The early bird problem is unsolvable in a one-dimensional cellular space with 4 states

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Legendi and Katona (1981) have shown that the early bird problem in a one-dimensional space is solvable with 5 states. The proof is based on a sophisticated concept of waves introduced by Vollmar. We will show that 5 states is a sharp bound for solvability.

1. Early bird problem

Vollmar (1977) defined the problem for a one-dimensional cellular space allowing more than one cell to be excited at a given time step. Only quiescent cells may be excited. Before the first time step at least one cell should be excited. After a certain period the first birds should be in a distinguished state while all the others in a different state.

2. Unsolvability with 4 states

Theorem. The early bird problem is unsolvable in a one-dimensional cellular space with 4 states.

Proof. Assume: There exists a four-state solution, say with a set of states $\{0, B, 2, 3\}$, where

O = initial state

B = bird state (arises only from state 0, spontaneously). Then there is a set of transitions — called A — solving the problem. After a certain period the first bird(s) should be in a distinguished state. The initial state 0 cannot be the distinguished state, because the space is unbounded and after a finite number of steps we obtain a finite configuration.

Case a: B is the distinguished state. There are no transitions $0B0 \to i$, $0BB \to i$, $BB0 \to i$, $BBB \to i$ $(i=0, 2, 3)$ in A, since a bird B cannot be generated by transitions. The set of transitions A must contain a transition $B0B \to 2$ or $B0B \to 3$, otherwise the initial configurations

$K_1 = \ldots 0BBBB\underset{\sim}{0}BBBB0\ldots$ and

$K_1' = \ldots 0BBBBBBBBB0\ldots$, where for K_1

a later bird (second step) occurs at the cell marked by $\sim$ would imply the same configuration sequence (after 3 steps). Without loss of generality we assume $B0B \to 2$ belongs to A. Then A contains no transition of the below defined set of transitions D, because a first bird would be killed.

$$D := \{BB2 \to i,\ 2BB \to i,\ 2B2 \to i,\ BB0 \to i,\ 0BB \to i,\ 0B0 \to i,\ BBB \to i \quad (i=0, 2, 3)\}.$$

Now it is investigated a case distinction. Let

$$L_2 := B00 \to 2, \quad R_2 := 00B \to 2,$$

$$L_3 := B00 \to 3, \quad R_3 := 00B \to 3.$$

Case 1: $L_2, R_2 \in A$

Let $K_1 = \ldots 0B000B0\ldots$ be an initial configuration. Then we obtain after one step $K_2 = \ldots 02B202B20\ldots$. In case of birth of a bird we have $K_2^* = \ldots 02B2B2B20\ldots$. Furthermore let $K_1' = \ldots 00B0B0B0\ldots$ be another initial configuration, then we obtain after one step $K_2' = \ldots 02B2B2B20\ldots$. We see that $K_2' = K_2^*$. This shows that a later bird survives. This is a contradiction.

Case 2: $L_3, R_2 \in A$

Let the initial configuration $K_1 = \ldots 0B0B00BB0\ldots$ be given. Then we get after one step $K_2 = \ldots 02B2B32BB30\ldots$. Thus we see that A does not contain the transitions

$2B2 \to i$
$2B3 \to i$ $(i=0, 2, 3)$ (otherwise a first bird is killed).
$BB3 \to i$

Now let (later birth of birds)

$$K_2' = \ldots 0B0B00B00\ldots 02B2B32BB30\ldots$$

then we obtain

$$K_3' = \ldots 02B2B32BB30\ldots i_1 \ldots i_{12} 0\ldots$$

for some $i_j \in \{0, B, 2, 3\}$. Eliminating these later birds is only possible from the right side. Since $BB3 \to i$, $BB2 \to i$, $BB0 \to i$ $(i=0, 2, 3)$ do not belong to A (see above and set D), the later birds cannot be killed. This is a contradiction.

Case 3: $L_2, R_3 \in A$ (analogue to case 2, symmetry)

Case 4: $L_3, R_3 \in A$

Let $K_1 = \ldots 0BB00B0B000B0\ldots$ be an initial configuration, then we get (after one step) $K_2 = \ldots 03BB33B2B303B30\ldots$. Thus we see that A does not contain the transitions

$3BB \to i$
$BB3 \to i$
$3B2 \to i$ $(i=0, 2, 3)$ (otherwise a first bird is killed).
$2B3 \to i$
$3B3 \to i$

Since $BB0 \leftarrow i$, $BB2 \rightarrow i$ $(i=0, 2, 3) \notin A$ (see set D) and above we have seen $BB3 \rightarrow i \notin A$ two later birds — left from the first birds — survive. This is a contradiction.

