t)$ having w as its first component. Clearly, $\|\mathrm{nds}(t)\|=r(t)$, the number of nodes in t.

Let $\mathfrak{A}=(\Sigma, \Delta, k, l, h, S)$ be an attributed tree transducer, fixed in the rest of the paper, $t\in\tilde{T}(\Sigma)^1_q$,

$$Z_t = \{x(\alpha, i), y(\alpha, m) | \alpha\in\mathrm{nds}(t), i\in[k], m\in[l]\}$$

a set of variable symbols. Construct a system $E_{t,h}$ of nondeterministic Δ-equations over the variables Z_t as follows

$$E_{t,h}=\{E_{x,h}(\alpha, i) | \alpha\in\mathrm{nds}(t)\setminus\mathrm{lvs}(t), i\in[k]\}\cup$$
$$\cup\{E_{y,h}(\alpha, m) | \alpha\in\mathrm{nds}(t)\setminus\{\langle\lambda, \mathrm{root}(t)\rangle\}, m\in[l]\},$$

where

(i) if $\alpha=\langle w, \sigma\rangle$ with $\sigma\in\Sigma_n$ and

$$h(\sigma) = \langle T_1, \ldots, T_k, Q_1, \ldots, Q_{l\cdot n}\rangle, \tag{2}$$

then the equation $E_x(\alpha, i)$ is of the form

$$x(\alpha, i) = T_i[x_{k\cdot(r-1)+p} \leftarrow x(\alpha_r, p), x_{k\cdot n+s} \leftarrow y(\alpha, s)|p\in[k], r\in[n], s\in[l]],$$

where $\leftarrow$ denotes variable substitution, $\alpha_r \in \mathrm{nds}(t)$ is the unique node having wr as first component. (We omitted the index h, which is fixed.)

(ii) If $\alpha = \langle wj, a\rangle$ with $a\in\Sigma\cup X_q$, then consider the unique node $\bar\alpha = \langle w, \sigma\rangle$, where $\sigma\in\Sigma_n$, $n\geqq j$, and the nodes $\bar\alpha_r$, $r\in[n]$. (Naturally $\bar\alpha_j = \alpha$.) Let $h(\sigma)$ be as (2) above. Then the equation $E_y(\alpha, m)$ looks as

$$y(\alpha, m) = Q_{l\cdot(j-1)+m}[x_{k\cdot(r-1)+p} \leftarrow x(\bar\alpha_r, p), x_{k\cdot n+s} \leftarrow y(\bar\alpha, s)|p\in[k], r\in[n], s\in[l]].$$

The variables

$$Z_t^1 = \{x(\alpha, i)|\alpha\in\mathrm{lvs}(t), i\in[k]\}\cup\{y(\langle\lambda, \mathrm{root}(t)\rangle, m)|m\in[l]\}$$

do not occur on the left-hand side of these equations, so they are considered as parameters. On the other hand, the variables

$$Z_t^2 = \{x(\langle\lambda, \mathrm{root}(t)\rangle, i)|i\in[k]\}\cup\{y(\alpha, m)|\alpha\in\mathrm{lvs}(t), m\in[l]\}$$

do not occur on the right-hand side of the equations. If we identify the elements of Z_t with the variables $x_1, \ldots, x_{(k+l)\cdot r(t)}$ by a bijection $\varepsilon_t\colon Z_t\to[(k+l)\cdot r(t)]$ so that the variables Z_t^1 get the highest and Z_t^2 the lowest indices, we get an $\omega'(t, \varepsilon_t)$: $(k+l)\cdot r(t)-(k\cdot q+l)\to(k+l)\cdot r(t)\in\mathrm{Rec}(\Delta)$ for which $\omega'(t, \varepsilon_t) = 0_{k+l\cdot q}\otimes\omega(t, \varepsilon_t)$ and $(\omega'(t, \varepsilon_t))^+ = E_t^+$ (with respect to ε_t). E_t^+ denotes the solution of E_t.

Lemma 3.4. Let R be a rational theory, $k\geqq 1, l\geqq 0, q\geqq 1, n\geqq 0, p\in[q]$ integers, $a\in R(k, l)_q^1$, $b\in R(k, l)_n^1$. Then

$$a\cdot(e_{p-1}\otimes b\otimes e_{q-p}) = \mu^{k+l\cdot(q-1+n)}\cdot(0_{k+l\cdot(q-1+n)}\otimes$$
$$\otimes(\varrho_{q,p,n}\cdot \langle\!\langle a\cdot\eta_{q,p,n}, b\cdot\zeta_{q,p,n}\rangle\!\rangle))^+, \tag{3}$$

where

$$\varrho_{q,p,n} = \langle\!\langle \mu^{k+l\cdot(p-1)}, \nu_{l\cdot n}, 0_{k+l\cdot p}\otimes\mu^{l\cdot(q-p)+k}, 0_{k+l\cdot(p-1)}\otimes\mu^l \rangle\!\rangle\colon$$
$$k+l\cdot(p-1)+l\cdot n+l\cdot(q-p)+k+l \to k+l\cdot(p-1)+l+l\cdot(q-p)+k+l\cdot n,$$

