

Locally synchronous cellular automata

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1. Cellular automaton and the synchronization problem*

The concept of cellular automaton was evolved by John von Neumann when he dealt with questions concerning the capabilities of machines, in particular the feasibility of a mechanical self-reproduction [6], see also [2]. A cellular automaton is a "macro automaton" or "polyautomaton" composed in a uniform manner out of a not a priori bounded number of "micro automata" called *cells*. These are practically thought of as being arranged at the integer lattice points of the Euclidian plane (or in general the Euclidian n -space). The cells are interconnected in a uniform local scheme called the *neighbourhood connection*, and the cells directly connected to a given cell are called its *neighbours*. Each of the cells can be in one of a finite number of *states* which can be changed at certain times. The next state of a cell depends on its actual state and those of its neighbours. This dependency is described by a finite set of *local transition rules* which is assumed to be the same for each individual cell.

A cellular automaton is thus characterized by a tuple $Z = (Z^2, N, Q, \Sigma)$ where Z^2 is the universe, N is the neighbourhood set given by a finite number of vectors, Q is the finite set of cell states having at least two elements, and Σ is the finite set of local rules. The basic idea is that in the simultaneous interplay "the whole is more than the sum of the parts", as was shown, for example, in universal computation, self-reproduction (of patterns figured by cell states), pattern recognition and -transformation. An essential topic which seems to be considered still in its first steps is *parallel computing*, particularly with regard to so-called myopic algorithms, the elementary operations of which take reference only to bounded, well-defined subsets of data, for example *cellular algorithms* as introduced by LEGENDI [4].

Such formal computation procedures can be conceived as *deductions* in a special type of a "more-dimensional" substitution calculus which in the following shall be called *cellular calculus*: The alphabet consists of the cells' states symbols, the dimension being given by the underlying space which can be thought of as a frame set of

* To avoid confusion it should be mentioned that quite a different type of "synchronization problems" is known in the literature such as the "firing squad synchronization problem" introduced by John Myhill.

symbol fields, and the basic substitution rules are given by the local transition rules. In the von Neumann two-dimensional cellular automaton and related cases with neighbourhood set $\{(0, 0), (1, 0), (0, -1), (-1, 0), (0, 1)\}$ these rules are of the following normed shape (a, b, c, d, e, a' denote state symbols):

$$\begin{array}{|c|} \hline e \\ \hline d | a | b \\ \hline c \\ \hline \end{array} \rightarrow a'$$

The cellular calculus operates on two-dimensional words over the state symbol alphabet which are called *patterns*. To determine the manner of rule application designating a deduction step in the calculus, the specification of a *meta rule* is needed: Here in the von Neumann concept, in each single step, the *simultaneous* substitution of the whole set of rule-shaped subwords is considered, according to the basic substitution rules. Wherever at adjacent symbol fields the rule premises are overlapping they must refer mutually to the state symbols given *before* application of a substitution rule, i.e.,

$$\begin{array}{|c|c|c|} \hline e & h & \\ \hline d | a | b | f \\ \hline c & g & \\ \hline \end{array} \text{ comes up to } a' b'$$

by the rules

$$\begin{array}{|c|} \hline e \\ \hline d | a | b \\ \hline c \\ \hline \end{array} \rightarrow a' \quad \text{and} \quad \begin{array}{|c|} \hline h \\ \hline a | b | f \\ \hline g \\ \hline \end{array} \rightarrow b'$$

Cellular calculus and cellular automaton can be regarded as being in the relationship “rules of a game” versus “game”; and to perform an *adequate* execution (according to the meta and basic rules — throughout this paper the term “adequate” shall be fixed for this notation) of the “game” by the automaton, it is a usual assumption that a *global clock* gives rise to a *synchronous* switching of state transitions of all automata cells. From this synchronicity assumption an organization problem arises which is here referred to as the *synchronization problem*: synchronizing a not a priori bounded number of cells, for a cellular automaton has to be considered as an unbounded, potentially infinite automaton (for detail see [9]).

Physically motivated objections against such a synchronicity assumption led to the development of so-called *asynchronous cellular automata* introduced by NAKAMURA [5], PRIESE [7] and GOLZE [3], where besides other things, put briefly, the meta rule of the underlying calculus is changed: The simultaneous application of the basic rules (and thereby the “grade of parallelism”) is more or less restricted. A synchronization problem as in the case of a “simultaneous cellular calculus” does not appear in these cases, see [8].

The approach we take in this paper is shown in the following questions.

— Cannot the original concept of simultaneous cellular calculus as a model for highly parallel information processing be kept, and the execution of such a calculus

by a cellular automaton be organized in such a manner that, for building a synchronization scheme, no reference to the whole number of cells is required?

— Can a cellular calculus in the sense of John von Neumann be executed in an adequate manner only by a synchronous automaton?

In other words:

— Is the *sufficient* condition of synchrony also *necessary*?

In the following paragraphs firstly a weaker assumption of “local synchrony” will be introduced and discussed and, secondly, a scheme will be designed which could lead to a “locally synchronous performance” of a cellular automaton by which a simultaneous cellular calculus can be adequately executed.

2. Synchronous and locally synchronous working

The main difference between regarding a calculus as an ideal system, and an automaton executing that calculus, seen as a physical device, is that the latter does while the former does not submit to certain physical restrictions; so that in the automaton case we have to consider physical limitations such as bounded signal velocity, delay, and bounded exactness of properties of materials — from this point the discussion of asynchronous automata has arisen.

Let us assume that clock signals are used each to initiate one deduction step in a simultaneous cellular calculus being executed by an appropriate cellular automaton. If accepted that signal transmission cannot instantaneously reach cells in a certain spatial distance from the signal outspring there arises at once the problem of time:

— At what time will a state transition of a given cell be initiated?

— Are all cells reached by the signal in a time interval small enough to perform an adequate execution of one deduction step as indicated above?

— At what delay after initiating the execution of one deduction step could the execution of another step be initiated?

All these problems are ignored when considering a discrete time scale \bar{t} by saying, as is usual, that time passes by in discrete steps $t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow \dots$; each time a time step has elapsed each cell must have executed one state transition. From this it is possible to consider the *configuration* (whole or global state) c of a cellular automaton which is simply defined by the infinite cartesian product of all cell states and can be formally denoted by a mapping $c: \mathbf{Z}^2 \rightarrow Q$, where Q is the set of state symbols.

Furthermore, a *global transformation* $\bar{\Sigma}$ — corresponding to a deduction step in the calculus given by the simultaneous application of the local rules in Σ to all neighbourhood-shaped patterns — can be considered which leads from a given configuration c to one out of a finite set of possible successor configurations, c' . If the rule set Σ is deterministic which means that no two rules with identical premises leading to different conclusions exist, $\bar{\Sigma}$ can be thought of as a *global transition function*.

Having the concept of “configuration”, and supposing that a synchronous state transition switching of all cells of a cellular automaton is given, it is possible to say that each time a time step has elapsed the cellular automaton has changed its global state (configuration) once; and a “run” of a cellular automaton is seen as a sequence of configurations, starting from an initial configuration.

As seen from this, however, a *concept of synchrony* is needed when speaking of a configuration and a global transition of a cellular automaton. To discuss such

a concept in relation to the above restrictions, we now introduce a continuous time scale to be thought of as an observation time. This refinement of the point of view will be seen later as an intermediate step towards the tool of explaining a locally synchronous performance.

Let us consider a totally ordered *time scale* $\ell = (\mathbf{R}, \cong, t_1)$, bounded from below and referring to the point of signal source, say the origin of \mathbf{Z}^2 . We call *instants* the elements of ℓ and *durations* the lengths of intervals out of ℓ . We denote the lattice points by $\alpha, \beta, \gamma, \dots$, thinking of them as being the cells. At certain instants t_1, t_2, t_3, \dots clock signals T_1, T_2, T_3, \dots shall be propagated from the origin.