Case 5: $L_2 \in A$, R_2, $R_3 \notin A$

Let the initial configuration be given

$K_1 = ...0B00B0...$, then we obtain (after one step)
$K_2 = ...00B20B20....$ Now let (birth of bird)
$K_2' = ...0B2BB20...$ and
$K_1^* = ...0B0BB0...$ another initial configuration.

Then we get after one step $K_2^* = ...0B2BB20....$ Since $K_2^* = K_2'$ the later bird survives. This is a contradiction.

Case 6: $R_2 \in A$, L_2, $L_3 \notin A$ (analogue to case 5, symmetry)

Case 7: $L_3 \in A$, R_2, $R_3 \notin A$

Let the initial configuration $K_1 = ...0BBB0...$ be given, then after one step we get $K_2 = ...0BBB30$. Thus we see that $BB3 \rightarrow i \notin A$ $(i=0, 2, 3)$. Since $BB2 \rightarrow i$, $BB0 \rightarrow i \in D$ and therefore not in A, later birds far enough left from the first birds survive. This is a contradiction.

For example:

$K_2' := 0...0\underset{\sim}{B}\underset{\sim}{B}\underset{\sim}{B}0...0...0BBB30...$ ($\sim$birth of birds)

then we get

$K_3' := 0...0BBB30...0i_1...i_{10}0...$ for some $i_j \in \{0, B, 2, 3\}$

Case 8: $R_3 \in A$, L_2, $L_3 \notin A$ (analogue to 7, symmetry)

Case 9: L_2, L_3, R_3, $R_3 \notin A$

Let $K_1 = ...0B00...$ be an initial configuration, then no transition is applicable to K_1. In case of birth of a bird $K_1' = ...0B000B00$, again we cannot apply a transition to K_1'. This is a contradiction, because K_1 and K_1' have the same configuration sequence. Altogether we have shown that the early bird problem is unsolvable with 4 states, where B is the distinguished state..

Next we will consider the distinguished states 2 or 3. Without loss of generality we assume

Case b: 2 is the distinguished state.

Before starting with a case distinction we will prove

Proposition 1. If the set of transitions A solves the problem, then

a) $\exists i_1, i_2 \in \{0, 3\}$: $(220 \rightarrow i_1 \in A,\ 022 \rightarrow i_2 \in A)$ and

$\forall i=0,2$: $(322 \rightarrow i \notin A,\ 223 \rightarrow i \notin A)$ or

b) $\exists i_1, i_2 \in \{0, 3\}$: $(223 \rightarrow i_1 \in A,\ 322 \rightarrow i_2 \in A)$ and

$\forall i=0,3$: $(022 \rightarrow i \notin A,\ 220 \rightarrow i \notin A)$.

Proof. Let us begin with an initial configuration $K_1 = \ldots 0BB0BB0BB0\ldots$ without later birds. After a finite number of steps we obtain a configuration.

$$K_n = \ldots i_2 i_1\, 22\, l\, 22\, m\, 22\, j_1 j_2 \ldots \quad \text{for some} \quad i_t, j_t, l, m \in \{0, 3\}$$

and state 2 remains in the next steps in these cells (no birth of birds).

Thus we see that $l22 \to i$, $22l \to i \notin A$ $(i=0, 3)$, otherwise the distinguished state 2 is changed.

IF $l=0$, then $022 \to i$, $220 \to i \notin A$.

IF $322 \to i$ resp. $223 \to i \notin A$ for $i=0, 3$, then it holds

$$\begin{matrix} 322 \to i & \text{resp.} & 223 \to i & \\ 222 \to i & & 222 \to i & \text{not in } A. \\ 022 \to i & & 220 \to i. & \end{matrix}$$

Thus we see that two later birds for enough right resp. left from the first birds survive. Therefore $322 \to i_1$ and $223 \to i_2 \in A$ for some $i_1, i_2 \in \{0, 3\}$. If $l=3$ the proof is similar.

