Maximal families of restricted subsets of a finite set

By H.-D. O. F. GRONAU and CHR. PROSKE

1. Introduction

Let R be the set of the first r natural numbers, i.e. $R=\{1, 2, \ldots, r\}$. Furthermore, let a and b be integers with $0 \leqq a \leqq b \leqq r$, $a \neq r$, $b \neq 0$ [1]. Finally let $\mathcal{F}$ be an n-tuple $(X_1, X_2, \ldots, X_n)$ of subsets of R satisfying $a \leqq |X_i| \leqq b$ $(i=1, 2, \ldots, n)$.

An ordered pair (X, Y) of subsets of R has the property

— **A**: if and only if there is a $v \in R: v \notin X, v \in Y$,
— **B**: if and only if there is a $v \in R: v \in X, v \notin Y$,
— **C**: if and only if there is a $v \in R: v \notin X, v \notin Y$,
— **D**: if and only if there is a $v \in R: v \in X, v \in Y$.

Let $\mathbf{P}=\mathbf{P}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be an arbitrary Boolean expression of **A**, **B**, **C**, **D**. $\mathcal{F}$ is said to be a **P**-family if and only if all ordered pairs (X_i, X_j), $1 \leqq i < j \leqq n$, satisfy the condition **P**. If there is a maximal value of n, we will denote this by $n_{a,b}(\mathbf{P}, r)$. Many well-known results in extremal set theory can be expressed in our concept. We will only mention the following two classical theorems.

1) SPERNER's theorem [13]: $$n_{0,r}(\mathbf{AB}, r)=\binom{r}{[r/2]}, \text{ [2]}$$

2) ERDŐS-KO-RADO-Theorem [3]: $$n_{0,k}(\mathbf{ABD}, r)=\binom{r-1}{k-1} \text{ if } k \leqq r/2.$$

In [5] the first-named author considered all 2^{16} possible Boolean expressions **P**, found those **P**'s for which $n_{0,r}(\mathbf{P}, r)$ [3] eixsts, and determined in all these cases $n_{0,r}(\mathbf{P}, r)$ exactly [4]. In the present paper we consider the same problem for all **P**'s and $n_{a,b}(\mathbf{P}, r)$.

The results in Sections 2 and 3 are close to the corresponding results for $n(\mathbf{P}, r)$. Thus, the proofs are sketched only or are omitted.

[1] For simplification we exclude the pathological cases $a=r$ resp. $b=0$.
[2] **AB** will be used in place of $\mathbf{A} \wedge \mathbf{B}$ and $\overline{\mathbf{A}}$ denotes *non* **A**.
[3] In [5] the notation $n(\mathbf{P}, r)$ is used for $n_{0,r}(\mathbf{P}, r)$.
[4] With exception of only one case, where bounds and the asymptotic are found.

2. Existence of $n_{a,b}(\mathbf{P}, r)$

We set $n_{a,b}(\mathbf{0}, r)=1$ for all a, b, where **0** denotes the empty condition. In all what follows let $\mathbf{P}\not\equiv\mathbf{0}$.

Then there is a nonempty canonical alternative normal form $CANF(\mathbf{P})$ of **P**. If $\mathbf{A}'$ is an elementary conjunction of $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$, then $\mathbf{A}'\in CANF(\mathbf{P})$ means that $\mathbf{A}'$ is one of the conjunctions of $CANF(\mathbf{P})$.

Since no pair (X, Y) satisfies $\overline{\mathbf{A}}\overline{\mathbf{B}}\overline{\mathbf{C}}\overline{\mathbf{D}}$ and the only pairs satisfying $\overline{\mathbf{A}}\overline{\mathbf{B}}\overline{\mathbf{C}}\mathbf{D}$ or $\overline{\mathbf{A}}\overline{\mathbf{B}}\mathbf{C}\overline{\mathbf{D}}$ are (R, R) or $(\emptyset, \emptyset)$, respectively, it follows

Lemma 1.
(i) $n_{a,b}(\overline{\mathbf{A}}\overline{\mathbf{B}}\overline{\mathbf{C}}\overline{\mathbf{D}}, r)=1,$
$n_{a,b}(\mathbf{P}\vee\overline{\mathbf{A}}\overline{\mathbf{B}}\overline{\mathbf{C}}\overline{\mathbf{D}}, r)=n_{a,b}(\mathbf{P}, r),$
(ii) $n_{a,b}(\overline{\mathbf{A}}\overline{\mathbf{B}}\overline{\mathbf{C}}\mathbf{D}, r)=1$ *if* $b<r$,
$n_{a,b}(\mathbf{P}\vee\overline{\mathbf{A}}\overline{\mathbf{B}}\overline{\mathbf{C}}\mathbf{D}, r)=n_{a,b}(\mathbf{P}, r),$
(iii) $n_{a,b}(\overline{\mathbf{A}}\overline{\mathbf{B}}\mathbf{C}\overline{\mathbf{D}}, r)=1$ *if* $0<a$,
$n_{a,b}(\mathbf{P}\vee\overline{\mathbf{A}}\overline{\mathbf{B}}\mathbf{C}\overline{\mathbf{D}}, r)=n_{a,b}(\mathbf{P}, r).$

Theorem 1. $n_{a,b}(\mathbf{P}, r)$ *does not exist if and only if*
(i) $\overline{\mathbf{A}}\overline{\mathbf{B}}\mathbf{C}\mathbf{D}\in CANF(\mathbf{P})$ *or*
(ii) $\overline{\mathbf{A}}\overline{\mathbf{B}}\overline{\mathbf{C}}\mathbf{D}\in CANF(\mathbf{P})$ *and* $b=r$ *or*
(iii) $\overline{\mathbf{A}}\overline{\mathbf{B}}\mathbf{C}\overline{\mathbf{D}}\in CANF(\mathbf{P})$ *and* $a=0$.