$$\eta_{q,p,n} = \langle\!\langle \nu^{k+l}_{k\cdot(p-1)}, \mu^k_{l+k\cdot(p+1)} \rangle\!\rangle\otimes\nu^{k\cdot n}_{k\cdot(q-p)+l}\colon$$
$$k\cdot(p-1)+k+k\cdot(q-p)+l \to k+l+k\cdot(p-1)+k\cdot n+k\cdot(q-p)+l,$$

$$\zeta_{q,p,n} = 0_k\otimes\langle\!\langle \nu^{l+k\cdot(p-1)}_{k\cdot n}, \mu^l_{k\cdot(p-1+n)} \rangle\!\rangle\otimes 0_{k\cdot(q-p)+l}\colon$$
$$k\cdot n+l \to k+l+k\cdot(p-1)+k\cdot n+k\cdot(q-p)+l.$$

(The left-hand side of (3) is a polynomial in $R(k, l)$, while the right-hand side is a polynomial in R.)

Instead of presenting a complete proof we only remark that it would be enough to prove the lemma for one special free rational theory, analogously to the proof of Theorem 2.2. Then the proof reduces to an easy computation that we do not preform here. The following lemma can be proved in the same way

Lemma 3.5. Let R be a rational theory; $n_1, n_2, n_3, p_1, p_2, p_3, m, r, s$ nonnegative integers,

$$f\colon n_1+m+n_3+s \to s+p_1+r+p_3 \in R,$$
$$g\colon r+n_2 \to p_2+m \in R.$$

Then

$$\mu^{n_1+n_2+n_3}\cdot\big(0_{n_1+n_2+n_3}\otimes(\varrho\cdot \nless \mu^{n_1+m+n_3}\cdot(0_{n_1+m+n_3}\otimes f)^+\cdot\eta,\ g\cdot\zeta)\big)^+ =$$
$$= \mu^{n_1+n_2+n_3}\cdot\big(0_{n_1+n_2+n_3}\otimes(\varrho_s\cdot \nless f\cdot\eta_s,\ g\cdot\zeta_s \ngtr)\big)^+, \tag{4}$$

where

$$\varrho = \nless \mu^{n_1}, \nu_{n_2}, 0_{n_1+m}\otimes\mu^{n_3+r}, 0_{n_1}\otimes\mu^m \ngtr\colon n_1+n_2+n_3+r+m \to n_1+m+n_3+r+n_2,$$

$$\varrho_s = \nless \mu^{n_1}, \nu_{n_2}, 0_{n_1+m}\otimes\mu^{n_3+s+r}, 0_{n_1}\otimes\mu^m \ngtr\colon$$
$$n_1+n_2+n_3+s+r+m \to n_1+m+n_3+s+r+n_2,$$

$$\eta = \nless \nu_{p_1}^{r+m}, \mu_{m+p_1}^{r} \ngtr \otimes \nu_{p_3}^{p_2}\colon p_1+r+p_3 \to r+m+p_1+p_2+p_3, \qquad \eta_s = \mathrm{id}_s\otimes\eta,$$

$$\zeta = 0_r\otimes \nless \nu_{p_2}^{m+p_1}, \mu_{p_1+p_2}^{m} \ngtr \otimes 0_{p_3}\colon p_2+m \to r+m+p_1+p_2+p_3,\ \zeta_s = 0_s\otimes\zeta.$$

Lemma 3.6. Let $q \geqq 0$, $t\in\tilde{T}(\Sigma)_q^1$, $t\neq x_1$. There exists a bijection $\varepsilon_t\colon Z_t \to [(k+l)\cdot r(t)]$ such that

(A) for arbitrary $i\in[k]$, $j\in[q]$, $m\in[l]$ and appropriate $w\in\mathbf{N}^*$

$$\varepsilon_t\big(x(\langle\lambda, \mathrm{root}\,(t)\rangle, i)\big)=i,$$
$$\varepsilon_t\big(y(\langle w, x_j\rangle, m)\big)=k+l\cdot(j-1)+m,$$
$$\varepsilon_t\big(x(\langle w, x_j\rangle, i)\big)=r(t)-(k\cdot q+l)+k\cdot(j-1)+i,$$
$$\varepsilon_t\big(y(\langle\lambda, \mathrm{root}\,(t)\rangle, m)\big)=r(t)-l+m;$$

(B) $\mu^{k+l\cdot q}\cdot\big(0_{k+l\cdot q}\otimes\omega(t,\varepsilon_t)\big)^+=h(t)$.