Now we write $XYZ \to$ instead of $\exists i \in \{0, B, 2, 3\} - \{Y\}$: $XYZ \to i$ and $XYZ \to \in A$ means $\exists i \in \{0, B, 2, 3\} - \{Y\}$: $XYZ \to i \in A$.

Let

$$E_1 := 323 \to \qquad E_3 := 023 \to$$

$$E_2 := 320 \to \qquad E_4 := 020 \to .$$

Next we will consider a case distinction.

Case 1: $E_1 \in A$; $E_2, E_3, E_4 \notin A$

Case 1a: $\{223 \to, 322 \to\} \subset A$, $220 \to$, $022 \to \notin A$ (see Prop. 1)

Let $K_1 = \ldots 0BB000\ldots$ be the initial configuration, then we obtain after a finite number of steps

$$K_n = \ldots i_1\, 22\, i_2 \ldots \quad \text{for some} \quad i_1, i_2 \in \{0, 3\}$$

and from hence cells with state 2 remain in state 2. Then $i_2 = i_1 = 0$, because $223 \to$, $322 \to \in A$. Since $E_3 = 023 \to$, $E_4 = 020 \to$, $022 \to \notin A$ two later birds far enough left from the early birds reach state 2 and remain in this state.

Case 1b: $\{220 \to, 022 \to\} \subset A$, $223 \to$, $322 \to \notin A$

If $K_1 = \ldots 0BB0B0BB0$ is an initial configuration, we obtain for some n, $K_n = \ldots i_1\, 22\, i_3\, \underset{\sim}{2} i_4\, 22\, i_5 \ldots$ where state 2 remains (no birth of birds) in these cells. Then $i_1 = i_3 = i_4 = i_5 = 3$, because $220 \to$, $022 \to \in A$. This is a contradiction to $323 \to \in A$. The cell marked by $\sim$ changes its state 2.

If K_1 is an initial configuration (with birds) and there are no later birds, then K_{n+1} denotes the configuration after n steps. If the bird-cells are in the distinguished state 2 and there is no change of states in these bird-cells in the following (without birth of birds) then the configuration is called K^*_{n+1}.

Case 2: $E_2 \in A$; $E_3, E_1, E_4 \notin A$

ad a: $\{220 \to, 022 \to\} \subset A$, $223 \to$, $322 \to \notin A$

If $K_1 = \ldots 0B0BB0B0\ldots$, then $\exists n$:

$K_n^* = \ldots i_1\, 2i_2\, 22i_3\, 2i_4$ for some $i_1, i_2, i_3, i_4 \in \{0, 3\}$.

Since $022\rightarrow$, $220\rightarrow \in A$, we see $i_2 = i_3 = 3$ and $i_4 = 3$, because $E_2 = 320\rightarrow \in A$. Since $E_1 = 323\rightarrow$, $E_3 = 023\rightarrow$, $223\rightarrow \notin A$, later birds ($0B0BB0B0$) far enough right from the first birds reach state 2 and remain in this state.

ad b: $\{223\rightarrow, 322\rightarrow\} \subset A$, $220\rightarrow$, $022\rightarrow \notin A$

If $K_1 = \ldots 0BB0B0\ldots$, then $\exists n$:

$K_n^* = \ldots i_1\, 22i_2\, 2i_3$ for some $i_1, i_2, i_3 \in \{0, 3\}$.

Since $223\rightarrow$, $322\rightarrow \in A$, it holds $i_1 = i_2 = 0$. Since $E_4 = 020\rightarrow$, $E_3 = 023\rightarrow$, $022\rightarrow \notin A$ later birds ($0BB0B$) far enough left from the first one reach state 2 and remain in this state.

Case 3: $E_3 \in A$; $E_1, E_2, E_4 \notin A$ (analogue to case 2)

Case 4: $E_4 \in A$; $E_1, E_2, E_3 \notin A$

ad a: $\{223\rightarrow, 322\rightarrow\} \subset A$, $220\rightarrow$, $022\rightarrow \notin A$

If $K_1 = \ldots 0BB0B0BB0\ldots$ then $\exists n$:

$K_n^* = \ldots i_1\, 22i_2\, 2i_3\, 22i_4 \ldots$ for some $i_1, \ldots, i_4 \in \{0, 3\}$.

