A note on the interconnection structure of cellular networks

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Using the concept of the structure automaton it is proved that every cellular automaton may be simulated by a cellular automaton realized by a cellular network of semigroup-type.

1. Introduction

The interconnection structures of infinite cellular automata, called also tesselation automata [3], are usually taken to be networks based on direct sum of infinite cyclic groups. Such networks have a great degree of uniformity [4]. Realizations of finite and infinite cellular automata by various types of uniform networks were described in [2]. It was shown there that such realizations may be described by the use of the theory of groups. The structure of cellular automata realized by nonuniform cellular networks has not been investigated because of the lack of the proper description method for such networks. In this paper a step in this direction is presented using the concept of the structure automaton. It is proved that every cellular automaton may be simulated by a cellular automaton realized by a cellular network of semigroup-type.

2. Preliminaries

Definition. *A cellular network* is a system $\mathcal{N}=(C, S, \delta, f)$ where C — is a count able *set of cells,* S — is a finite set of *cell-states,* $\delta\colon S^k \to S$ — is a *cell transition function,* $f\colon C \to C^k$ — is the *neighbourhood function.*

Definition. *The cellular automaton* (CA) realized by a cellular network $\mathcal{N}$ is a pair $\mathcal{A}(\mathcal{N})=(S^C, F)$ where $S^C=\{h|h\colon C\to S\}$ — is *the set of* CA *configurations* $F\colon S^C \to S^C$ is the *global map* defined by $\forall_{c\in C}\ F(h(c))=\delta\cdot h^k\cdot f(c)$ where $h^k(c_1, c_2, \ldots, c_k)=\big(h(c_1), h(c_2), \ldots, h(c_k)\big)$.

Let $\mathcal{N}_1=(C_1, S_1, \delta_1, f_1)$ and $\mathcal{N}_2=(C_2, S_2, \delta_2, f_2)$ be two cellular networks with $f_1\colon C_1 \to C_1^k$ and $f_2\colon C_2 \to C_2^k$. A network $\mathcal{N}_1$ is a *realization* of the network $\mathcal{N}_2$ when there exists a pair of functions (φ, ψ), $\varphi\colon C_1 \to C_2$, $\varphi(C_1)=C_2$, $\psi\colon S_1 \to S_2$ such that $\forall_{c_1\in C_1}\ \varphi^k(f_1(c_1))=f_2(\varphi(c_1))$ and $\forall_{s_1, s_2, \ldots s_k\in S_1}\ \psi\cdot\delta_1(s_1, s_2, \ldots, s_k)=\delta_2(\psi(s_1), \psi(s_2), \ldots, \psi(s_k))$. These equalities mean that (φ, ψ) is a *homomorphism* of $\mathcal{N}_1$ onto $\mathcal{N}_2$.

In [2] the following theorem was proved.

Theorem 1. *If a cellular network $\mathcal{N}_1$ realizes the cellular network $\mathcal{N}_2$ then the cellular automaton $\mathcal{A}(\mathcal{N}_1)$ simulates the cellular automaton $\mathcal{A}(\mathcal{N}_2)$ i.e. a function H from $S_1^{C_1}$ onto $S_2^{C_2}$ exists such that*

$$\forall_{s^c \in S_1^{C_1}} H(F_1(s^c)) = F_2(H(s^c)).$$

Simulations of CA realized by various types of cellular networks having a high degree of uniformity were described in [2]. It was shown there that the simulation of CA realized by such networks is essentially a problem of the group homomorphism and, in some cases, a problem of permutation groups generators.

No attempts were reported on the simulation of CA realized by nonuniform networks. Particularly interesting question is whether there is an algebraic structure for the description similarily as the theory of groups in the case of uniform networks. The answer for this question is given here. It states that simulations of all CA may be described by the use of the theory of semigroups.

3. Results

Let $\mathcal{N}=(C, S, \delta, f)$ be a cellular network defined as above. For notational convenience we label cell inputs $1, 2, \ldots, k$ of the cells in $\mathcal{N}$ by $x_1, x_2, \ldots, x_k$.

Definition. *A structure automaton* of the cellular network $\mathcal{N}$ is a triple $\mathcal{N}_A = (X, C, \omega)$ where X — is the *imput alphabet of cell input labels, C* — is a countable *set of cells of* $\mathcal{N}$ $\omega\colon X\times C \to C$ is a *transition function* defined by $\forall_{x_i \in X} \forall_{c_k \in C} \omega(x_i, c_k) = c_l \Leftrightarrow$ the i-th component of the neighbourhood function value $f(c_l)$ is equal to c_k.

It is easy to see that every cellular network may be described by some structure automaton. Classical results [1] obtained in the theory of autcmata may be now applied to the description of cellular networks. For example we can generalize the classification of networks as follows: (For the notions below [1] may be consulted).