Hence, if $n_{a,b}(\mathbf{P}, r)$ exists, $\overline{\mathbf{A}}\overline{\mathbf{B}}\overline{\mathbf{C}}\overline{\mathbf{D}}$, $\overline{\mathbf{A}}\overline{\mathbf{B}}\overline{\mathbf{C}}\mathbf{D}$, and $\overline{\mathbf{A}}\overline{\mathbf{B}}\mathbf{C}\overline{\mathbf{D}}$ can be omitted in $CANF(\mathbf{P})$.

3. Some reductions

The following table gives an equivalent description of some conditions **P** in terms of ordered pairs (X, Y).

P	$\leftrightarrow$	(X, Y)
$\mathbf{A}\overline{\mathbf{B}}\overline{\mathbf{C}}\overline{\mathbf{D}}$		$(\emptyset, R)$
$\overline{\mathbf{A}}\mathbf{B}\overline{\mathbf{C}}\overline{\mathbf{D}}$		$(R, \emptyset)$
$\mathbf{A}\overline{\mathbf{B}}\mathbf{C}\overline{\mathbf{D}}$		$(\emptyset, Z)$
$\overline{\mathbf{A}}\mathbf{B}\mathbf{C}\overline{\mathbf{D}}$		$(Z, \emptyset)$
$\mathbf{A}\overline{\mathbf{B}}\overline{\mathbf{C}}\mathbf{D}$		(Z, R)
$\overline{\mathbf{A}}\mathbf{B}\overline{\mathbf{C}}\mathbf{D}$		(R, Z)

(1)

where $Z\subseteq R$, $Z\neq\emptyset$, $Z\neq R$. The remaining 6 conditions $\mathbf{ABCD}$, $\mathbf{ABC}\overline{\mathbf{D}}$, $\mathbf{AB}\overline{\mathbf{C}}\mathbf{D}$, $\mathbf{A}\overline{\mathbf{B}}\mathbf{CD}$, $\overline{\mathbf{A}}\mathbf{BCD}$, $\overline{\mathbf{A}}\mathbf{B}\overline{\mathbf{C}}\overline{\mathbf{D}}$ are conditions for pairs (X, Y) with $\{X, Y\}\cap\{\emptyset, R\}=\emptyset$.

If $a=0$ and $b=r$ we refer to [5]. Let $a>0$ or $b<r$.

Then no pair (X, Y) can satisfy $\mathbf{A}\overline{\mathbf{B}}\overline{\mathbf{C}}\overline{\mathbf{D}}$ or $\overline{\mathbf{A}}\mathbf{B}\overline{\mathbf{C}}\overline{\mathbf{D}}$ and we may omit these conjunctions in **P**. Let $\mathbf{P}=\mathbf{P}'\vee\mathbf{P}''$, where $\mathbf{P}''$ contains exactly those conjunctions which are in (1).

Theorem 2.

	$\mathbf{P}'\equiv 0$	$\mathbf{P}'\not\equiv 0$
$a>0$ $b=r$	$n_{a,r}(\mathbf{P},r)=\begin{cases}2^{(*)}\\ 1 \textit{ otherwise}\end{cases}$	$n_{a,r}(\mathbf{P},r)=\begin{cases} n_{a,r-1}(\mathbf{P}',r)+1 & \textit{if } \mathbf{A\bar{B}\bar{C}D}\in CANF(\mathbf{P}'') \textit{ or } \mathbf{\bar{A}B\bar{C}D}\in CANF(\mathbf{P}'')\\ n_{a,r-1}(\mathbf{P}',r) & \textit{otherwise}\end{cases}$
$b<r$ $a=0$	$n_{0,b}(\mathbf{P},r)=\begin{cases}2^{(**)}\\ 1 \textit{ otherwise}\end{cases}$	$n_{0,b}(\mathbf{P},r)=\begin{cases} n_{1,b}(\mathbf{P}',r)+1 & \textit{if } \mathbf{A\bar{B}C\bar{D}}\in CANF(\mathbf{P}'') \textit{ or } \mathbf{\bar{A}BC\bar{D}}\in CANF(\mathbf{P}'')\\ n_{1,b}(\mathbf{P}',r) & \textit{otherwise}\end{cases}$
$a>0$ $b<r$	$n_{a,b}(\mathbf{P},r)=1$	$n_{a,b}(\mathbf{P},r)=n_{a,b}(\mathbf{P}',r)$

(*) if $\mathbf{A\bar{B}\bar{C}D}\in CANF(\mathbf{P}'')$ or $\mathbf{\bar{A}B\bar{C}D}\in CANF(\mathbf{P}'')$;
(**) if $\mathbf{A\bar{B}C\bar{D}}\in CANF(\mathbf{P}'')$ or $\mathbf{\bar{A}BC\bar{D}}\in CANF(\mathbf{P}'')$.