Since $223\rightarrow$, $322\rightarrow \in A$, it holds $i_1 = i_2 = i_3 = i_4 = 0$, but $E_4 = 020\rightarrow \in A$ changes in the next step state 2. This is a contradiction to K_n^*.

ad b: $\{220\rightarrow, 022\rightarrow\} \subset A$, $223\rightarrow$, $322\rightarrow \notin A$

If $K_1 = \ldots 0B0BB0BB0\ldots$ then $\exists n$:

$K_n^* = \ldots i_0 2 i_1 22 i_2 22 i_3 \ldots$ for some $i_0, \ldots, i_4 \in \{0, 3\}$.

Since $220\rightarrow$, $022\rightarrow \in A$, it holds $i_1 = i_2 = i_3 = 3$. Since $E_1 = 323\rightarrow$, $E_3 = 023\rightarrow$, $223\rightarrow \notin A$ later birds ($0B0BB0BB0$) far enough right from the first one reach state 2 and remain in this state.

Case 5: $E_3, E_4 \in A$; $E_1, E_2 \notin A$

If $K_1 = \ldots 0B0BB0B0\ldots$ then $\exists n$:

$K_n^* = \ldots i_1\, 2i_2\, 22i_3\, 2i_4 \ldots$ for some $i_1, \ldots, i_4 \in \{0, 3\}$.

ad a: $\{220\rightarrow, 022\rightarrow\} \subset A$; $223\rightarrow$, $322\rightarrow \notin A$. Then it holds $i_2 = i_3 = 3$ and because of $023\rightarrow \in A$, $i_1 = 3$ holds. Since $E_1 = 323\rightarrow$, $E_2 = 320\rightarrow$, $322\rightarrow \notin A$ later birds far enough left from the first one reach state 2 and state 2 cannot be changed.

ad b: $\{223\rightarrow, 322\rightarrow\} \subset A$; $220\rightarrow$, $022\rightarrow \notin A$. Then it holds $i_2 = 0 = i_3$, but $E_3 = 023\rightarrow$, $E_4 = 020\rightarrow \in A$. Thus for one bird-cell (left from i_4) state 2 is changed in the next step.

Case 6: $E_2, E_4 \in A$; $E_1, E_3 \notin A$ (analogue to case 5)

Case 7: $E_2, E_3 \in A$; $E_1, E_4 \notin A$

ad a: $\{223\rightarrow, 322\rightarrow\} \subset A$; $220\rightarrow$, $022\rightarrow \notin A$

If $K_1 = \ldots 0B0BB00BB0B0B0BB0B0\ldots$ then $\exists n$:

$K_n = \ldots i_1\, 2i_2\, 22i_3\, i_4\, 22i_5\, 2i_6\, 2i_7\, 22i_8\, 2i_9 \ldots$ for some $i_1, \ldots, i_8 \in \{0, 3\}$.

Since $223\to$, $322\to\in A$ and $E_3=023\to$, $E_2=320\to\in A$ it holds $i_j=0$ $(1\leqq j\leqq 9)$ and $200\to$, $002\to$, $202\to\notin A$. Birds must send out signals to the right or to the left. So we can assume that a cell in state 3 is left or right from the cell with state i_1 or i_9 — say left —. Since $002\to$, $202\to\notin A$ it holds $302\to\in A$, otherwise later birds (0*B*0*BB*) far enough right from the first birds survive (in state 2). Now let $K_n^*= =\dots 3\underbrace{0\dots 0}_{m}2022002202020 22020$ be and $m=1$. This leads to a contradiction. If $302\to 3\in A$, then $E_3=320\to\in A$ eliminates state 2. If $302\to 2\in A$, then a new state 2 occurs. Now let $m>1$. Because $200\to\notin A$ and later birds (double the configuration K_1) far enough right from the first ones must be eliminated, the set A contains the transition $300\to$. This leads to a contradiction, because in case of $300\to 3$ we reach case $m=1$ and in case of $300\to 2$ a new distinguised state 2 occurs.

ad b: $\{220\to, 022\to\}\subset A$; $223\to$, $322\to\notin A$

If $K_1=\dots 0B0BB00BB0$ then $\exists n$:

$$K_n^*=\dots i_1\,2i_2\,22i_3\,i_4\,22i_5\dots \quad \text{for some} \quad i_1,\dots,i_5\in\{0,3\}.$$

Then it holds $i_1=i_2=i_3=i_4=i_5=3$, because $220\to$, $022\to$, $E_3=023\to$ and $E_2= =320\to\in A$.