We write $\langle t_k^\alpha \rangle$ to fix the *time-spatial event* that at cell $\alpha (\alpha \in \mathbf{Z}^2)$ at time t_k^α ($t_k^\alpha \in \ell$) a unique clock pulse appears initiating a k -th state transition of α , assuming that, at this time, α is in a well-defined state q ($q \in Q$). We shall refer to an event $\langle t_k^\alpha \rangle$ as the k -th *clock pulse* at α .

As indicated above, there may be objections when assuming that an *unbounded* number of cells could be synchronized while this would be certainly possible for any given fixed number of cells. Furthermore, an *absolutely* synchronous cell state switching would not be possible in a physical device and therefore a small amount of phase variations not leading to mishaps should be admissible. From that it is carried out the following definition.

Definition 1. A finite set of clock-pulsed cells is said to be *synchronously working*, if for any given clock signal T_k , in the *whole* set of cells the pairwise difference in time of k -th clock pulses of cells is at most ε s.t.

- (i) the correct execution of local transitions is not affected,
- (ii) ε is not exceeded after an arbitrary number of clock steps.

Since it will take a nonzero but bounded duration s until the state transition of any given cell is executed, say

$$0 < s_{\min} \leq s \leq s_{\max} < \infty \quad (1)$$

in all cases, the minimum signal distance in a synchronously working cellular automaton, leading with certainty to a one-to-one execution of each global transition by reason of one signal, is given by

$$\varepsilon + s_{\max} \quad (2)$$

In opposition to this, *synchronicity* of clock pulse events would usually be defined, as an *equivalence relation*, in the following way.

Definition 2. Two events $\langle t_k^\alpha \rangle$ and $\langle t_l^\beta \rangle$, where $t_k^\alpha, t_l^\beta \in \ell$, are said to be *synchronous* iff $t_k^\alpha = t_l^\beta$. To denote that these two events are in the synchronicity relation we shall write $\langle t_k^\alpha \rangle \text{ syn } \langle t_l^\beta \rangle$.

Surely, in a synchronously working set of cells, if $\langle t_k^\alpha \rangle \text{ syn } \langle t_l^\beta \rangle$ then $k=l$. Consider, for the moment, the case that all cells of an unbounded cellular automaton Z undergo synchronous state transitions by reason of clock pulses occurring synchronously at each individual cell, without discussion whether or not this is possible. In this case, for a given k , and with 0 denoting the origin of \mathbf{Z}^2 , the set $\{\alpha | \langle t_k^\alpha \rangle \text{ syn } \langle t_k^0 \rangle\}$ consists of the whole set of cells of Z and thus at time t_k^0 a well-defined configuration of Z , resp. the initiation of the k -th global transition of Z , can be considered ($k=1, 2, 3, \dots$).

The relation between the continuous time scale $t = (\mathbf{R}, \cong, t_1)$ and a discrete time scale \vec{t} is worked out in the following. Denote the times of input of clock signals T_1, T_2, \dots at the origin 0 of a cellular automaton Z by t_1^0, t_2^0, \dots , and the events arising from this by $\langle t_1^0 \rangle, \langle t_2^0 \rangle, \dots$. Regarding (2), let, for all k , $|t_{k+1}^0 - t_k^0| \cong \varepsilon + s_{\max}$. Thus, if two clock pulse events are synchronous, they are in particular localized within the same time interval $[t_k^0, t_{k+1}^0)$ out of t . Let $t_1 = t_1^0$. We can now define a discrete time scale by a totally ordered set of instants which is bounded from below:

$$\vec{t} = (\{t_k^0\}_{k \in \mathbf{N}}, \cong, t_1^0).$$

Now, when saying that time passes by in steps $t_1^0 \rightarrow t_2^0 \rightarrow \dots$, we can assume that each time a time step has elapsed Z has once changed its global state. Because of the synchronicity of t_k^0 , for all α , we could instead consider $\{t_k^\alpha\}_{k \in \mathbf{N}}$ as set of instants, for any α ; indeed, the simultaneous consideration of all t_k^α , $\alpha \in \mathbf{Z}^2$, given by the equivalence relation of synchronicity, leads to a globally applicable discrete time concept.

We now return to the questions formulated in the first paragraph and firstly look at the necessity of synchronicity for adequate execution of a simultaneous cellular calculus.

In a synchronous cellular automaton, at any individual global transition there takes place only a *local* information processing at each cell, namely, the state of a cell is requested by its *finite* number of neighbouring cells for their computation of a successor cell state, and vice versa. Only by way of a state changing, can inputs to the cell effect its output which will, however, be only requested by the neighbours at the next global transition. (We agree that cells are Moore-type automata; if considering non-deterministic rule sets Σ , this case can easily be generalized, see [8].)

Hence, the effect of the information processing of an individual cell during a global transition is restricted to the region of neighbourhood of the cell. Thus, for the correct execution of the state transitions of each cell, it is only necessary that every two neighbouring cells work synchronously. This leads us to the following definition.

Definition 3. A set of clock-pulsed cells is said to be *locally synchronously working*, if for any given clock signal T_k , the pairwise difference in time of k -th clock pulses of *neighbouring* cells is at most ε s.t.

- (i) the correct execution of local transitions is not affected,
- (ii) ε is not exceeded after an arbitrary number of clock steps.

Obviously, synchronous working includes locally synchronous working, the opposite of which is not true since, for example, three adjacent cells working locally synchronously allow the clock pulses of the outer cells to differ in time by 2ε . Thus locally synchronous working is a weaker concept than synchronous working of a set of cells, and note that a reference to a boundation of the number of cells is not required in this case.

As it is easy to see, in a cellular automaton Z working locally synchronously the same successor cell states are (or, in case of an indeterministic rule set, could be) generated as when working synchronously. That means, if it is assumed that all cells have had one but not more clock pulses and have finished execution of the induced state transition, in each case the same result would (resp. could) be obtained, namely, the result of one deduction step in the basing cellular calculus.

To get an adequate execution of a simultaneous cellular calculus by way of a locally synchronous performance, it must be guaranteed that by each command to the automaton to execute one deduction step each cell will perform, at some time, one and only one corresponding state transition whereby neighbouring cells do this synchronously. Again, this gives rise to an *organization problem*: to achieve such a locally synchronous performance. It appears that two subproblems are to be solved:

- (P1) How to perform the adequate execution of one individual deduction step?
- (P2) If (P1) is solved, how to proceed on sequences of deduction steps?

3. The concept of *T-net*

For the solution of these problems we now introduce an organization scheme called *T-net*. A *T-net* is a device to be added to a cellular automaton to effect an appropriate distribution of clock signals such that a locally synchronous working can be performed. It consists of uniform type components which are thought to be integrated each to one automaton cell and hence allow the building-up of the automaton simply by the proper arranging of the cells and interconnecting them neighbour-to-neighbour. Thus an automaton in realization can be extended, including the organization scheme, by simply subjoining the required number of cells.

Signal transmission in a *T-net* will intentionally be considered as a propagation in a plane lattice such that each lattice point will receive exactly one signal offspring. Since in a physical device a loss of energy has to be supposed it is assumed that a signal regeneration occurs in the *T-net* components. The delay arising between the appearance of the signal (as a clock pulse) at a given cell and its neighbours is considered as being small enough to allow proper working of the cells, according to definition 3.