Hence, we have to consider only alternatives $\mathbf{P}$ over $\{\mathbf{ABCD}, \mathbf{ABC\bar{D}}, \mathbf{AB\bar{C}D}, \mathbf{A\bar{B}CD}, \mathbf{\bar{A}BCD}, \mathbf{AB\bar{C}\bar{D}}\}$ and we may assume $a>0$, $b<r$.

Lemma 2.

(i) $n_{a,b}(\mathbf{P}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}),r)=n_{a,b}(\mathbf{P}(\mathbf{B},\mathbf{A},\mathbf{C},\mathbf{D}),r)$,
(ii) $n_{a,b}(\mathbf{P}(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}),r)=n_{r-b,r-a}(\mathbf{P}(\mathbf{A},\mathbf{B},\mathbf{D},\mathbf{C}),r)$,
(iii) $n_{a,b}((\mathbf{A\bar{B}}\vee\mathbf{\bar{A}B})\mathbf{P}'(\mathbf{C},\mathbf{D}),r)=n_{a,b}(\mathbf{A\bar{B}}\mathbf{P}'(\mathbf{C},\mathbf{D}),r)$,
(iv) $n_{a,b}((\mathbf{A}\vee\mathbf{B})\mathbf{P}'(\mathbf{C},\mathbf{D}),r)=n_{a,b}(\mathbf{A}\mathbf{P}'(\mathbf{C},\mathbf{D}),r)$,
(v) $n_{a,b}(\mathbf{P}''\vee\mathbf{A\bar{B}CD}\vee\mathbf{\bar{A}BCD},r)=n_{a,b}(\mathbf{P}''\vee\mathbf{A\bar{B}CD},r)$,
(vi) $n_{a,b}(\mathbf{P}''\vee\mathbf{\bar{A}BCD},r)=n_{a,b}(\mathbf{P}''\vee\mathbf{A\bar{B}CD},r)$,
(vii) $n_{a,b}(\mathbf{P}''',r)=n_{r-b,r-a}(\mathbf{P}''',r)$,
(viii) $n_{a,b}(\mathbf{P}'''\vee\mathbf{AB\bar{C}D},r)=n_{r-b,r-a}(\mathbf{P}'''\vee\mathbf{ABC\bar{D}},r)$,
(ix) $n_{a,b}(\mathbf{P}'''\vee\mathbf{AB\bar{C}D}\vee\mathbf{ABC\bar{D}},r)=n_{r-b,r-a}(\mathbf{P}'''\vee\mathbf{AB\bar{C}D}\vee\mathbf{ABC\bar{D}},r)$,

where $\mathbf{P}'$ is an arbitrary Boolean function in 2 arguments,
$\mathbf{P}''$ is any alternative over $\{\mathbf{ABCD}, \mathbf{AB\bar{C}D}, \mathbf{ABC\bar{D}}, \mathbf{AB\bar{C}\bar{D}}\}$, and
$\mathbf{P}'''$ is any alternative over $\{\mathbf{ABCD}, \mathbf{AB\bar{C}\bar{D}}, \mathbf{A\bar{B}CD}\}$.

4. $n_{a,b}(\mathbf{P},r)$ for the reduced P's

For simplification we use $\mathbf{MN}\vee\mathbf{M\bar{N}}=\mathbf{M}$ and $\mathbf{M}\vee\mathbf{MN}=\mathbf{M}$. Now we consider the three general cases:

1) $a\leqq b<r/2$,
2) $a\leqq r/2\leqq b$,
3) $r/2<a\leqq b$.

The third case may be reduced to the first one using Lemma 2, (ii). If $a \leqq b < r/2$, then obviously no pair (X, Y) of $\mathcal{F}$ can satisfy $\mathbf{AB\bar{C}D}$ or $\mathbf{AB\bar{C}\bar{D}}$, i.e., these two conjunctions may be omitted, or if $CANF(\mathbf{P})$ has only conjunctions of these ones, $n_{a,b}(\mathbf{P}, r) = 1$ follows immediately.

Case 1. $a \leqq b < r/2$. $CANF(\mathbf{P})$ contains only conjunctions of $\{\mathbf{ABCD}, \mathbf{A\bar{B}CD}, \mathbf{ABC\bar{D}}\}$. Thus, only 7 **P**'s are possible ($\mathbf{P} \equiv \mathbf{0}$ was excluded at the beginning).

No.	P	$n_{a,b}(\mathbf{P}, r)$	reference/remark
1.1	$\mathbf{A\bar{B}CD}$	$b-a+1$	$\mathcal{F}$ forms a chain.
1.2	$\mathbf{ABCD}$	$\binom{r-1}{b-1}$	Erdős, Ko, Rado [3] or Greene, Katona, Kleitman [4].
1.3	$\mathbf{ABC\bar{D}}$	$\left[\frac{r}{a}\right]$	The sets of $\mathcal{F}$ have to be disjoint.
1.4	$\mathbf{ACD}$	$\sum_{i=a}^{b} \binom{r-1}{i-1}$	Hilton [7].
1.5	$\mathbf{A\bar{B}CD} \vee$ $\vee \mathbf{ABC\bar{D}}$	$2r-]r/b[$ *if* $a=1$ $r-(a-1)]r/b[$ *if* $a \geqq 2$, $a \leqq r-b(]r/b[-1)$ $(b-a+1)(]r/b[-1)$ *if* $a \geqq 2$, $a > r-b(]r/b[-1)$	see Section 5.
1.6	$\mathbf{ABC}$	$\binom{r}{b}$	Lubell [9], Meshalkin [10], Yamamoto [14].
1.7	$\mathbf{ABC} \vee$ $\vee \mathbf{ACD}$	$\sum_{i=a}^{b} \binom{r}{i}$	Every pair (X, Y), $a \leqq \lvert X \rvert \leqq \lvert Y \rvert \leqq b$, satisfies the condition.