Proof. If $t=\sigma(x_1, \dots, x_q)$ for some $\sigma\in\Sigma_q$, then ε_t is completely determined by (A). Obviusly, $\omega(t,\varepsilon_t)=h(t)$, so (B) is trivially satisfied. Now let $t=t_0\cdot(\mathrm{id}_{p-1}\otimes \sigma(x_1, \dots, x_n)\otimes\mathrm{id}_{q-p})$, where $q\geqq 1$, $p\in[q]$, $t_0\in\tilde{T}(\Sigma)_q^1$, $t_0\neq x_1$, $n\geqq 0$, $\sigma\in\Sigma_n$, and suppose the lemma is true for t_0. Let $s=(k+l)\cdot\|\mathrm{nds}\,(t_0)\setminus\mathrm{lvs}\,(t_0)\|-(k+l)$. Using the sets $V_1, \dots, V_5$ introduced in the construction of nds (t), we define ε_t as follows. If $\alpha\in V_1$, $\alpha=\langle w, x_j\rangle$, then for arbitrary $i\in[k]$ and $m\in[l]$

$$\varepsilon_t\big(x(\alpha, i)\big)=k+l\cdot(p-1)+l\cdot n+l\cdot(q-p)+s+k+l+k\cdot(j-1)+i,$$
$$\varepsilon_t\big(y(\alpha, m)\big)=k+l\cdot(j-1)+m.$$

If $\alpha\in V_2$, $\alpha=\langle w, x_j\rangle$, then

$$\varepsilon_t\big(x(\alpha, i)\big)=$$
$$=k+l\cdot(p-1)+l\cdot n+l\cdot(q-p)+s+k+l+k\cdot(p-1)+k\cdot n+k\cdot(j-1)+i,$$
$$\varepsilon_t\big(y(\alpha, m)\big)=k+l\cdot(p-1)+l\cdot n+l\cdot(j-1)+m.$$

If $\alpha\in V_3$, $\alpha=\langle wj, x_{p+j-1}\rangle$, then

$$\varepsilon_t\big(x(\alpha, i)\big)=k+l\cdot(p-1)+l\cdot n+l\cdot(q-p)+s+k+l+k\cdot(p-1)+k\cdot(j-1)+i,$$
$$\varepsilon_t\big(y(\alpha, m)\big)=k+l\cdot(p-1)+l\cdot(j-1)+m.$$

If $\alpha\in V_4$ and $\alpha=\langle\lambda, \operatorname{root}(t)\rangle$, then

$$\varepsilon_t(x(\alpha, i))=i,$$
$$\varepsilon_t(y(\alpha, m))=$$
$$=k+l\cdot(p-1)+l\cdot n+l\cdot(q-p)+s+k+l+k\cdot(p-1)+k\cdot n+k\cdot(q-p)+m,$$

else

$$\varepsilon_t(x(\alpha, i))=\varepsilon_{t_0}(x(\alpha, i))+l\cdot n-l,$$
$$\varepsilon_t(y(\alpha, m))=\varepsilon_{t_0}y(\alpha, m))+l\cdot n-l.$$

If $\alpha\in V_5$, then $\alpha=\langle w, \sigma\rangle$ and

$$\varepsilon_t(x(\alpha, i))=k+l\cdot(p-1)+l\cdot n+l\cdot(q-p)+s+i,$$
$$\varepsilon_t(y(\alpha, m))=k+l\cdot(p-1)+l\cdot n+l\cdot(q-p)+s+k+m,$$

It is easy to see that ε_t is a bijection and satisfies (A). To prove (B), apply Lemma 3.6 for $R=\operatorname{Rec}(\Delta), f=\omega(t_0, \varepsilon_{t_0}), g=h(\sigma), n_1=k+l\cdot(p-1), n_2=l\cdot n, n_3=l\cdot(q-p)$, $m=l, r=k, p_1=k\cdot(p-1), p_2=k\cdot n, p_3=k\cdot(q-p)$, (and $s=s$). Observe that $\varrho_s\cdot \nleftarrow f\cdot\eta_s, g\cdot\zeta_s \nrightarrow =\omega(t, \varepsilon_t)$, and the right-hand side of (4) equals to $\mu^{k+l\cdot(q-1+n)}\cdot$ $\cdot(0_{k+l\cdot(q-1+n)}\otimes\omega(t, \varepsilon_t))^+$. So we must prove that the left-hand side of (4) equals to $h(t)$. By the inductive hypothesis $\mu^{n_1+m+n_3}\cdot(0_{n_1+m+n_3}\otimes f)^+=h(t_0)$, so we have to see that

$$h(t) = h(t_0)\cdot(e_{p-1}\otimes h(\sigma)\otimes e_{q-p}) =$$
$$=\mu^{k+l\cdot(q-1+n)}\cdot(0_{k+l\cdot(q-1+n)}\otimes(\varrho\cdot \nleftarrow h(t_0)\cdot\eta, h(\sigma)\cdot\zeta \nrightarrow))^+.$$

This is exactly the statement of Lemma 3.5, so we are through.

Replacing Σ by Σ_s we get

Corollary 3.7. For each $t\in\tilde{T}(\Sigma)_0^1$, $\tau_{\mathfrak{A}}(t)$ equals to the $x(\langle\lambda, S\rangle, 1)$ component of $E^+_{S(t)}$.

This result links our work to [4], where the same technic was used to define the semantics of attribute grammars.