1. *Connected networks* described by connected structure automata.
2. *Strongly connected networks* described by strongly connected structure automata.
3. *Balanced networks* [2] described by connected permutation automata.
4. *Uniform networks* [2] described by quasi-perfect automata.
5. *Arrays* [2] described by perfect automata.

The enumeration above is done according to the generality of specific class of networks. The first two classes are important from the point of information flow in CA. In the cellular network described by the connected structure automaton there are some parts from (or to) which information flows in only one direction, and there are no such parts in the cellular network described by the strongly connected structure automaton.

Definition. *A cellular network* $\mathcal{N}$ is of *semigroup-type* if there is ono-to-one correspondence α between the set of cells C and certain semigroup J with operation

$*$ such that for some subset $L=\{l_1, l_2, \ldots, l_k\}\subset J$,

$$\underset{c\in C}{\forall} \alpha^k(f(c)) = (l_1 * \alpha(c), l_2 * \alpha(c), \ldots, l_k * \alpha(c)).$$

Theorem 2. If $\mathcal{N}$ is a cellular network described by the strongly connected structure automaton $\mathcal{N}_A$ then there exist a cellular network $\mathcal{M}$ of semigroup-type such that the CA $\mathcal{A}(\mathcal{M})$ simulates the CA $\mathcal{A}(\mathcal{N})$.

Proof. By Theorem 1, it is sufficient to consider cellular network realizations. Let $\mathcal{N}_A=(X, C, \omega)$. With every imput symbol x_i we can associate a transformation $\omega_{x_i}: C\to C$ by taking $\omega(x_i, c)$ for all $c\in C$. Let J be the transformation semigroup generated by all ω_{x_i} for $x_i\in X$.

We define the following structure automaton $\mathcal{M}_J=(\Omega, J, m)$, where $\Omega=\{\omega_{x_i}|x_i\in X\}$ — input alphabet, $m(\omega_{x_i}, j)=\omega_{x_i}\cdot j$ — transition function defined as a composition of mappings in J.

Let $H: J\to C$ be a function defined by $\underset{j\in J}{\forall} H(j)=j(c_0)$ for some fixed $c_0\in C$. H is onto C because the semigroup J is transitive. We shall prove that H is a homomorphism of the structure automaton $\mathcal{M}_J(\Omega, J, m)$ onto the structure automaton $\mathcal{N}_A=(X, C, \omega)$. We have

$$\underset{\omega_{x_i}\in\Omega}{\forall}\ \underset{j\in J}{\forall} H(m(\omega_{x_i}, j)) = H(\omega_{x_i}\cdot j) = \omega_{x_i}\cdot j(c_0) = \omega(x_i, H(j)).$$

From Theorem 1 it follows that the CA realized by the semigroup-type network $\mathcal{M}$ described by the structure automaton $\mathcal{M}_J$ simulates the CA realized by the network $\mathcal{N}$. □

Now, we will extend Theorem 2 to cellular networks described by connected structure automata. In this case the transformation semigroup J is not transitive.

An *extension of the semigroup* J will be defined in two steps. First, when there is no identity, we add an identity e to the semigroup J obtaining the semigroup $J\cup e=J_e$. Let $C_g\subset C$ be the set (possibly with the smallest cardinality) such that

$$J_e(C_g) = C.$$

In the second step, let the elements of C_g be numbered $c_1, c_2, \ldots, c_i, \ldots$. For each element $c_i\in C_g$ a set of vectors is constructed

$$[J_e]_i = \{[0, 0, \ldots, 0, j_e, 0, \ldots]\} \quad \text{for all} \quad j_e\in J$$

where j_e is on the i-th position and 0 is an element such that $0\cdot 0=0\cdot j_e=j_e\cdot 0=0$.

Let $[J_e]=\bigcup_i [J_e]_i$. It is easy to see that the set $[J_e]$ together with componentwise multiplication forms a semigroup.

Let H_e be a function $H_e: J_e\to C$ such that for each $c_i\in C_g$, $H([J_e]_i)$ is defined as

$$H([0, 0, \ldots, 0, j_e, 0, \ldots]) = j(c_i).$$

H is onto C because of the definition of the set C_g and vector semigroup $[J_e]$.

From these constructions we finally have

Theorem 3. Any cellular automaton may be simulated by a cellular automaton realized by a cellular network of semigroup-type.

4. Conclusion

It was proved that semigroupe-type cellular networks are universal in the sense that any cellular automaton may be simulated by a network of such type. This result may compared with the simulation power of the group-type and Abelian group-type networks [2]. Note, that in the case of connected networks with infinite number of cells the semigroup for simulation may be not finitely generated, which gives a new level of complexity in the theory of cellular automata. Further investigation is needed in two directions. First, on the computational capability of the cellular automata realized by semigroup-type networks comparing to tesselation automata. Second, in the finite case, on the algebraic characterization of the structure of finite cellular automata using finite structure automata.

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