Case 2. $a \leqq r/2 \leqq b$.

Using the statements of Lemma 2 we may reduce all possible conditions to 23 types. More precisely, by Lemma 2 (v) and (vi), we may omit $\mathbf{\bar{A}BCD}$ if $\mathbf{A\bar{B}CD} \in CANF(\mathbf{P})$ or replace $\mathbf{\bar{A}BCD}$ by $\mathbf{A\bar{B}CD}$ if $\mathbf{A\bar{B}CD} \notin CANF(\mathbf{P})$. Furthermore, if $\mathbf{AB\bar{C}D} \in CANF(\mathbf{P})$ and $\mathbf{ABC\bar{D}} \notin CANF(\mathbf{P})$, $\mathbf{AB\bar{C}D}$ can be replaced by $\mathbf{ABC\bar{D}}$ according to Lemma 2 (viii). This procedure is the same as in [5]. We also use this notation. Many results are well-known, others are very simple. But there are some really new problems.

Most of them have been solved. The proofs are given in Sections 5 and 6. Finally some open problems are presented in Section 7.

No.	P	$n_{a,b}(\mathbf{P}, r)$	reference/remark
2.1	**ACD**	?	see Section 7.
2.2	**AB**	$\binom{r}{[r/2]}$	SPERNER [13], LUBELL [9], MESHALKIN [10], or YAMAMOTO [14].
2.3	**ABC**	$\binom{r}{[(r-1)/2]}$	MILNER [11] or GREENE, KATONA, KLEITMAN [4].
2.4	**ABCD**	$\binom{r-1}{[(r-2)/2]}$	KATONA [8], SCHÖNHEIM [12], or GRONAU [6].
2.5	**ABC**∨ ∨**ABD**	$\binom{r-1}{[(r-1)/2]}$	CLEMENTS, GRONAU [1].
2.6	**ABC**∨ ∨**ACD**∨ ∨$\mathbf{AB\bar{D}}$	$\begin{cases} \sum_{i=a}^{r/2} \binom{r}{i} & \text{if } r \text{ is even} \\ \sum_{i=a}^{(r-1)/2} \binom{r}{i} + \binom{r-1}{(r+1)/2} & \text{if } r \text{ is odd} \end{cases}$ if $a \leqq r-b$; ? if $a > r-b$	see Section 6. see Section 7.
2.7	$\mathbf{AB\bar{C}\bar{D}}$	2	clear.
2.8	$\mathbf{A\bar{B}CD}$	$b-a+1$	$\mathscr{F}$ forms a chain.
2.9	$\mathbf{A\bar{B}CD}$∨ ∨$\mathbf{AB\bar{C}\bar{D}}$	$\begin{cases} 2 & \text{if } a = b = r/2 \\ b-a+1 & \text{otherwise} \end{cases}$	It follows by 2.7 and 2.8.
2.10	$\mathbf{AB\bar{D}}$	$[r/a]$	The sets of $\mathscr{F}$ are disjoint.
2.11	$\mathbf{ABC\bar{D}}$	$\begin{cases} [r/a] & \text{if } a \neq r/2 \\ 1 & \text{if } a = r/2 \end{cases}$	
2.12	$\mathbf{AB\bar{C}}$∨ ∨$\mathbf{AB\bar{D}}$	$[r/c]$, $c = \min(a, r-b)$	In [5] it was proved that $\mathscr{F}$ satisfies $\mathbf{AB\bar{C}}$ ($\mathbf{AB\bar{C}D}$) or $\mathscr{F}$ satisfies $\mathbf{AB\bar{D}}$ ($\mathbf{ABC\bar{D}}$). 2.10 resp. 2.11 implies the result.
2.13	$\mathbf{AB\bar{C}D}$∨ ∨$\mathbf{ABC\bar{D}}$	$\begin{cases} 1 & \text{if } a = b = r/2 \\ [r/c], c = \min(a, r-b) & \text{otherwise} \end{cases}$	
2.14	**ABC**∨ ∨**ABD**∨ ∨**ACD**	$\begin{cases} \sum_{i=a}^{r-b-1} \binom{r}{i} + \frac{1}{2} \sum_{i=r-b}^{b} \binom{r}{i} & \text{if } a \leqq r-b \\ \frac{1}{2} \sum_{i=a}^{r-a} \binom{r}{i} + \sum_{i=r-a+1}^{b} \binom{r}{i} & \text{if } a > r-b \end{cases}$	$\mathscr{F}$ contains no set and its complement.