A *T-net* component is an automata network able to handle a number of clock pulses running in parallel. For the description of this, first a special type of automaton is needed:

An *asynchronous parallel automaton* (APA) A is a system $A = (S_A, I_A, O_A, R_A)$ of pairwise disjoint finite sets S_A , I_A , and O_A , and a subset R_A of $(S_A \times \mathcal{P}(I_A)) \times (S_A \times \mathcal{P}(O_A))$ where $\mathcal{P}(M)$ denotes the set of all subsets of a given set M . The elements of S_A , I_A , and O_A are called *states*, *inputs* resp. *outputs* of A . R_A is called the *transition relation* of A .

This concept was introduced by PRIESE [7]. It allows the description of the behaviour of automata under the simultaneous occurrence of several inputs or outputs, indeterminacy, and, in addition, the feasibility of state transitions independent of inputs or outputs. Concerning the sets I_A and O_A , note that, instead of considering distinct input and output signals on *one* channel, it is here assumed that one type of signals appears at *distinct* channels. Thus we shall call the elements of I_A and O_A input resp. output *places* or likewise *entrances* resp. *exits*. Since the sets are disjoint no confusion will arise. An APA can hence be named a *directed* automaton.

An APA network, shortly *net*, is simply the result of any junction of several APA in such a way that in no case is one output place connected to more than one input place, and vice versa. Input and output places remaining unconnected in such a process will be called *input* resp. *output places of the net*. Again, the junction of nets in the indicated way gives a net. We will imagine in the sequel that signals are very

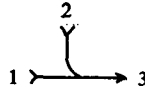
short pulses distinct from a quiescent state which are “running along wires through the net”.

A *T-net component* is figured out of four types of APA called *K, A, P, I*. The elements of their relations are given by transitions

$$(s, \{i_1, \dots, i_j\}) \rightarrow (s', \{\sigma_1, \dots, \sigma_k\})$$

to be read as follows: “in state *s* under input at places i_1, \dots, i_j go over to state s' under output at $\sigma_1, \dots, \sigma_k$ ”.

The module *K* gets the symbol



and is defined by

$$K := (\{0\}, \{1, 2\}, \{3\}, R_K)$$

$$R_K := \{(0, \{1\}) \rightarrow (0, \{3\}), (0, \{2\}) \rightarrow (0, \{3\}), (0, \{1, 2\}) \rightarrow (0, \{3\})\}$$

which means that signal inputs appearing at place 1 or 2 will come out at place 3; if two inputs synchronously appear at 1 and 2 a “united” output signal will appear at place 3.

The module *A* gets the symbol

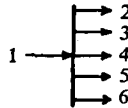


and is defined by

$$A := (\{0\}, \{1\}, \emptyset, \{(0, \{1\}) \rightarrow (0, \emptyset)\}).$$

It describes the total absorbing of signals.

The module *P* gets the symbol

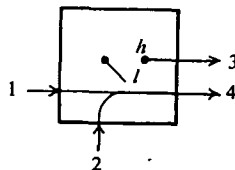


and is defined by

$$P := (\{0\}, \{1\}, \{2, 3, 4, 5, 6\}, \{(0, \{1\}) \rightarrow (0, \{2, 3, 4, 5, 6\})\}).$$

A signal input at 1 generates five parallel outputs.

The behaviour of the following “pulse injecting” module *I* depends on its internal state. *I* gets the symbol



and is defined by

$$I := (\{l, h, \uparrow, \downarrow\}, \{1, 2\}, \{3, 4\}, R_I)$$

$$R_I := \{(l, \{1\}) \rightarrow (l, \{4\}), (l, \{1\}) \rightarrow (l, \{4\}), (l, \emptyset) \rightarrow (h, \{3\}),$$

$$(l, \{1\}) \rightarrow (h, \{3, 4\}), (h, \{1\}) \rightarrow (h, \{4\}), (h, \{2\}) \rightarrow (l, \{4\}), (l, \emptyset) \rightarrow (l, \emptyset)\}$$

I has two stable states, l and h , and two unstable states, \uparrow and \downarrow . At the moment when I reaches state h ("high") a newly created signal is put out at 3; causal for a switching from l to h , and hence for this output, is an input signal reaching the module in state l . For the duration of this switching a "transition state", denoted by \uparrow (resp. \downarrow for the switching from h to l), is considered¹; in the relational description it is expressed that h can be reached, not influenced by an occurrence of further inputs at 1. State-input combinations not listed above will not appear in the nets considered in the sequel.

The indeterministic description of the I -module contains the stated behaviour only in principle in the first instance. To make sure that it will actually happen within a given time period we fix in addition layers for the *duration of switching*, denoted by the delay that will occur between the initiation by a matching input and the moment of the occupation of the new state:

- (i) from l to h : \underline{v} ($0 < \underline{v}_{\min} \leq \underline{v} \leq \underline{v}_{\max} < \infty$)
- (ii) from h to l : \bar{v} ($0 < \bar{v}_{\min} \leq \bar{v} \leq \bar{v}_{\max} < \infty$).

In the case under consideration (two-dimensional cellular space with von Neumann neighbourhood) a T -net component is compounded from three K -modules and always one module I , A , and P , which are interconnected to a net as indicated in Fig. 1. Such a net has five input places and five output places: always four *external* places for the reception and emission of clock signals and one *internal* input resp. output place for the junction of always one T -net component and one cell. The signal transmission channels will be called *wires* for short. A heuristic description of the functioning of a T -net component follows now.

Consider a T -net component with no signals running on any wire, in the starting position, and with the I -module being in state l . Signals appearing at the external input places of the net pass the K -gates and reach place 1 of the I -module as a *sequence of signals*, the number of which may have diminished by the possible case of synchronous checking into the two input places of a K -module. The signals leave the I -module at 4 and become absorbed at the A -module. Only the first signal of the sequence initiates a switching of I from the stable state l to the stable state h whereby, after the delay \underline{v} , a (newly created) signal occurs at place 3. This reaches the P -module whereby five signals run in parallel:

- (i) to all external output places of the T -net component
- (ii) to the internal output place CLOCK

and leave the net. Incoming signals are thus separated by the I -module: When I is in state l only the first signal of a sequence is able to influence the net. Only when a

¹ When assuming a *duration* is going along with a switching process it is possible, as in the considered case of an APA, that during a switching further signal inputs occur. It is thus necessary, for a complete description, to take account of such an unstable transition state.

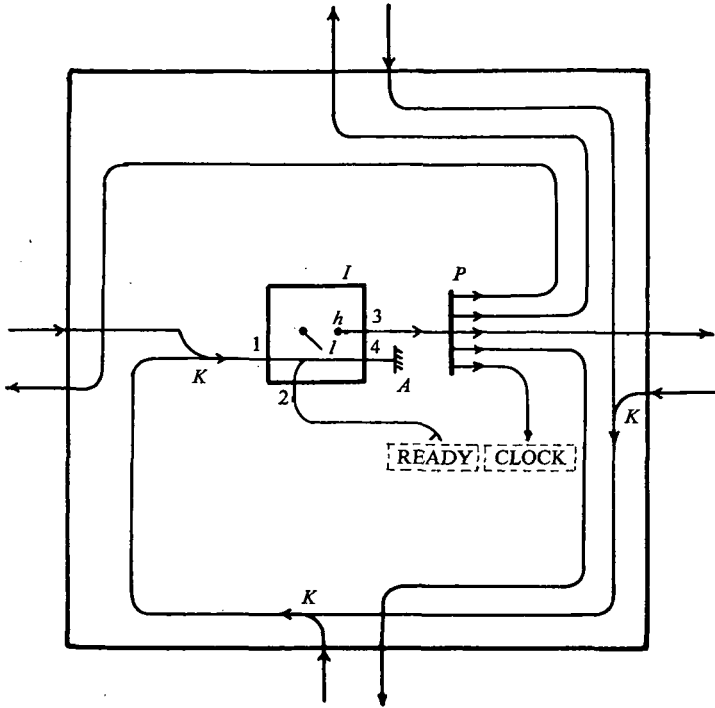


Fig. 1

transition of I back to state l is initiated and fulfilled, by way of a signal incoming at the internal READY entrance, will a signal reaching place 1 of the I -module cause a new signal output at 3.