Now we turn our attention to the domain of $\tau_{\mathfrak{A}}$, that is the set $D\tau_{\mathfrak{A}}=\{t\in\tilde{T}(\Sigma)_0^1|$ for some $u\in T(\Delta)_0^1\langle t, u\rangle\in\tau_{\mathfrak{A}}\}$. Let $G(k, l)$ be the following finite set

$G(k, l)=\{(G; V_{1,1}, V_{1,2}, V_{2,1}, V_{2,2})|G=(V, E)$ is a directed acyclic bipartite graph, and

(i) $V=V_1\cup V_2$, $V=[k+l]$, $V_1=[k]$, $V_2=V\setminus V_1$, $E=E_1\cup E_2$, $\operatorname{dom}(E_1)\subseteqq V_1$, $\operatorname{dom}(E_2)\subseteqq V_2$;

(ii) $V_1=V_{1,1}\cup V_{1,2}$, $V_{1,1}\cap V_{1,2}=\emptyset$; $V_2=V_{2,1}\cup V_{2,2}$, $V_{2,1}\cap V_{2,2}=\emptyset$;

(iii) for each $j\in V_{2,1}$ there exists an $i\in V_{1,1}$ such that $\langle i, j\rangle\in E_1$ and the vertices $V_{1,2}\cup V_{2,2}$ are isolated.$\}$

(A vertex is called isolated if there are no edges entering or leaving it.)

We construct a finite state top-down tree automaton $\mathfrak{B}$ that operates nondeterministically on $\tilde{T}(\Sigma)^1$ with states $A=G(k, l)$. Let $t\in D\tau_{\mathfrak{A}}$, α a node in t and suppose that $\mathfrak{B}$ passes through α in state $(G; V_{1,1}, \ldots, V_{2,2})$. The synthesized (inherited) attributes of α are represented as the nodes in V_1 (V_2, respectively). $V_{1,1}\cup V_{2,1}$ will contain the indices of those attributes that take part in the computation of $\tau_{\mathfrak{A}}(t)$. The edges of G will show how these "useful" attributes depend on each other. A similar construction was used in [5] for testing circularity of attribute grammars.

The fact that, starting from state a_0, $\mathfrak{B}$ is able to reach the vector of states $\langle a_1, \ldots, a_q\rangle$ on input $t \in \tilde{T}(\Sigma)_q^1$ will be denoted by $a_0 t \overset{*}{\underset{\mathfrak{B}}{\vdash}} t(a_1, \ldots, a_q)$. If for some $\sigma \in \Sigma_q$, $t = \sigma(x_1, \ldots, x_q)$, we simply write $a_0\sigma \overset{*}{\underset{\mathfrak{B}}{\vdash}} \sigma(a_1, \ldots, a_q)$.

Let $\sigma \in (\Sigma_S)_n$, $h(\sigma) = \langle T_1, \ldots, T_{k+l\cdot n}\rangle$, $I_\sigma = \{i \in [k+l\cdot n] | T_i = \emptyset\}$. The set of alternatives of σ is

$$A[\sigma] = \{\langle t_1, \ldots, t_{k+l\cdot n}\rangle | \text{ if } i \in I_\sigma, \text{ then } t_i = \perp, \text{ else } t_i \in T_i\}.$$

We say that $c \in A[S]$ realizes the initial state $a = (G_c; V_{1,1}^c, \ldots, V_{2,2}^c)$ if the following conditions are satisfied:

(a) If $j \in V_{2,1}^c$, then $\langle j, i\rangle \in E_2^c$ if and only if x_i occurs in t_j.

(b) $V_{1,1}^c \supseteq Q = \{i \in [k] | x_i$ occurs in $t_1\}$, and for each $i \in V_{1,1}^c \setminus Q$ there exists an $i' \in Q$ such that $i' \overset{+}{\underset{G_c}{\vdash}} i \cdot \overset{+}{\underset{G_c}{\vdash}}$ denotes the transitive closure of $\underset{G_c}{\vdash} = E^c$.

(c) $V_{2,2}^c \supseteq \{j > k | j \in I_S\}$.

Define the set of initial states of $\mathfrak{B}$ as $A_0 = \{a \in A | a$ is realized by some $c \in A[S]\}$.

Let $n \geqq 0, \sigma \in \Sigma_n, a_0, \ldots, a_n \in A$, $a_m = (G_m; V_{1,1}^m, \ldots, V_{2,2}^m)$ for each $0 \leqq m \leqq n$, and $c = \langle t_1, \ldots, t_{k+l\cdot n}\rangle \in A[\sigma]$. Construct the graph $G[c, a_0, \ldots, a_n]$ by adding the edges $E[c, a_0, \ldots, a_n]$ to the disjoint union of graphs G_m, $0 \leqq m \leqq n$. An edge $\langle\langle i, m_1\rangle, \langle j, m_2\rangle\rangle \in E[c, a_0, \ldots, a_n]$ if and only if one of the following conditions is satisfied:

(i) $m_1 = m_2 = 0$, $i \in V_{1,1}^0$, $j > k$ and $x_{k\cdot n+(j-k)}$ occurs in t_i;