No.	P	$n_{a,b}(\mathbf{P},r)$	reference/remark
2.15	$\mathbf{ABC}\vee$ $\vee\mathbf{ACD}$	$\begin{cases} \sum_{i=a}^{[\frac{r-1}{2}]} \binom{r}{i} + \begin{cases} 0 & \text{if } r \text{ is odd} \\ \binom{r-1}{r/2-1} & \text{if } r \text{ is even} \end{cases} & \text{if } a \leqq r-b \\ \sum_{i=a}^{b} \binom{r-1}{i} & \text{if } a > r-b \end{cases}$	see Section 6. HILTON [7].
2.16	$\mathbf{AB}\vee$ $\vee\mathbf{ACD}$	$\sum_{i=a}^{b} \binom{r}{i}$	Every pair satisfies this condition.
2.17	$\mathbf{ABC}\vee$ $\vee\mathbf{AB\bar{D}}$	$\binom{r}{[r/2]}$	$\mathscr{F}$ satisfies **AB** too. Indeed, $\mathscr{F}=\{X\colon X\subseteqq R, \lvert X\rvert=[r/2]\}$ has that cardinality.
2.18	$\mathbf{ABCD}\vee$ $\vee\mathbf{AB\bar{C}\bar{D}}$	$2\binom{r-1}{[(r-2)/2]}$	Omitting complements $\mathscr{F}$ satisfies **ABCD** (see 2.4). $\mathscr{F}=\{X\colon X\subseteqq R, v\in X, \lvert X\rvert=[r/2]\}\cup\{X\colon X\subseteqq R, v\notin X, \lvert X\rvert=\{[(r+1)/2]\}$, where $v\in R$ is fixed, has the desired cardinality.
2.19	$\mathbf{A\bar{B}CD}\vee$ $\vee\mathbf{AB\bar{D}}$	$2r-2$ if $a=1$ $r-2a+2$ if $2\leqq a\leqq r-b$ $b-a+1$ if $2\leqq a>r-b$	see Section 5.
2.20	$\mathbf{A\bar{B}CD}\vee$ $\vee\mathbf{ABC\bar{D}}$	$2r-3$ if $a=1$ $r-2a+1$ if $2\leqq a<r-b$ $b-a+1$ if $2\leqq a\geqq r-b$	see Section 5.
2.21	$\mathbf{A\bar{B}CD}\vee$ $\vee\mathbf{AB\bar{C}D}\vee$ $\vee\mathbf{ABC\bar{D}}$	$2r-3$ if $a=1$ or $b=r-1$ $r-2a+1$ if $a\geqq 2$, $b\leqq r-2$, $a\leqq r-b$ $2b-r+1$ if $a\geqq 2$, $b\leqq r-2$, $a\geqq r-b$	see Section 5.
2.22	$\mathbf{A\bar{B}CD}\vee$ $\vee\mathbf{AB\bar{C}}\vee$ $\vee\mathbf{AB\bar{D}}$	$4r-6$ if $a=1, b=r-1$ $r+2b-2$ if $a=1, b<r-1$ $3r-2a-2$ if $a>1, b=r-1$ $2(b-a+1)$ if $a>1, b<r-1$	see Section 5.
2.23	$\mathbf{ACD}\vee$ $\vee\mathbf{AB\bar{C}\bar{D}}$	?	see Section 7.

5. Proofs of 1.5, 2.19, 2.20, 2.21, 2.22

In order to give examples of maximal families we use the following notations

$$\mathscr{D}_1=\{X\colon X=\{t\},\ 1\leqq t\leqq r\},$$
$$\mathscr{D}_2=\{X\colon R\setminus X\in\mathscr{D}_1\},$$
$$\mathscr{D}_3(p,q,s)=\{X\colon X=\{p+1,p+2,\ldots,p+t\},\ q\leqq t\leqq s,\ p+t\leqq r\},$$
$$\mathscr{D}_4(p,q)=\{X\colon X=\{t+1,t+2,\ldots,r\},\ t=p,p-1,p-2,\ldots,q\}.$$

For all these conditions we have

$$n_{1,b}(\mathbf{P}, r) \leqq n_{2,b}(\mathbf{P}, r)+r. \tag{2}$$

Thus, $n_{a,b}(\mathbf{P}, r)$, $a \geqq 2$, implies an upper bound for $n_{1,b}(\mathbf{P}, r)$.

5.1. Let $\mathbf{P}=\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{D}}$ (2.19, 1.5).

Denote by $\mathscr{F}$ a maximal $(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{D}})$-family with $a \leqq |X| \leqq b$ for all $X \in \mathscr{F}$. Then for every pair (X, Y) of $\mathscr{F}$ we have $X \subset Y$ or $X \cap Y=\emptyset$. Hence, there is a unique subfamily $\mathscr{G}(\mathscr{F}) \subseteqq \mathscr{F}$ satisfying

— $X \cap Y=\emptyset$ for all pairs (X, Y) with $X, Y \in \mathscr{G}(\mathscr{F})$,
— for all $X \in \mathscr{F} \setminus \mathscr{G}(\mathscr{F})$ there is an element $Y \in \mathscr{G}(\mathscr{F})$ with $X \subset Y$.

If $X \in \mathscr{G}(\mathscr{F})$, then $\mathscr{H}(X)=\{Y: Y \in \mathscr{F}, Y \subset X\}$. Thus,

— $\mathscr{F}=\mathscr{G}(\mathscr{F}) \cup \bigcup_{X \in \mathscr{G}(\mathscr{F})} \mathscr{H}(X)$,
— $\mathscr{H}(X)$ satisfies $\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{D}}$, and
— all $Y \in \mathscr{H}(X)$ satisfy $a \leqq |Y| \leqq |X|-1$.