In respect of the internal exit CLOCK that means: Signals coming from outside into a T -net component can effect a signal output at CLOCK *only once in a certain time period*. This period contains in particular the duration of the event lying between a signal output to CLOCK and a signal input from READY.

Concerning the total time period taken by a process in a T -net component the delay v will be significant which appears between the earliest input of a signal, causally for an output at place 3 of the I -module, and the resulting output of five parallel signals at the exits of the T -net component ($\underline{v} < v$). Considering a number of T -net components, as is done in the following, it is assumed that the following condition holds:

$$0 < v_{\min} \cong v \cong v_{\max} < \infty \tag{3}$$

However, concerning repetitions of the described process, we focus attention throughout this paper on the case that

$$\textit{component delays do not vary in time} \tag{4}$$

It will be subject of a future paper [9] to examine the implications given by the assumption of time-variant component delays.

We call *T-net* an APA net originating when a set of copies of the described *T*-net component is compounded at neighbouring integer lattice points of the Euclidian plane and interconnected, in a canonical way, by identifying corresponding output and input places. With respect to its connection scheme and its components a *T-net* is thus *homogeneously* compounded and can, assuming that it consists of a finite number of components, be *boundlessly* extended by repetition of the same process: attaching another *T*-net component at a free lattice point adjacent to the border of the hitherto existing net; admitted are rotations by $k \cdot 90^\circ$, $k \in \{1, 2, 3\}$, in the plane. A section of a *T-net* is shown in Fig. 2.

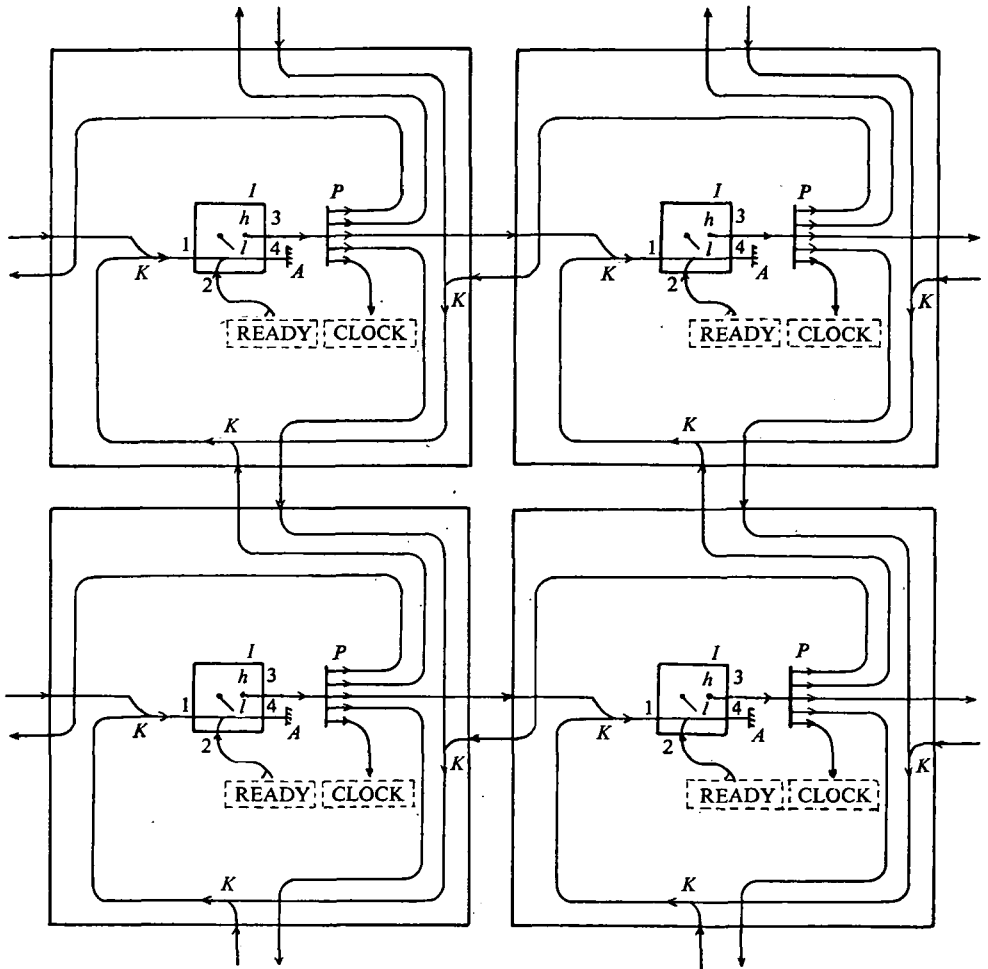


Fig. 2

This completes the description of the construction of a T -net. As pointed out, the concept of APA net here is extended by a time concept to be applied locally as will be worked out later. This gives the possibility of referring to the durations of subprocesses in the T -net. To integrate, for the moment, the starting and ending points of such processes in a T -net component into the time scale t (to be thought of as an observation time), we introduce the following notations. For cells α which contain a T -net component we say that:

at time t , α is in T -state s iff the I -module of α is in state s at that time, $s \in \{l, \dagger, h, \downarrow\}$;

at time t , α has a T -output iff at one or several of the external exits of the T -net component a signal is appearing at that time;

at time t , α has a T -input iff at one or several of the external entrances of the T -net component a signal is appearing at that time;

at time t , α has a causal T -input iff α has a T -input at that time and the earliest signal subsequently appearing at place 1 of the I -module happens to reach the I -module being in state l .

A causal T -input will thus give rise to the switching from l to h whereby, in particular, at a time t' ($t' > t$) a signal will appear at the CLOCK exit; as stated above, the delay occurring between t and t' is v .

In the following, the interplay between the T -net components and the cells of a cellular automaton will be considered. A clock pulse signal, leaving a certain T -net component by the CLOCK exit, will initiate a state transition of the cell which, after the delay s (see (1)), will terminate with a "ready"-signal entering the T -net component by the READY entrance whereby the I -module is re-enabled, i.e. switched into state l . To distinguish some characteristic steps in such a process we mark by indices the corresponding instants. For cell α , denote by

t_k^α the instant of the k -th causal T -input at α

t_k^α the instant of the subsequent k -th CLOCK output at α

r_k^α the instant of the k -th READY input at α

r_k^α the instant of the subsequent k -th re-enabling of the I -module of α .

The events $\langle t_k^\alpha \rangle$, $\langle r_k^\alpha \rangle$, and $\langle r_k^\alpha \rangle$ occurring at those instants are ordered in time in the manner shown; each is represented by the appearance of a signal at a certain section place in the net, except for the last event in such a chain, $\langle r_k^\alpha \rangle$, which is represented by the transition from T -state \downarrow to T -state l . Note that a causal T -input could be represented by several signals entering the T -net component at different times (and places) but resulting in one signal appearing at place 1 of the I -module. Thus we fix, in addition, $\langle t_k^\alpha \rangle$ to be represented by the earliest signal in question.