(ii) $m_1 = 0$, $m_2 \geqq 1$, $i \in V_{1,1}^0$, $j \leqq k$ and $x_{k\cdot(m_2-1)+j}$ occurs in t_i;

(iii) $m_1 \geqq 1$, $m_2 = 0$, $i \in V_{2,1}^{m_1}$, $j > k$ and $x_{k\cdot n+(j-k)}$ occurs in $t_{k+l\cdot(m_1-1)+(i-k)}$;

(iv) $m_1 \geqq 1$, $m_2 \geqq 1$, $i \in V_{2,1}^{m_1}$, $j \leqq k$ and $x_{k\cdot(m_2-1)+j}$ occurs in $t_{k+l\cdot(m_1-1)+(i-k)}$.

$G'[c, a_0, \ldots, a_n]$ can be obtained from $G[c, a_0, \ldots, a_n]$ by leaving the edges $E_1^0 \cup \left(\bigcup_{m=1}^{n} E_2^m\right)$. We say that c realizes the transition $a_0\sigma \underset{\mathfrak{B}}{\vdash} \sigma(a_1, \ldots, a_n)$ if the following conditions are satisfied. (The mark $[c, a_0, \ldots, a_n]$ will be omitted from the right of G, G' and E.)

(A) Let $i \in I_\sigma$. If $i \leqq k$, then $i \in V_{1,2}^0$, else if for some $m \in [n]$ and $k < j \leqq k+l$, $i = l\cdot(m-1)+j$, then $j \in V_{2,2}^m$.

(B) For each $m \in [n]$, $i \in V_{1,1}^m$ if and only if there exists an $i' \in V_{1,1}^0$ such that $\langle i', 0\rangle \overset{+}{\underset{G}{\vdash}} \langle i, m\rangle$.

(C) For each $0 \leqq m \leqq n$, $\overset{+}{\underset{G'}{\vdash}} | G_m = \overset{+}{\underset{G_m}{\vdash}}$.

Now for each $\sigma \in \Sigma_n$, $a_0\sigma \underset{\mathfrak{B}}{\vdash} \sigma(a_1, \ldots, a_n)$ if and only if this transition is realized by some $c \in A[\sigma]$.

Let $q \geqq 0$, $t \in \tilde{T}(\Sigma)_q^1$. A deterministic part of $E_{S(t)}$ can be chosen as follows. Replace the equations of the form $z = \emptyset$ by $z = z$, then for each $z \in Z_{S(t)} \setminus Z_{S(t)}^1$ replace the right-hand side of the equation $z = T_z$ by an arbitrary $t_z \in T_z$. Further on $DE_{S(t)}$ will always denote a deterministic part of $E_{S(t)}$. For each $z \in Z_{S(t)} \setminus Z_{S(t)}^1$, $\pi(z) \cdot E_{S(t)}^+ \neq \emptyset$ if and only if there exists a $DE_{S(t)}$ such that $\pi(z) \cdot DE_{S(t)}^+ \neq \emptyset$. ($\pi(z)$. means the selection of the component z.) Let $\underset{}{\vdash} DE_{S(t)}$ denote the dependence relation among the variables $Z_{S(t)}$ in a deterministic part of $E_{S(t)}$, that is, $z_1 \vdash DE_{S(t)} z_2$ if and only if z_2 occurs in t_{z_1}. It is clear that $\pi(z) \cdot DE_{S(t)} \neq \emptyset$ if and only if $z \overset{*}{\vdash} DE_{S(t)} z'$ implies $z' \overset{+}{\not\vdash} DE_{S(t)} z'$.

For each $n \in [l]$ take a new symbol γ_n, and construct the ranked alphabet $\Gamma = \bigcup_{n=1}^{l} \Gamma_n$ with $\Gamma_n = \{\gamma_n\}$. Let $q \geqq 0$, $t \in \tilde{T}(\Sigma)^1_q$, $a_1, \ldots, a_q \in A$, $a_j = (G_j; V^j_{1,1}, \ldots, V^j_{2,2})$ for each $j \in [q]$. By $E_t[a_1, \ldots, a_q]$ we mean the following system of equations

$$E_t[a_1, \ldots, a_q] = \{x(\langle w, x_j\rangle, i) = \gamma_n(y(\langle w, x_j\rangle, m_1), \ldots, y(\langle w, x_j\rangle, m_n)) \mid j \in [q], \langle w, x_j\rangle \in \text{lvs}\,(t),\ i \in [k] \text{ and } m_1, \ldots, m_n \text{ are all the possible values of such an } m \text{ for which } \langle i, k+m\rangle \in E^j_1\}.$$

Lemma 3.8. Let $q \geqq 0$, $t \in \tilde{T}(\Sigma)^1_q$, $a_1, \ldots, a_q \in A$ and for each $j \in [q]$, $a_j = (G_j; V^j_{1,1}, \ldots, V^j_{2,2})$. There exists an $a \in A_0$ for which $at \overset{*}{\underset{\mathfrak{B}}{\vdash}} t(a_1, \ldots, a_q)$ if and only if a $DE_{S(t)}$ can be chosen such that