Then

$$|\mathscr{F}| \leqq |\mathscr{G}(\mathscr{F})|+\sum_{X \in \mathscr{G}(\mathscr{F})} n_{a,|X|-1}(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{D}}, |X|). \tag{3}$$

Now we prove

Lemma 3. $n_{a,r-1}(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{D}}, r)=r-a$ *for* $3 \leqq a+1 \leqq r$.

Proof. The proof is given by induction on r for arbitrary, but fixed a, $r \geqq a+1$.
1. $r=a+1$. The statement is true, clearly.
2. We obtain for every maximal family $\mathscr{F}$, by (3),

$$|\mathscr{F}| \leqq |\mathscr{G}(\mathscr{F})|+\sum_{X \in \mathscr{G}(\mathscr{F})} (|X|-a).$$

If $|\mathscr{G}(\mathscr{F})| = 1$, then $\sum_{X \in \mathscr{G}(\mathscr{F})} |X| \leqq r-1$ and $|\mathscr{F}| \leqq r-a$.

If $|\mathscr{G}(\mathscr{F})| \geqq 2$, then $\sum_{X \in \mathscr{G}(\mathscr{F})} |X| \leqq r$ and

$$|\mathscr{F}| \leqq r-(a-1)|\mathscr{G}(\mathscr{F})| \leqq r-2(a-1) \leqq r-a.$$

Indeed, $\mathscr{D}_3(0, a, r-1)$ is a $(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{D}})$-family with the desired cardinality. □

Now we return to the general a, b-case.
Lemma 3 and (3) yield for a maximal $(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{D}})$-family $\mathscr{F}$

$$|\mathscr{F}| = \sum_{X \in \mathscr{G}(\mathscr{F})} |X|-(a-1)|\mathscr{G}(\mathscr{F})|.$$

If $|\mathscr{G}(\mathscr{F})| \leqq]r/b[-1$, then $|X| \leqq b$ implies

$$|\mathscr{F}| \leqq (b-a+1)|\mathscr{G}(\mathscr{F})| \leqq (b-a+1)(]r/b[-1). \tag{4}$$

If $|\mathscr{G}(\mathscr{F})| \geqq]r, b[$, then $\sum_{X \in \mathscr{G}(\mathscr{F})} |X| = \left|\bigcup_{X \in \mathscr{G}(\mathscr{F})} X\right| \leqq r$

and

$$|\mathscr{F}| \leqq r-(a-1)|\mathscr{G}(\mathscr{F})| \leqq r-(a-1)]r/b[. \tag{5}$$

Simple verification shows that the upper estimation of (4) is not smaller than that one of (5) iff

$$a > r-b(]r/b[-1).$$

Indeed, $\mathscr{D}_5(a,b)=\bigcup_{t=0,1,\dots}\mathscr{D}_3(bt,a,b)$ confirms in both cases that these upper bounds are the desired results. Thus, 1.5 is proven if $a\geqq 2$ remarking that **C** is always satisfied. The case $a=1$ follows by (2) and the example $\mathscr{D}_1\cup\mathscr{D}_5(1,b)$. Moreover, 2.19 is proven, note $]r/b[=2$ for $b\geqq r/2$. Also here the case $a=1$ follows by (2) and the example $\mathscr{D}_1\cup\mathscr{D}_5(1,b)$.

5.2. Let $\mathbf{P}=\mathbf{A\bar{B}CD}\vee\mathbf{ABC\bar{D}}$ (2.20).

In analogy to the preceding case a special subfamily $\mathscr{G}(\mathscr{F})$ exists and we obtain for a maximal family $\mathscr{F}$

$$|\mathscr{F}|\leqq|\mathscr{G}(\mathscr{F})|+\sum_{X\in\mathscr{G}(\mathscr{F})}n_{a,|X|-1}(\mathbf{A\bar{B}CD}\vee\mathbf{AB\bar{D}},|X|). \tag{6}$$

We remark that $\mathscr{H}(X)$ satisfies $\mathbf{A\bar{B}CD}\vee\mathbf{AB\bar{D}}$, not necessarily $\mathbf{A\bar{B}CD}\vee\mathbf{ABC\bar{D}}$.

Lemma 3 implies

$$|\mathscr{F}|\leqq\sum_{X\in\mathscr{G}(\mathscr{F})}|X|-(a-1)|\mathscr{G}(\mathscr{F})|.$$

If $|\mathscr{G}(\mathscr{F})|=1$, then $\sum_{X\in\mathscr{G}(\mathscr{F})}|X|\leqq b$ and $|\mathscr{F}|\leqq b-a+1$.

If $|\mathscr{G}(\mathscr{F})|=2$, then $\sum_{X\in\mathscr{G}(\mathscr{F})}|X|\leqq r-1$ (since $\mathscr{G}(\mathscr{F})$ contains no complementary sets) and $|\mathscr{F}|\leqq r-1-2(a-1)=r-2a+1$.

If $|\mathscr{G}(\mathscr{F})|\geqq 3$, then $\sum_{X\in\mathscr{G}(\mathscr{F})}|X|\leqq r$ and

$$|\mathscr{F}|\leqq r-3(a-1)\leqq r-2a+1.$$

Hence,

$$|\mathscr{F}|\leqq\max(b-a+1,r-2a+1)=\begin{cases}b-a+1 & \text{if } a\geqq r-b,\\ r-2a+1 & \text{if } a<r-b.\end{cases}$$

Indeed,

$$\mathscr{D}_6(a,b)=\begin{cases}\mathscr{D}_3(0,a,b) & \text{if } a\geqq 2, a\geqq r-b,\\ \mathscr{D}_3(0,a,b)\cup\mathscr{D}_3(b,a,r-b-1) & \text{if } a\geqq 2, a<r-b\end{cases}$$

is an example which confirms that the upper bound is the desired result for $a\geqq 2$. (2) and $\mathscr{D}_1\cup\mathscr{D}_6(1,b)$ yield the result for $a=1$.