Together with the CLOCK signal, appearing with an event $\langle t_k^\alpha \rangle$, four signals appear, in parallel, at the external output places of the T -net component, possibly but not necessarily at the same time, say at the instants $\bar{t}_{1,k}^\alpha$, $\bar{t}_{2,k}^\alpha$, $\bar{t}_{3,k}^\alpha$, $\bar{t}_{4,k}^\alpha$. Since the construction of the T -net is done by identifying corresponding external input/output places of neighbouring T -net components, each of the associated events $\langle \bar{t}_{1,k}^\alpha \rangle$, ... represents a T -input for a neighbour cell of α . It is then possible that such a T -output, say $\langle \bar{t}_{1,k}^\alpha \rangle$, is a causal T -input for some neighbour cell β of α : $\langle t_l^\beta \rangle$. This identity of such two events is denoted by $\langle t_l^\beta \rangle \equiv \langle \bar{t}_{1,k}^\alpha \rangle$, i.e. in particular $t_l^\beta = \bar{t}_{1,k}^\alpha$.

It is assumed in the sequel that, in each case, the appearance of a CLOCK signal is the event latest in that parallelism, i.e.

$$t_k^a = \max \{t_k^a, \bar{t}_{1,k}^a, \dots, \bar{t}_{4,k}^a\} \tag{5}$$

and, in addition,

$$\max_i |t_k^a - \bar{t}_{i,k}^a| < v_{\min} \tag{6}$$

If no confusion arises the k -index will be suppressed in the following. To distinguish the notations for a cellular calculus and the corresponding device we write Z in the first case and in the second case \hat{Z} .

4. The structure (\hat{Z} , T)

By use of the concept of T -net we want to show how to perform adequate execution of deduction steps in simultaneous cellular calculi.

Assume a cellular automaton $\hat{Z} = (Z^2, N, Q, \Sigma)$ with von Neumann neighbourhood is given, executing such a calculus in case all cells undergo synchronous state transitions. A state transition of a cell α will be initiated by a clock pulse signal $\langle t^a \rangle$, and a "ready"-signal shall appear at the moment of its termination. Following ARBIB [1, p. 375] we will assume here that a state transition of cell α is completed in three phases (see Fig. 3):

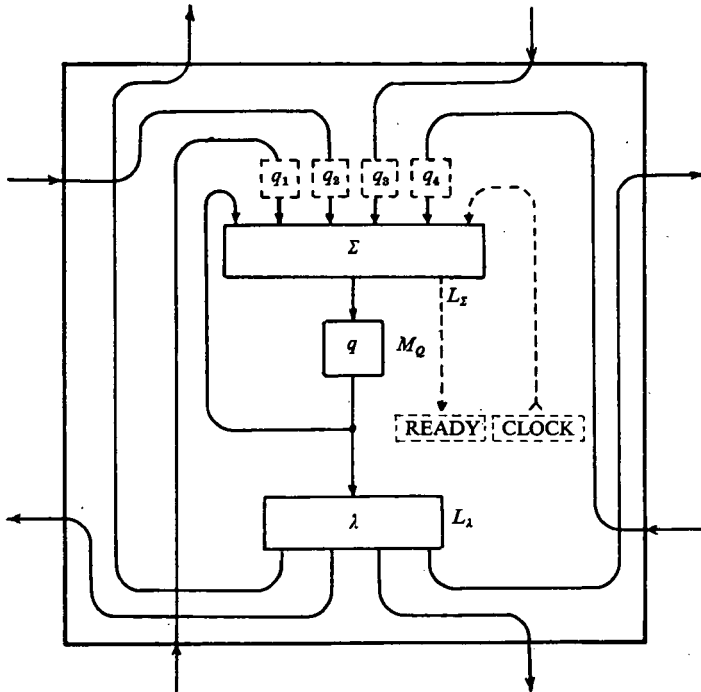


Fig. 3

(i) 'POOL': The relevant neighbourhood information given at time t^a : q_1, q_2, q_3, q_4 , is pooled in suitable registers;

(ii) 'EXECUTE': Based on these and its own state q an applicable rule is executed, by a logic L_x ;

(iii) 'MOVE': The resulting successor state symbol q' is moved to a memory register, M_Q , whereat the predecesing state symbol q is deleted.

From this it follows that the "old" state symbol of the cell is displayed to its neighbours, by an output logic L_x , till the end of phase (ii), i.e., always from the end of phase (iii) of a state transition to the end of phase (ii) of the *next* transition, a cell is in a *well-defined* state.

Consider now identical copies of a cell of this type and the T -net component, as introduced above, being one-to-one compounded by identification of the internal places CLOCK and READY, and then, as usual, being arranged at the lattice points of the plane. While denoting the cellular automaton, bare of the T -net components, by \hat{Z} , the new structure originating from this process is named (\hat{Z}, T) . It is of *homogeneous* compound, as sketched in Fig. 4 (dotted lines indicate the junctions of T -net components).

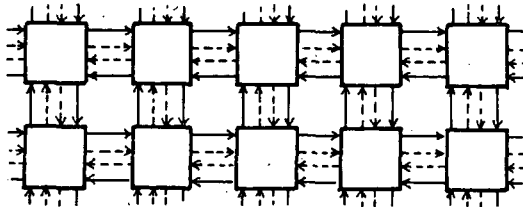


Fig. 4

The interplay of a T -net component and the associated cell is now explained. From a causal T -input $\langle t^a \rangle$ at a cell α being in T -state l a clock pulse signal output $\langle t^a \rangle$ will result which, by way of the CLOCK junction, effects the initiation and execution of a state transition of α , whereby the T -net component is in T -state h . On the termination of the state transition, by way of the READY junction, a READY input $\langle r^a \rangle$ effects the removing to the T -state l , occurring as event $\langle r^a \rangle$, whereat the cell is enabled for a new causal T -input.

In a schematic way, taking into account the durations s , v , and \bar{v} of subprocesses, as introduced earlier, the mode of operation of a cell in the (\hat{Z}, T) -structure is outlined below in Fig. 5.

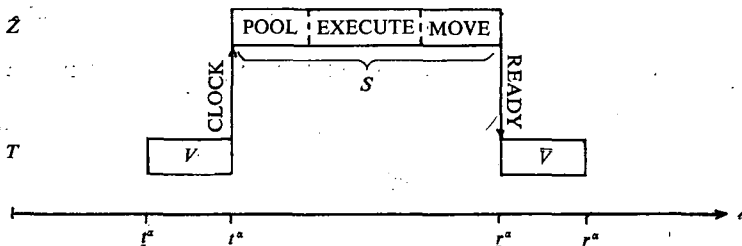


Fig. 5

On considering the \hat{Z} -structure, we see that the operation mode of a cell is characterized by two time values:

- a) the bounded switching duration² s
- b) ε , the maximum amount admissible for clock pulses to differ in time, according to definition 3.

For illustration of this, Fig. 6 is given where α and β are neighbouring cells; in phase (i) cell β sees α in a well-defined state, and vice versa.

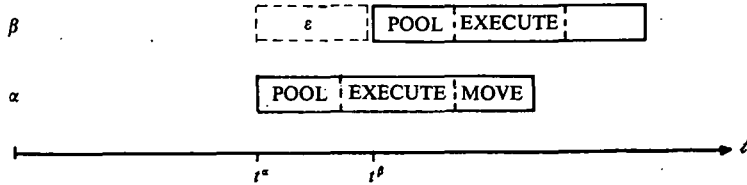


Fig. 6

On considering the T -net structure, we note that the values characterizing a T -net component are

- c) the bounded delay v occurring between a causal T -input and the clock pulse (i.e. the CLOCK output) effected by this;
- d) the bounded delay \bar{v} which occurs with the re-enabling of the T -net component, i.e. the switching from T -state h to l .

These four time values s , ε , v , and \bar{v} have to be considered as a whole in the (\hat{Z}, T) -structure. For the following, it is required that

$$v_{\max} \leq \varepsilon \quad (7)$$

which means that the maximum delay occurring in a T -net component does not exceed the admissible difference of the clock pulse instants of neighbouring cells, and

$$v_{\max} \leq s_{\min} + \bar{v}_{\min} \quad (8)$$

which is in particular satisfied by (7).