(i) $\pi(x(\langle \lambda, S\rangle, 1)) \cdot (DE_{S(t)} \cup E_t[a_1, \ldots, a_q])^+ \neq \emptyset$;

(ii) for each $j \in [q]$, $\langle w, x_j\rangle \in \text{lvs}\,(S(t))$, $i \in [k]$, $x(\langle \lambda, S\rangle, 1) \overset{+}{\vdash} x(\langle w, x_j\rangle, i)$ holds in $DE_{S(t)} \cup E_t[a_1, \ldots, a_q]$ if and only if $i \in V^j_{1,1}$;

(iii) for each $m+k \in V^j_{2,1}$, $y(\langle w, x_j\rangle, m) \overset{+}{\vdash} x(\langle w, x_j\rangle, i)$ if and only if $m+k \overset{+}{\underset{G_j}{\vdash}} i$.

Proof. Only if: If $t = x_1$, then $a = a_1 \in A_0$. In this case $E_{S(t)}$ is the same as $h(S)$, written in the form of equations, so (i), (ii) and (iii) follow from the conditions (a), (b) and (c) that must hold for $a \in A_0$. Let $q \geqq 1$, $p \in [q]$, $n \geqq 0$, $\sigma \in \Sigma_n$, $t_0 \in \tilde{T}(\Sigma)^1_q$ and $t = t_0 \cdot (\text{id}_{p-1} \otimes \sigma(x_1, \ldots, x_n) \otimes \text{id}_{q-p})$. If $at \overset{+}{\underset{\mathfrak{B}}{\vdash}} t(a^1, \ldots, a^{p-1}, a_1, \ldots, a_n, a^{p+1}, \ldots, a^q)$, then there exists an $a_0 \in A$ such that $at_0 \overset{*}{\underset{\mathfrak{B}}{\vdash}} t_0(a^1, \ldots, a^{p-1}, a_0, a^{p+1}, \ldots, a^q)$ and $a_0\sigma \underset{\mathfrak{B}}{\vdash} \sigma(a_1, \ldots, a_n)$. Suppose the *Only if* part is true for t_0 and states $a^1, \ldots, a^{p-1}$, $a_0, a^{p+1}, \ldots, a^q$, and the transition $a_0\sigma \underset{\mathfrak{B}}{\vdash} \sigma(a_1, \ldots, a_n)$ is realized by $c = \langle t_1, \ldots, t_{k+l\cdot n}\rangle \in A[\sigma]$. Then there exists an appropriate $DE_{S(t_0)}$ satisfying the three conditions. For all $i \in [k]$ and $m \in [l]$, replace the variables $x(\langle w, x_p\rangle, i)$ and $y(\langle w, x_p\rangle, m)$ in $DE_{S(t_0)}$ by $x(\langle w, \sigma\rangle, i)$ and $y(\langle w, \sigma\rangle, m)$, respectively, and add the set of equations

$$\begin{aligned}&\{x(\langle w, \sigma\rangle, i) = t_i[x_{k\cdot(j-1)+r} \leftarrow x(\langle wj, x_{j+p-1}\rangle, r),\\ &x_{k\cdot n+s} \leftarrow y(\langle w, \sigma\rangle, s),\ \bot \leftarrow x(\langle w, \sigma\rangle, i) \mid j \in [n], r \in [k], s \in [l]] \mid i \in [k]\} \cup\\ &\cup \{y(\langle wj, x_{j+p-1}\rangle, m) = t_{k+l\cdot(j-1)+m}[x_{k\cdot(u-1)+r} \leftarrow x(\langle wu, x_{u+p-1}\rangle, r),\\ &x_{k\cdot n+s} \leftarrow y(\langle w, \sigma\rangle, s),\ \bot \leftarrow y(\langle wj, x_{j+p-1}\rangle, m) \mid u \in [n], r \in [k], s \in [l]] \mid j \in [n], m \in [l]\}\end{aligned}$$

to obtain $DE_{S(t)}$. For $0 \leqq m \leqq n$ let $\alpha_m \in \text{nds}\,(S(t))$ such that $\alpha_m = \langle w, \sigma\rangle$ if $m = 0$, else $\alpha_m = \langle wm, x_{m+p-1}\rangle$. If $i \in [k+l]$, then $z(\langle i, m\rangle)$ will denote the following variable of $Z_{S(t)}$

$$z(\langle i, m\rangle) = \begin{cases} x(\alpha_m, i) & \text{if } i \in [k], \\ y(\alpha_m, i-k) & \text{if } i > k. \end{cases}$$

By the inductive hypothesis and conditions (A), (B), (C) imposed on the transitions of $\mathfrak{B}$ we have

(*) $\langle i_1, m_1\rangle \overset{+}{\underset{G}{\vdash}} \langle i_2, m_2\rangle$ if and only if for $j=1, 2, i_j \in (V_{1,1}^{m_j} \cup V_{2,1}^{m_j})$ and $z(\langle i_1, m_1\rangle) \overset{+}{\vdash} DE_{S(t)} \cup E_t[a^1, \dots, a^{p-1}, a_1, \dots, a_n, a^{p+1}, \dots, a^q]\, z(\langle i_2, m_2\rangle)$ are both satisfied $(G=G[c, a_0, \dots, a_n])$.