5.3. Let $\mathbf{P}=\mathbf{A\bar{B}CD}\vee\mathbf{AB\bar{C}D}\vee\mathbf{ABC\bar{D}}$ (2.21).

5.3.1. $a\leqq r-b$.

If $\mathscr{F}$ is a maximal family, consider

$$\mathscr{F}'=\left\{X:\begin{cases}X & \text{if } X\in\mathscr{F}, |X|<r/2,\\ X & \text{if } X\in\mathscr{F}, |X|=r/2, 1\notin X,\\ R\setminus X & \text{if } X\in\mathscr{F}, |X|=r/2, 1\in X,\\ R\setminus X & \text{if } X\in\mathscr{F}, |X|>r/2.\end{cases}\right\}.$$

Since $\mathscr{F}$ contains no complementary sets, $|\mathscr{F}'|=|\mathscr{F}|=n_{a,b}(\mathbf{P}, r)$. Obviously, $\mathscr{F}'$ satisfies $\mathbf{\bar{A}BCD} \vee \mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}D} \vee \mathbf{ABC\bar{D}}$. No pair of $\mathscr{F}'$ satisfies $\mathbf{AB\bar{C}D}$. By Lemma 2 (v), we have now that $\mathscr{F}'$ satisfies $\mathbf{A\bar{B}CD} \vee \mathbf{ABC\bar{D}}$. Thus a maximal family of 2.20 is also a maximal family here. Hence, 2.21 follows by 2.20 if $a \leqq r-b$, $a=1$ or $a \geqq 2$.

5.3.2. $a>r-b$. Then $r-b<r-(r-a)$.

We apply Lemma 2 (ix) and the results of 5.3.1, and get

$$n_{a,b}(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}D} \vee \mathbf{ABC\bar{D}}, r) =$$

$$= n_{r-b,r-a}(\mathbf{A\bar{B}CD} \vee \mathbf{ABC\bar{D}}, r) = \begin{cases} 2r-3 & \text{if} \quad r-b = 1, \\ r-2(r-b)+1 & \text{if} \quad r-b \geqq 2. \end{cases}$$

5.4. Let $\mathbf{P} = \mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}} \vee \mathbf{AB\bar{D}}$ (2.22).

If $\mathscr{F}$ is a maximal family, we split $\mathscr{F}$ into two subfamilies $\mathscr{F}_1$ and $\mathscr{F}_2$ by

if $X \in \mathscr{F}$, $R \setminus X \notin \mathscr{F}$, then $X \in \mathscr{F}_1$,

if $X \in \mathscr{F}$, $R \setminus X \in \mathscr{F}$, then $X \in \mathscr{F}_1$, $R \setminus X \in \mathscr{F}_2$ or $X \in \mathscr{F}_2$, $R \setminus X \in \mathscr{F}_1$.

Then $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively, satisfy $\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}D} \vee \mathbf{ABC\bar{D}}$. Since $X \in \mathscr{F}$, $R \setminus X \in \mathscr{F}$ can hold only if $c \leqq |X| \leqq r-c$, $c=\max(a, r-b)$. We obtain immediately

$$|\mathscr{F}_1| \leqq n_{a,b}(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}D} \vee \mathbf{ABC\bar{D}}, r)$$

and

$$|\mathscr{F}_2| \leqq n_{c,r-c}(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}D} \vee \mathbf{ABC\bar{D}}, r).$$

Hence,

$$|\mathscr{F}| \leqq \begin{cases} r-2a+1+r-2(r-b)+1 & \text{if} \quad a \leqq r-b, \\ 2b-r+1+r-2a+1 & \text{if} \quad a > r-b, \end{cases}$$

i.e.

$$|\mathscr{F}| \leqq 2(b-a+1).$$

Similarly, it follows by (2)

$$|\mathscr{F}| \leqq \begin{cases} r+2b-2 & \text{if} \quad a = 1, \quad b \leqq r-2, \\ 3r-2a-2 & \text{if} \quad a \geqq 2, \quad b = r-1, \\ 4r-6 & \text{if} \quad a = 1, \quad b = r-1. \end{cases}$$

Finally, we complete the proof by following examples

$\mathscr{D}_7(a, b) = \mathscr{D}_3(0, a, b) \cup \mathscr{D}_4(r-a, r-b)$	*if* $a \geqq 2$, $b \leqq r-2$,
$\mathscr{D}_1 \cup \mathscr{D}_7(1, b)$	*if* $a=1$, $b \leqq r-2$,
$\mathscr{D}_7(a, r-1) \cup \mathscr{D}_2$	*if* $a \geqq 2$, $b=r-1$,
$\mathscr{D}_1 \cup \mathscr{D}_7(1, r-1) \cup \mathscr{D}_2$	*if* $a=1$, $b=r-1$.

Remark. A family satisfying $\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}} \vee \mathbf{AB\bar{D}}$ may be interpreted as a family without qualitatively independent sets (see also KATONA [8]).