We now investigate the distribution of clock signals in a given polyautomaton (\hat{Z}, T) underlying the assumptions stated above, with a given initial configuration of \hat{Z} , i.e. all cells are in an initial state which is stored in M_0 . It is assumed that the T -net is clear of signals, and that all cells are in T -state l .

At first, (\hat{Z}, T) will be considered under *input of a single clock signal*. If an *adequate* execution of the corresponding simultaneous calculus Z is expected, it must be shown that, from this single signal, *one* deduction step in the calculus is put into execution (P1).

At time $t_1=0$ (referring to the origin) a solitary T -input shall be given to the origin cell, from an external clock. This T -input is causal for a clock pulse occurring after a delay v by which a state transition of the origin cell is initiated, while all

² In the non-deterministic case it is assumed that an applicable rule is executed within that duration.

neighbours (being in the T -state l) have previously had a T -input, which, at each cell, again gives rise to a clock pulse etc. This process can be interpreted as the propagation of a clock signal, introduced into the T -net at the origin, by signal offsprings which appear as an individual clock pulse for each cell α , at a certain time t^α . Whether a cell at which has already occurred a clock pulse event can have a repeated causal T -input, is to be elucidated.

By way of the T -net components an original signal is thus propagated in all directions of the plane, whereby in the cells passed, state transitions are initiated. At individual cells different delay, lying between v_{\min} and v_{\max} , can thereby appear (3). The actual course of a signal frontage (of first occurrence of a clock pulse) at a given time t is located within bounds determined by v_{\min} , v_{\max} , and t , as follows. If defining the distance between two cells α and β with coordinates (a_1, a_2) resp. (b_1, b_2) by

$$d(\alpha, \beta) = |a_1 - b_1| + |a_2 - b_2|$$

then the maximum distance from the origin 0 of cells α which had a clock pulse $\langle t^\alpha \rangle$ until time $t (t^\alpha \cong t)$ lies between

$$d_{\max}^t(\alpha, 0) = \left[\frac{t}{v_{\min}} \right] - 1 \quad (t \cong v_{\min})$$

$$d_{\min}^t(\alpha, 0) = \left[\frac{t}{v_{\max}} \right] - 1 \quad (t \cong v_{\max})$$

where $[]$ denotes the greatest integer $z: z < \frac{t}{v}$. In the following example an actual signal distribution possible under the above assumptions will be demonstrated.

Consider the T -net of a (Z, T) -structure for which it is assumed that $v_{\min} = 1$ and $v_{\max} = 3$. A causal T -input $\langle t^\alpha \rangle$ for a cell α is given by the earliest T -output $\langle t_v^\beta \rangle$ of a neighbour β of α . We assume here, for simplicity, that, for all β , $t^\beta = i_1^\beta = \dots = i_4^\beta$. Thus we have, in the above case, $i_v^\beta = t^\beta = t^\alpha$. In Fig. 7 a section of the cellular plane is shown where the numbers in a field for a cell α represent the following values:



instant of a causal T -input at α delay occurring between t^α and t^α instant of the clock pulse at α

From a T -input at time 0 at the origin, and by occurrence of the specified delays, the process displayed in Fig. 7 will arise. The sequence of the appearances of signal offsprings, represented by clock pulses at the individual cells, can be read along the clock pulse instants.

We denote by F_t the set of all cells that have a clock pulse up to time t ; $F_t = \{ \alpha | \exists t^\alpha (t^\alpha \cong t \& \langle t^\alpha \rangle) \}$. From the way of signal distribution it is clear that, for any t , F_t — conceived as a subset of the cellular plane — is connected and that, for $t_1, t_2 \in t, t_1 < t_2$ implies $F_{t_1} \subseteq F_{t_2}$. From the above-stated example, secondly, it follows immediately that F_t can be of a “genus” greater than zero (consider F_7 in the example).

0 ① 1	1 ① 2	2 ① 3	3 ① 4	4 ① 5	5 ① 6
1 ① 2	2 ③ 5	3 ③ 6	4 ① 5	5 ① 6	
2 ① 3	3 ③ 6	6 ③ 9	5 ① 6		
3 ① 4	4 ① 5	5 ① 6			
4 ① 5	5 ① 6				
5 ① 6					

Fig. 7

5. Locally synchronous performance

We now proceed to state the main result of this paper.

Theorem. For a given simultaneous cellular calculus $Z=(Z^2, N, Q, \Sigma)$ with von Neumann neighbourhood, let (\hat{Z}, T) be the corresponding polyautomaton, and let all cells of (\hat{Z}, T) be in T -state l . At time 0 of the observation time $t=(\mathbf{R}, \cong, 0)$ a clock signal is introduced as T -input into the origin cell. Then it follows that:

- For each cell of (\hat{Z}, T) there exists an instant t where it has a clock pulse.
- The earliest clock pulses (within t) of every two neighbouring cells of (\hat{Z}, T) differ in time at most by v_{\max} .
- No cell of (\hat{Z}, T) has more than one clock pulse.

Proof. (a) From the foregoing discussion it follows that a cell α with coordinates (x, y) has a clock pulse at the latest at time $t=(|x|+|y|+1) \cdot v_{\max}$, where $0 < v_{\max} < \infty$.

(b) Sketch: Of any two neighbouring cells, each of which still has not had a clock pulse, the one which has a clock pulse first gives a T -input to the other which will be causal (and thus will lead, within the delay v_{\max} , to a clock pulse of the second cell), unless this one had an earlier causal T -input from elsewhere (which would then lead to an earlier occurring clock pulse). The clock pulses differ in time then at most by v_{\max} .

Let now α and β be any two neighbouring cells in T -state l , none of which has already had a clock pulse. From (a) there exists, for each cell, a clock pulse instant.

1st case. α and β have clock pulses at the same time. Then the difference is $0 < v_{\max}$.

2nd case. Without loss of generality, let α have a clock pulse first, at time t^α . Thus, by (5), at the latest at the same time as $\langle t^\alpha \rangle$ a T -output of α will appear which is

a T -input for β . In the sequel, we denote such a T -output of α for β by $\langle \bar{t}_{j(\beta)}^\alpha \rangle$. According to (6), $|t^\alpha - \bar{t}_{j(\beta)}^\alpha| < v_{\min}$.

Hence, β has a clock pulse, at a time t^β ($t^\alpha < t^\beta$), for which either (i) the T -output $\langle \bar{t}_{j(\beta)}^\alpha \rangle$, linked with the event $\langle t^\alpha \rangle$, is causal, i.e. $\langle t^\beta \rangle \equiv \langle \bar{t}_{j(\beta)}^\alpha \rangle$, or (ii) a T -output of another neighbour cell of β , having occurred earlier than $\langle \bar{t}_{j(\beta)}^\alpha \rangle$, or (iii) several synchronous T -outputs of neighbours of β other than α having occurred earlier than $\langle \bar{t}_{j(\beta)}^\alpha \rangle$, or (iv) $\langle \bar{t}_{j(\beta)}^\alpha \rangle$ and one or more T -outputs of neighbours of β appearing at the same time as $\langle \bar{t}_{j(\beta)}^\alpha \rangle$; see the representation of an event $\langle t_k^\alpha \rangle$ as pointed out earlier.

The clock pulse of β will occur at the latest after the delay v_{\max} following a causal T -input, thus $t^\beta \leq \bar{t}_{j(\beta)}^\alpha + v_{\max}$.

Cases (i) and (iv) apply to: $t^\beta = \bar{t}_{j(\beta)}^\alpha$ ($\bar{t}_{j(\beta)}^\alpha \leq t^\alpha$),
 then $t^\beta + v_{\max} \leq t^\alpha + v_{\max}$,
 and thus $t^\beta = t^\alpha + v_{\max}$,
 equivalent to $t^\beta - t^\alpha \leq v_{\max}$.
 Hence, by $t^\alpha < t^\beta$, $|t^\beta - t^\alpha| \leq v_{\max}$.