To prove (i) suppose that $x(\langle\lambda, S\rangle, 1) \overset{*}{\vdash} z$ and $z \overset{+}{\vdash} z$ hold in $DE_{S(t)} \cup \cup E_t[a^1, \dots, a^{p-1}, a_1, \dots, a_n, a^{p+1}, \dots, a^q]$ for some $z \in Z_{S(t)}$. By the inductive hypothesis we can assume that $z=z(\langle i, m\rangle)$ for some $i \in [k+l]$, $0 \le m \le n$. Using (*) and (C) we conclude that G_m contains a cycle, which is a contradiction.

Let $\alpha=\langle u, x_j\rangle \in \text{lvs}\,(S(t))$. By (B) and (*), $i \in V_{1,1}^j$ if and only if there exists a $j' \in [q]$ and an $i' \in V_{1,1}^{j'}$ such that

$$x(\alpha_{j'}, i') \overset{*}{\vdash} DE_{S(t)} \cup E_t[a^1, \dots, a^{p-1}, a_1, \dots, a_n, a^{p+1}, \dots, a^q]x(\alpha, i),$$

where $\alpha_{j'}=\langle w, \sigma\rangle$ if $j'=p$, else $\alpha_{j'}=\alpha$. Let $\bar{\alpha}_{j'}=\langle w, x_p\rangle$ if $j'=p$, else $\bar{\alpha}_{j'}=\alpha$. By the inductive hypothesis $i' \in V_{1,1}^{j'}$ if and only if $x(\langle\lambda, S\rangle, 1) \overset{+}{\vdash} x(\bar{\alpha}_{j'}, i')$ holds in $DE_{S(t_0)} \cup E_{t_0}[a^1, \dots, a^{p-1}, a_0, a^{p+1}, \dots, a^q]$, which is equivalent to

$$x(\langle\lambda, S\rangle) \overset{+}{\vdash} DE_{S(t)} \cup E_t[a^1, \dots, a^{p-1}, a_1, \dots, a_n, a^{p+1}, \dots, a^q]\, x(\alpha_{j'}, i').$$

Thus, $i \in V_{1,1}^j$ if and only if $x(\langle\lambda, S\rangle, 1) \overset{+}{\vdash} x(\alpha, i)$, which proves (ii).

Let us remark that (iii) is already proved for $p \le j < p+n$ as a special case of (*). It is easy to prove it for other values of j, too.

If: The case $t=x_1$ is again trivial. Let $t=t_0 \cdot (\text{id}_{p-1} \otimes \sigma(x_1, \dots, x_n) \otimes \text{id}_{q-p})$ as above, and suppose the *If* part is true for t_0 and any appropriate states $b_1, \dots, b_q$. Let $DE_{S(t)}$ and the states $a^1, \dots, a^{p-1}, a_1, \dots, a_n, a^{p+1}, \dots, a^q$ satisfy (i), (ii) and (iii). Split $DE_{S(t)}$ into $DE_{S(t_0)}$ and a part that can be derived from $c=\langle t_1, \dots, t_{k+l \cdot n}\rangle \in \in A[\sigma]$. Let $a_0=(G_0; V_{1,1}^0, \dots, V_{2,2}^0)$ be the following state

$i \in V_{1,1}^0$ if and only if $x(\langle\lambda, S\rangle, 1) \overset{+}{\vdash} x(\langle w, \sigma\rangle, i)$ holds in $DE_{S(t)} \cup \cup E_t[a^1, \dots, a^{p-1}, a_1, \dots, a_n, a^{p+1}, \dots, a^q]$, where w is the first component of the node $\langle w, x_p\rangle$ in t_0;

$\langle i, j\rangle \in E_1^0$ if and only if $i \in V_{1,1}^0$ and $x(\langle w, \sigma\rangle, i) \overset{+}{\vdash} y(\langle w, \sigma\rangle, j-k)$, $V_{2,1}^0=\{j|$ for some $i \in V_{1,1}^0 \langle i, j\rangle \in E_1^0\}$;

$\langle j, i\rangle \in E_2^0$ if and only if $j \in V_{2,1}^0$ and $y(\langle w, \sigma\rangle, j-k) \overset{+}{\vdash} x(\langle w, \sigma\rangle, i)$.

It is clear that $DE_{S(t_0)}$ and states $a^1, \dots, a^{p-1}, a_0, a^{p+1}, \dots, a^q$ satisfy (i), (ii) and (iii), hence, by the inductive hypothesis $at_0 \overset{*}{\underset{\mathfrak{B}}{\vdash}} t_0(a^1, \dots, a^{p-1}, a_0, a^{p+1}, \dots, a^q)$ for some $a \in A_0$. On the other hand it can easily be checked that $a_0\sigma \underset{\mathfrak{B}}{\vdash} \sigma(a_1, \dots, a_n)$ is realized by c, so we are through.