Furthermore we see that $232\to$, $332\to$, $233\to\notin A$ (otherwise a new state 2 arises or a state 2 is eliminated one or two steps later). Since $232\to$, $332\to\notin A$ and later birds (0*B*0*BB*00*BB*0) far enough right from the first birds must be killed, the transition $032\to$ *belongs to* A.

Now we consider $K_1'=\dots 0BB000BB0\dots$ then $\exists t$:

$$K_t'^*=\dots j_1\,22j_2\,j_3\,j_4\,22j_5 \quad \text{for some} \quad j_i\in\{0,3\}\ (1\leqq i\leqq 5).$$

Then it holds $j_i=3$ $(i\neq 3)$ $(220\to$, $022\to\in A)$. If $j_3=0$ then $032\to\in A$ and $022\to\in A$ lead to a contradiction (new state 2 or elimination). Thus we see that $j_3=3$ and $333\to\notin A$.

Going back to K_1 it holds

$$K_n^*=\dots 0\underbrace{3\dots 3}_{m}\,232223 3223\dots$$

If $m=1$ and if $032\to 0\in A$, then $E_2=023\to\in A$ eliminates the distinguished state 2 and if $032\to 2\in A$ we reach a new state 2. Let $m>1$. It holds $033\to\notin A$, otherwise we obtain after some steps the situation $m=1$ or a new state 2. Altogether we get $333\to$, $033\to$, $233\to\notin A$ and therefore a later bird (state 2) (0*B*0*BB*0) far enough from the first birds survive.

Case 8: $E_1, E_4\in A$; $E_2, E_3\notin A$

If $K_1=\dots 0BB0B0BB0\dots$ then $\exists n$:

$$K_n^*=\dots i_1\,22i_2\,2i_3\,22i_4 \quad \text{for some} \quad i_1,\dots,i_4\in\{0,3\}.$$

ad a: $\{220\to, 022\to\}\subset A$; $223\to$, $322\to\notin A$, then it holds $i_1=i_2=i_3=i_4=3$, but $E_1=323\to\in A$ contradicts K_n^*.

ad b: $\{223\to, 322\to\}\subset A$; $220\to$, $022\to\notin A$, then it holds $i_1=i_2=i_3=i_4=0$, but $E_4=020\to\in A$ contradicts K_n^*.

Case 9: $E_1, E_2\in A$; $E_3, E_4\notin A$

If $K_1 = \dots 0BB0B0BB0B0\dots$ then $\exists n$:

$K_n^* = \dots i_1\, 22 i_2\, 2 i_3\, 22 i_4\, 2 i_5 \dots$ for some $i_1, \dots, i_5 \in \{0, 3\}$.

ad a: $\{220\rightarrow, 022\rightarrow\} \subset A$; $223\rightarrow$, $322\rightarrow \notin A$, then it holds $i_1 = i_2 = i_3 = i_4 = 3$, but $E_1 = 323 \rightarrow \in A$ contradicts K_n^*.

ad b: $\{223\rightarrow, 322\rightarrow\} \subset A$; $220\rightarrow$, $022\rightarrow \notin A$, then it holds $i_1 = i_2 = i_3 = i_4 = 0$. Since $E_4 = 020\rightarrow$, $E_3 = 023\rightarrow$, $022\rightarrow \notin A$ later birds ($0BB0B0BB0B0$) far enough left from the first one reach state 2 and remain in this state.

Case 10: $E_1, E_3 \in A$; $E_2, E_4 \notin A$ (analogue to case 9)

Case 11: $E_1, E_2, E_4 \in A$; $E_3 \notin A$
If $K_1 = \dots 0B0B0B0\dots$ then $\exists n$

$K_n^* = \dots i_1\, 2 i_2\, 2 i_3\, 2 i_4 \dots$ for some $i_1, \dots, i_4 \in \{0, 3\}$.

If $i_2 = 3$, then $i_3 = 0$ or 3, but $E_1 = 323\rightarrow$, $E_2 = 320 \rightarrow \in A$ contradicts K_n^*. If $i_2 = 0$, then $i_3 = 3$ and then $i_4 = 0$ or 3, but $E_1 = 323\rightarrow$, $E_2 = 320 \rightarrow \in A$ contradicts K_n^*.