Cases (ii) and (iii) apply to: $t^\beta < \bar{t}_{j(\beta)}^\alpha$ ($\bar{t}_{j(\beta)}^\alpha \leq t^\alpha$).
 Here we have $t^\beta + v_{\max} < t^\alpha + v_{\max}$
 and it follows analogously: $|t^\beta - t^\alpha| < v_{\max}$.

Hence, in each case the earliest clock pulse instants of every two neighbouring cells differ at most by v_{\max} , what had to be shown for (b).

Before proving (c) we formulate with the results obtained up to now the *rules for the distribution of clock signals by a T -net*:

$$(T0) \quad \exists t^0 (0 + v_{\min} \leq t^0 \leq 0 + v_{\max} \ \& \ \langle t^0 \rangle)$$

$$(T1) \quad \text{For } t^\alpha > t^0:$$

$$\langle t^\alpha \rangle \Rightarrow \exists \beta (\beta \in N(\alpha) \setminus \{\alpha\} \ \& \\ \exists t^\beta (t^\beta < t^\alpha \ \& \ \langle t^\beta \rangle \ \& \ 0 < |t^\alpha - t^\beta| \leq v_{\max} \ \& \ \langle \bar{t}_{j(\alpha)}^\beta \rangle \equiv \langle t^\alpha \rangle))$$

$$(T2) \quad \langle t^\alpha \rangle \Rightarrow \forall \beta (\beta \in N(\alpha) \setminus \{\alpha\} \Rightarrow \exists j(\beta) (j(\beta) \in \{1, \dots, 4\} \ \&$$

$$\langle \bar{t}_{j(\beta)}^\alpha \rangle \text{ is } T\text{-input for } \beta \ \& \ (\beta \text{ in } T\text{-state } l \text{ at } \bar{t}_{j(\beta)}^\alpha) \Rightarrow$$

$$\exists t^\beta (t^\alpha < t^\beta \ \& \ \langle t^\beta \rangle \ \& \ 0 < |t^\alpha - t^\beta| \leq v_{\max})))$$

(T3) Only by (T0), (T1), and (T2) can there occur clock pulse events.

(T0) means that the origin cell 0 has, at time t^0 , a clock pulse event (by hypothesis of the theorem that there is given an external T -input to the origin at time 0, to be interpreted as the input of a clock signal).

(T1) concerns the *cause* of the appearance of a clock signal at a given cell α and means that, for the occurrence of a clock pulse event at α (with exception of $\alpha=0$), a foregoing (at most by v_{\max}) clock pulse event of a neighbour cell different from α is necessary which is linked with a causal T -input for α .

(T2) concerns the propagation of a clock signal which has appeared at α , and means that, succeeding a clock pulse event $\langle t^\alpha \rangle$ there occurs, at the latest by time v_{\max} , a clock pulse event $\langle t^\beta \rangle$ at all cells β being in the T -state l at time $\bar{t}_{j(\beta)}^\alpha$ (whereby

not necessarily $\langle \bar{t}_{j(\beta)}^\alpha \rangle \equiv \langle t^\beta \rangle$, see (ii) in the second case above, it being allowed that $t^\beta < \bar{t}_{j(\beta)}^\alpha$.

By way of the construction of T -net and the assumptions made, only in accordance with (T0), (T1), and (T2) can there occur clock pulses. The rules are therefore completed by (T3).

We now proceed to prove (c). For the occurrence of a clock pulse at a cell α a foregoing causal T -input is necessary which, by (T1), appears as T -output of a distinct neighbour cell of α (except for $\langle t^0 \rangle$). It will be shown for (c) that no cell having had a first clock pulse can receive a new causal T -input.

Consider, for arbitrary $t \in \mathcal{t}$, the set G_t of cells which have *exactly* one clock pulse until time t :

$$G_t = \{ \alpha \mid \exists t^\alpha (t^\alpha \leq t \ \& \ \langle t^\alpha \rangle) \ \& \ \neg \exists t_1^\alpha (t_1^\alpha \leq t \ \& \ t_1^\alpha \neq t^\alpha \ \& \ \langle t_1^\alpha \rangle) \}.$$

First we show:

Lemma 1. No cell β having a first clock pulse at time t^β can thereby yield a causal T -input for a cell $\alpha \in G_{t^\beta}$.

For proving this we distinguish three cases, for any $\beta \in Z^2$ with the first clock pulse at t^β .

1st case. $\alpha \in G_{t^\beta}$ & $\alpha \notin N(\beta)$

No T -input for α is linked with $\langle t^\beta \rangle$, see (T2), (T3).

2nd case. $\alpha \in G_{t^\beta}$ & $\alpha = \beta$

The case that β yields a T -input for itself is excluded, see (T2), (T3), particularly in case $\beta = \alpha = 0$.

3rd case. $\alpha \in G_{t^\beta}$ & $\alpha \in N(\beta) \setminus \{ \beta \}$

By (T2) at the latest with $\langle t^\beta \rangle$ a T -input for α has appeared, as T -output of β : $\langle \bar{t}_{j(\alpha)}^\beta \rangle$. As $\alpha \in G_{t^\beta}$, there exists exactly one t^α : $t^\alpha \leq t$ & $\langle t^\alpha \rangle$. We show that $\langle \bar{t}_{j(\alpha)}^\beta \rangle$ is neither causal (i) for $\langle t^\alpha \rangle$ nor (ii) for clock pulse events possibly occurring later.

(i) By $t^\alpha \leq t^\beta$:
 from (6) it follows $t^\alpha - v_{\min} \leq t^\beta - v_{\min}$
 and from (3) $t^\alpha - v_{\max} \leq t^\alpha \leq t^\alpha - v_{\min}$
 Hence $t^\alpha < \bar{t}_{j(\alpha)}^\beta$
 and thus $\langle \bar{t}_{j(\alpha)}^\beta \rangle \neq \langle t^\alpha \rangle$, i.e. $\langle \bar{t}_{j(\alpha)}^\beta \rangle$ is not causal for $\langle t^\alpha \rangle$.

Remark. A clock pulse $\langle t^\beta \rangle$ cannot therefore be causal for a clock pulse $\langle t^\alpha \rangle$ occurring either at the same or an earlier time. If conversely $\langle \bar{t}_{j(\alpha)}^\beta \rangle \equiv \langle t^\alpha \rangle$ holds, it follows that $t^\alpha - t^\beta > 0$. This fact, ensured by (6), is already considered in the above formulation or rules (T1), (T2).

For (ii) we show that $\langle \bar{t}_{j(\alpha)}^\beta \rangle$ is not causal for a $\langle t_1^\alpha \rangle$, $t^\alpha < t_1^\alpha$. As assumed, $\langle t^\alpha \rangle$ and $\langle t^\beta \rangle$ are the earliest clock pulses of the neighbouring cells α and β . From $t^\alpha \leq t^\beta$ it follows by (b):

$$\begin{aligned} 0 &\leq t^\beta - t^\alpha \leq v_{\max} \\ \text{and upon (8)} \quad 0 &\leq t^\beta - t^\alpha < s_{\min} + \bar{v}_{\min} \\ \text{i.e.} \quad t^\alpha &\leq t^\beta < t^\alpha + s_{\min} + \bar{v}_{\min} \end{aligned}$$

However, α is *not* in the T -state l up to the re-enabling of its I -module, at the least until time $t^\alpha + s_{\min} + \bar{v}_{\min}$ and, hence, at time t^β (see Fig. 5); i.e. the T -input

for $\alpha: \langle t_{j(\alpha)}^\beta \rangle$ occurring at the latest with $\langle t^\beta \rangle$ is not causal for a new clock pulse event at α . This ends the proof of Lemma 1.