6. Proofs of 2.6 and 2.15

6.1. Let $\mathbf{P}=\mathbf{ABC}\vee\mathbf{ACD}$ (2.15) and let $a\leqq r-b$.

Let $\mathcal{F}$ be an arbitrary maximal family. Then $\mathcal{F}$ contains no complementary sets, i.e.

$$|\mathcal{F}|\leqq \sum_{i=a}^{r-b-1}\binom{r}{i}+\frac{1}{2}\sum_{i=r-b}^{b}\binom{r}{i}.$$

An example for a maximal family is

$$\{X\colon X\subseteqq R,\ a\leqq |X|<r/2\quad or\quad (|X|=r/2\quad and\quad 1\notin X)\}.$$

6.2. Let $\mathbf{P}=\mathbf{ABC}\vee\mathbf{ACD}\vee\mathbf{AB\bar{D}}$ (2.6) and let $a\leqq r-b$.

If $\mathcal{F}$ is a maximal family, then we split $\mathcal{F}$ into two subfamilies $\mathcal{F}_1$ and $\mathcal{F}_2$ by the same procedure as in Section 5, 4. Thus, $\mathcal{F}_1$ satisfies $\mathbf{ABC}\vee\mathbf{ACD}$, i.e.

$$|\mathcal{F}_1|\leqq n_{a,b}(\mathbf{ABC}\vee\mathbf{ACD},r).$$

$\mathcal{F}_2$ satisfies $\mathbf{ABC}\vee\mathbf{ACD}$. Moreover, for arbitrary sets $X, Y\in\mathcal{F}_2$ also $(R\setminus X, Y)$, $(X, R\setminus Y)$, and $(R\setminus X, R\setminus Y)$ satisfy $\mathbf{ABC}\vee\mathbf{ACD}=\mathbf{ABCD}\vee\mathbf{A\bar{B}CD}\vee\mathbf{ABC\bar{D}}$. Hence, (X, Y) satisfies $\mathbf{ABCD}\vee\mathbf{A\bar{B}CD}\vee\mathbf{ABC\bar{D}}$ as well as $\mathbf{ABCD}\vee\mathbf{\bar{A}BCD}\vee\mathbf{AB\bar{C}D}$, i.e. $\mathbf{ABCD}$. 2.15 implies

$$n_{a,b}(\mathbf{ABC}\vee\mathbf{ACD}\vee\mathbf{AB\bar{D}},r)=\sum_{i=a}^{\left[\frac{r-1}{2}\right]}\binom{r}{i}+\binom{r-1}{[r/2]-1}+\begin{cases}0 & \text{if } r \text{ is odd,}\\ \binom{r-1}{r/2} & \text{if } r \text{ is even.}\end{cases}$$

Indeed, $\{X\colon X\subseteqq R,\ a\leqq |X|\leqq r/2$ *and, if* r *is odd,* $|X|=[r/2]+1,\ 1\notin X\}$ is a maximal family.

7. Open problems

In this section we give explicitely the open problems in usual notation. Also some estimations are presented.

1. *Problem* (2.1) $n_{a,b}(\mathbf{ACD},r)=?$

Remember that $\mathbf{ACD}$ means $(X\cap Y\neq\emptyset)\wedge(X\cup Y\neq R)$ for all $X, Y\in\mathcal{F}$. It is known only that

$$\sum_{i=a}^{b}\binom{r-2}{i-1}\leqq n_{a,b}(\mathbf{ACD},r)\leqq\begin{cases}\sum_{i=a}^{b}\binom{r-1}{i} & \text{if}\quad a\leqq r-b,\\ \sum_{i=a}^{b}\binom{r-1}{i-1} & \text{if}\quad a>r-b\end{cases}$$

by 2.15 and Lemma 2 (viii).

Equality occurs, for example, in the left hand side if $a=r-b=1$, and in the right hand side if $a=b=r/2$.

2. *Problem* (2.6, $a>r-b$) $n_{a,b}(\mathbf{ABC}\vee\mathbf{ACD}\vee\mathbf{AB\bar{D}},r)=?$ *if* $a>r-b$.

Remember that this condition means that $\mathcal{F}$ contains no not-complementary sets with union R. The investigations in the case $a \leqq r-b$ yield immediately

$$n_{a,b}(\mathbf{ABC} \vee \mathbf{ACD}, r) \leqq n_{a,b}(\mathbf{ABC} \vee \mathbf{ACD} \vee \mathbf{AB\overline{D}}, r) \leqq n_{a,b}(\mathbf{ABC} \vee \mathbf{ACD}, r) + \binom{r-1}{[r/2]-1}.$$

3. *Problem* (2.23) $n_{a,b}(\mathbf{ABC} \vee \mathbf{AB\overline{C}\overline{D}}, r) = ?$

In this case also $n_{1,r-1}(\mathbf{P}, r)$ is unknown. Bounds are in analogy to [5] given by

$$n_{a,b}(\mathbf{ACD}, r) \leqq n_{a,b}(\mathbf{ACD} \vee \mathbf{AB\overline{C}\overline{D}}, r) \leqq n_{a,b}(\mathbf{ACD}, r) + \binom{r-1}{[r/2]-1}.$$

WILHELM-PIECK-UNIVERSITÄT
SEKTION MATHEMATIK
DDR—25 ROSTOCK
UNIVERSITÄTSPLATZ 1

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(Received June 11, 1980)