Case 12: $E_1, E_3, E_4 \in A$; $E_2 \notin A$ (analogue to case 11)

Case 13: $E_1, E_2, E_3 \in A$; $E_4 \notin A$
ad a: $\{220\rightarrow, 022\rightarrow\} \subset A$; $223\rightarrow$, $322\rightarrow \notin A$
If $K_1 = \dots 0BB0B0BB0\dots$ then $\exists n$:

$K_n^* = \dots i_1\, 22 i_2\, 2 i_3\, 22 i_4$ for some $i_1, \dots, i_4 \in \{0, 3\}$.

Since $220\rightarrow$, $022 \rightarrow \in A$, it holds $i_1 = i_2 = i_3 = i_4 = 3$. But $E_1 = 323 \rightarrow \in A$ contradicts K_n^* $(i_2\, 2 i_3)$.
ad b: $\{223\rightarrow, 322\rightarrow\} \subset A$; $220\rightarrow$, $022\rightarrow \notin A$
If $K_1 = \dots 0B0BB00BB0B0\dots$ then $\exists n$:

$K_n^* = \dots i_1\, 2 i_2\, 22 i_3\, i_4\, 22 i_5\, 2 i_6 \dots$ for some $i_1, \dots, i_6 \in \{0, 3\}$.

Since $223\rightarrow$, $322 \rightarrow \in A$, it holds $i_2 = i_3 = i_4 = i_5 \rightarrow 0$ and because of $E_3 = 023\rightarrow$, $E_2 = 320 \rightarrow \in A$ it holds $i_1 = i_6 = 0$. Furthermore $i_1 = \dots = i_6 = 0$ implies $202\rightarrow$, $200\rightarrow$, $002 \rightarrow \notin A$, otherwise a state 2 is changed. Birds must send out signals to the right or to the left. Therefore we have a transition $300 \rightarrow 3$ or $003 \rightarrow 3$ in A.

Suppose: $300 \rightarrow 3, 003 \rightarrow 3 \in A$.

Case b1: Left from the cell with state i_1 in K_n^* state 3 occurs.

$K_n^* = \dots 3\underbrace{0\dots 0}_{m}20220022020\dots$

Let $m = 1$: $302 \rightarrow 3$ or $2 \in A$ contradicts K_n^*, because of $E_2 = 320 \rightarrow \in A$ resp. a new state 2 occurs. Thus $302 \rightarrow \notin A$ and because $202\rightarrow$, $002 \rightarrow \notin A$ a later bird marked by $\sim$ ($0\tilde{B}0BB0$) far enough right from the first remains in state 2.

If $m > 1$, we reach after $m-1$ steps case $m = 1$, because $300 \rightarrow 3 \in A$.

Case b2: Right from the cell with state i_6 in K_n^* state 3 occurs. This leads to a contradiction similar to case b1.

Thus we see that $300\to 3\notin A$ or $003\to 3\notin A$. Without loss of generality we assume $300\to 3\in A$ and $003\to 3\notin A$. Then there is no cell left from the cell with state i_1 in K_n^* which has state 3 (apply case b_1 again). Therefore a cell right from the cell i_8 must have state 3.

$$\ldots 020220022020\underbrace{\ldots}_{m}03$$

If $m=1$ for all further steps, then $203\to 3\notin A$, because $E_3=023\to\ \in A$ and $203\to 2\notin A$, because a distinguished state 2 arises. Altogether we obtain $203\to$, $200\to$, $202\to\ \notin A$. This shows that later birds ($0BB0B0$) far enough left from the first reach state 2 and remain in this state.

If $m>1$, then $003\to 2\in A$, otherwise there is no feedback from a meeting with birds far enough right from the origin, but $003\to 2$ generates new distinguished states in case of no birth of birds. This is a contradiction.