Taking $q=0$ in the lemma we get

Theorem 3.9. The domain of attributed tree transformations is a regular tree language.

However, Lemma 3.8 is worth some further considerations. It can be seen that Lemma 3.8 remains valid if we require the states of $\mathfrak{B}$ not contain any redundant edges. (An adge $\langle i, j\rangle$ is redundant if there is another path from i to j containing more than one edge.) Let $\mathfrak{A}$ be deterministic, and suppose the states of $\mathfrak{B}$ satisfy the above additional requirement. The following statement can be proved by a bottom-up type induction combined with Lemma 3.8.

Proposition 3.10. Let $t \in D\tau_{\mathfrak{A}}$, $t = t_0 \cdot u$ with $t_0 \in \tilde{T}(\Sigma)^1_1$. There exists a unique $a \in A$ such that for some $a_0 \in A_0$ we have $a_0 t_0 \overset{*}{\underset{\mathfrak{B}}{\vdash}} t_0(a)$ and $au \overset{+}{\underset{\mathfrak{B}}{\vdash}} u$. This unique $a = (G; V_{1,1}, \ldots, V_{2,2})$ is the following: $V_{1,1} \cup V_{2,1} = Z_\alpha = \{z \in Z_{S(t)} |$ the "node" index of z is $\alpha = \mathrm{root}(u)$ and $x(\langle\lambda, S\rangle, 1) \overset{+}{\vdash} DE_{S(t)} z\}$, and $\overset{+}{\underset{G}{\vdash}} = \overset{+}{\vdash} DE_{S(t)} | Z_\alpha$. (Obviously, $DE_{S(t)}$ is unique in this case.)

As an application of Proposition 3.10 we finally show how to decide the K-visit property for deterministic attributed tree transducers. (Alternative proofs can be derived from [6] and [7].) Let $t \in D\tau_{\mathfrak{A}}$, $\alpha \in \mathrm{nds}(t)$. Proposition 3.10 shows that the state $a = (G_\alpha; V^\alpha_{1,1}, \ldots, V^\alpha_{2,2})$ in which $\mathfrak{B}$ passes through α during the recognition of t is uniquely determined, and it describes the dependence relation among the useful attributes of α. If p is a path in G_α $(p \in \mathrm{path}(G_\alpha))$, then let $v_p = \|\{i \in V^\alpha_{1,1} | p$ passes through $i\}\|$, $v_\alpha = \max\{v_p | p \in \mathrm{path}(G_\alpha)\}$. v_α shows how many times we must "enter" the subtree having root α to ask for the value of certain attributes. (Supposing an optimal, maximally paralleled evaluation of the useful attributes.) Define

$$v_{\mathfrak{A}} = \max\{v_\alpha | \alpha \in \mathrm{nds}(t) \quad \text{for some} \quad t \in D\tau_{\mathfrak{A}}\}.$$

Since this set is finite, it is easy to give an algorithm that computes $v_{\mathfrak{A}}$, and obviously, $\mathfrak{A}$ is K-visit if and only if $v_{\mathfrak{A}} \leqq K$. Moreover, it follows from the construction that

$$\text{if } l < k, \quad \text{then} \quad v_{\mathfrak{A}} \leqq l+1, \quad \text{else} \quad v_{\mathfrak{A}} \leqq k.$$

A trivial consequence of this statement is the known fact that every deterministic attributed tree transducer is K-visit for some K.

Abstract

A general concept of attributed transformation is introduced by means of magmoids and rational theories. It is shown that the domain of attributed tree transformations is a regular tree language, and an alternative proof is given for the decidability of the K-visit property of deterministic attributed tree-transducers.

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References

[1] ARNOLD, A. & M. DAUCHET, Theorie des magmoides, Preliminary work to the authors' theses, Univ. de Lille, France, 1977.
[2] DAUCHET, M., Transductions de forets, bimorphismes de magmoides, These, Univ. de Lille, France, 1977.
[3] WRIGHT, J. B., J. W. THATCHER, E. G. WAGNER & J. A. GOGUEN, Rational algebraic theories and fixed-point solutions, 17-th IEEE Symposium on Foundations of Computing, Houston, 1976, pp. 147—158.
[4] CHIRICA, L. M. & D. F. MARTIN, An order-algebraic definition of Knuthian semantics, *Math. Systems Theory,* v. 13, 1979, pp. 1—27.
[5] KNUTH, D. E., Semantics of context-free languages, *Math. Systems Theory,* v. 2, 1968, pp. 127—145; Correction, v. 5, 1971, pp. 95—96.
[6] RIIS, H. & S. SKYUM, *k*-visit attribute grammars, DAIMI PB-121, Aarhus University, Denmark, 1980.
[7] FÜLÖP, Z., Attribute grammars and attributed tree transducers, 15-th National Scientific Conference for Students, 1981, Budapest, Hungary.

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