Case 14: $E_2, E_3, E_4\in A$; $E_1\notin A$

ad a: $\{223\to,\ 322\to\}\subset A$; $220\to$, $022\to\ \notin A$

If $K_1=\ldots 0BB0B0BB0\ldots$ then $\exists n$:

$$K_n^* = i_1\,22 i_2\,2 i_3\,22 i_4 \ldots \quad \text{for some } i_1, \ldots, i_4\in\{0, 3\}.$$

Because $223\to$, $322\to\ \in A$ it holds $i_1=i_2=i_3=i_4=0$, but $E_4=020\to\ \in A$ contradicts K_n^*.

ad b: $\{220\to,\ 022\to\}\subset A$; $223\to$, $322\to\ \notin A$

If $K_1=\ldots 0B0BB00BB0B0BB0B0\ldots$ then $\exists n$:

$$K_n = i_1\,2 i_2\,22 i_3\, i_4\,22 i_5\,2 i_6\,22 i_7\,2 i_8 \quad \text{for some } i_1, \ldots, i_8\in\{0, 3\}.$$

Because $220\to$, $022\to\ \in A$ it holds $i_j=3$ $(1\leqq j\leqq 8)$. This implies $232\to$, $332\to$, $233\to\ \notin A$.

If $K_1'=\ldots 0BB000BB0\ldots$ then $\exists n$:

$$K_n^* = j_1\,22 j_2\, j_3\, j_4\,22 j_5 \ldots \quad \text{for some } j_1, \ldots, j_5\in\{0, 3\}.$$

Because of $220\to$, $022\to\ \in A$ it holds $032\to\ \in A$, otherwise a later bird ($0B0BB0$) far enough right from the first bird (starting with K_1) reaches state 2 and remains in this state. This implies $j_3=3$, because $022\to\ \in A$ and $E_3=023\to\ \in A$. Furthermore it follows from $j_2=j_3=j_4=3$ that $333\to\ \notin A$. Now we consider K_1 and K_n^* again.

$$K_n^* = \ldots 03\underbrace{\ldots}_{m}3\ 2 i_2\,22 i_3\, i_4\,22 i_5\,2 i_6\,22 i_7\,2 i_8 \ldots$$

$m=1$ implies a contradiction, because $023\to 0\in A$ (then $023\to\ \in A$ eliminates state 2) or $032\to 2\in A$ (then a new distinguished state arises). Let $m>1$. Since $333\to\ \notin A$ and $233\to\ \notin A$ and later birds ($0B0BB0$) far enough from the first birds must be killed (state 2 changed) the transition $033\to$ belongs to A.

This transition leads to a configuration K_t^* (from K_n^*)

$$\ldots 0\tilde{3}2322332232322323\ldots$$

and in the next step we obtain a new state 2 ($032\to 2$) or a state 0 ($032\to 0$) in the cell marked by $\sim$ and then we eliminate state 2, because of $E_3=023\to\ \in A$. This shows that case 14 is impossible.

Case 15: $E_1, E_2, E_3, E_4 \in A$

If $K_1 = \dots 0B0\dots$, then $\exists n$: $K_n^* = \dots i_1 2 i_2 \dots$ for some $i_1, i_2 \in \{0, 3\}$, but $E_i \in A$ $(1 \leqq i \leqq 4)$ contradicts K_n^*.

Case 16: $E_i \notin A$ $(1 \leqq i \leqq 4)$

If $K_1 = \dots 0B0BB0B0\dots$ then $\exists n$:

$$K_n^* = \dots i_1 2 i_2 22 i_3 2 i_4 \dots \text{ for some } i_1, \dots, i_4 \in \{0, 3\}.$$

ad a: $\{220\rightarrow, 022\rightarrow\} \subset A$; $223\rightarrow, 322\rightarrow \notin A$, then it holds $i_2 = i_3 = 3$. Since $E_2 = 320\rightarrow$, $E_1 = 323\rightarrow$, $322\rightarrow \notin A$ later birds $0B0BB0B0$ far enough left from the first birds reach state 2 and remain in this state.

ad b: $\{223\rightarrow, 322\rightarrow\} \subset A$; $220\rightarrow, 022\rightarrow \notin A$, then it holds $i_2 = i_3 = 0$. Since $E_4 = 020\rightarrow$, $E_3 = 023\rightarrow$, $022\rightarrow \notin A$ later birds $(0B0BB0B0)$ far enough left from the first birds reach state 2 and remain in this state.

Altogether we have proved that the one-dimensional early bird problem is unsolvable with set of states $\{0, B, 2, 3\}$ and with distinguished state 2. □

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