In Lemma 1, it is referred to sets G_{t^β} . Certainly, for all t , $G_t \subseteq F_t$. However, for certain t , $G_t \subset F_t$ could be, i.e. nothing can be said about whether the Lemma regards, in every case, all cells which had a clock pulse up to time t^β generally. It is thus to be shown that no cell of F_t can have more than one causal T -input. By hypothesis of the theorem, only one external T -input is given to the origin cell; i.e. for any cell α only such possibly causal T -inputs need to be considered which appear in the neighbourhood of α as T -outputs, linked with clock pulse events, see (T0), (T1). In the following, we denote by $N(S)$ the *extended neighbourhood set* of a finite subset S of \mathbf{Z}^2 : $N(S) = \bigcup_{\alpha \in S} N(\alpha)$, where $N(\alpha)$ is the set of neighbours of a single cell α as given above (no confusion will arise by using the same symbol for both).

Lemma 2. For every t : $F_t = G_t$

Proof. We define, within \mathcal{t} , special instants t_1, t_2, t_3, \dots as follows.

$$t_1 = t^0$$

$$t_{n+1} = \min \{t^\alpha | t_n < t^\alpha \ \& \ \exists \alpha \langle t^\alpha \rangle\}$$

Each instant t_n is thus a clock pulse instant of one or several cells in \mathbf{Z}^2 , and, for all n , $F_{t_n} \subseteq F_{t_{n+1}}$. The above minimum always exists arising from the following fact: In each case, only a finite number of cells α comes into question for the next clock pulse (within \mathcal{t}) caused by the cells β of F_{t_n} , namely, the cells $\alpha: \alpha \in N(F_{t_n})$. Thereby F_{t_n} and thus $N(F_{t_n})$ are finite sets, at each instant, because from each clock pulse of any cell only clock pulse events of its finite number of neighbour cells can be caused, see (T2). So $(t_n)_{n \in \mathbf{N}}$ is well-ordered within $\mathcal{t}: t_1 < t_2 < \dots$, and $(F_{t_n})_{n \in \mathbf{N}}$ is well-ordered concerning ' \subseteq '; and the following statement holds:

$$F_{t_1} = \{0\}; \quad \bigcup_n F_{t_n} = \mathbf{Z}^2.$$

We now claim

$$\forall n: F_{t_n} = G_{t_n} \quad (n \in \mathbf{N}).$$

Proof by induction on n . Obviously, $F_{t_1} = G_{t_1} = \{0\}$ (base of induction).

Induction hypothesis: Let $F_{t_n} = G_{t_n}$.

Induction step. As $G_{t_{n+1}} \subseteq F_{t_{n+1}}$, it remains to be shown that

$$F_{t_{n+1}} \setminus G_{t_{n+1}} = \emptyset.$$

Supposing there exists $\alpha: \alpha \in F_{t_{n+1}} \setminus G_{t_{n+1}}$, then α has exactly one clock pulse $\langle t^\alpha \rangle$ up to time t_n , for it is $F_{t_n} \subseteq F_{t_{n+1}}$ and, by induction hypothesis, $F_{t_n} = G_{t_n}$, i.e. $\alpha \in G_{t_n}$; and α has a second clock pulse $\langle t_1^\alpha \rangle$ at time t_{n+1} , for by construction of (t_n) a clock pulse cannot exist between t_n and t_{n+1} . Hence, by (T1),

$$\exists \beta (\beta \in N(\alpha) \setminus \{\alpha\} \ \& \ \exists t^\beta (\langle t^\beta \rangle \ \& \ t^\beta \cong t_n < t_1^\alpha \ \& \\ |t_1^\alpha - t^\beta| \cong v_{\max} \ \& \ \langle t_{j(\alpha)}^\beta \rangle \equiv \langle t_1^\alpha \rangle).$$

Thus α must have received a causal T -input before $t_{n+1} (t_{n+1} = t_1^\alpha)$, linked with a clock pulse of a neighbour cell β .

1st case. $\langle t^\beta \rangle$ is not the first clock pulse of $\beta (t^\beta \cong t_n)$.
That would mean: $\beta \in F_{t_n} \setminus G_{t_n}$ which is a contradiction of the induction hypothesis.

2nd case. $\langle t^\beta \rangle$ is the first clock pulse of $\beta (t^\beta \cong t_n)$.

(i) $t^\beta = t_n$. From Lemma 1 it follows that no cell of G_{t^β} , $G_{t^\beta} = G_{t_n}$, can get a causal T -input by $\langle t^\beta \rangle$. But from the above assumption it follows that $\langle t_{j(\alpha)}^\beta \rangle \equiv \langle t_1^\alpha \rangle$ & $\alpha \in G_{t_n}$ which is impossible by Lemma 1 and hence results in a contradiction.

(ii) $t^\beta < t_n$. Then $\alpha \in G_{t^\beta}$ which is shown as follows. Between two clock pulses of α , $\langle t^\alpha \rangle$ and $\langle t_1^\alpha \rangle$, there are located the events $\langle r^\alpha \rangle$, and $\langle t_1^\alpha \rangle$, as was pointed out in the third section. Thus, it is valid for the distance in time of the two clock pulses:

$$|t^\alpha - t_1^\alpha| > s_{\min} + \bar{v}_{\min} + v_{\min}$$

for it must be $r^\alpha < t_1^\alpha$; furthermore:

$$|t^\beta - t_1^\alpha| \leq v_{\max}$$

and by $t^\beta < t_1^\alpha$:

$$t_1^\alpha - t^\beta \leq v_{\max}$$

resp.

$$-v_{\max} \leq t^\beta - t_1^\alpha$$

Thus, by $t^\alpha < t_1^\alpha$:

$$s_{\min} + \bar{v}_{\min} + v_{\min} < t_1^\alpha - t^\alpha$$

Addition of the last two inequalities yields

$$s_{\min} + \bar{v}_{\min} + v_{\min} - v_{\max} < t^\beta - t^\alpha$$

and by (8):

$$0 < t^\beta - t^\alpha$$

thus $t^\alpha < t^\beta$ whence $\alpha \in G_{t^\beta}$. From Lemma 1 then again a contradiction results. Hence, $F_{t_{n+1}} = G_{t_{n+1}}$. From this, the above assertion is proved: For all n , $F_{t_n} = G_{t_n}$ is valid.

The proof of Lemma 2 is now completed as follows. By construction of (t_n) :

$$\forall n \forall t (t_n \leq t < t_{n+1} \Rightarrow F_t = F_{t_n} \& G_t = G_{t_n})$$

It holds that: $\bigcup_n [t_n, t_{n+1}) = \{t | t \in \ell \& t_1 \leq t\}$. But there is no clock pulse before t_1 , it is $F_t = G_t = \emptyset$ for $t \in [0, t_1)$. Thus, for all $t \in \ell$, $F_t = G_t$. This ends the proof of Lemma 2.

Altogether we have now:

$$\bigcup_{t \in \ell} G_t = \bigcup_{n \in \mathbb{N}} G_{t_n} = \bigcup_{n \in \mathbb{N}} F_{t_n} = \mathbf{Z}^2$$

i.e. every cell $\alpha \in \mathbf{Z}^2$ has exactly one clock pulse, which concludes the proof of the theorem.

From this, the above stated organization problem (P1) is solved, and it is now to be examined how to proceed on sequences of deduction steps to perform an adequate execution of a simultaneous cellular calculus by a (\hat{Z}, T) -structure (P2). Thus, (\hat{Z}, T) will now be considered under *input of a sequence of clock signals*.

As was assumed here (4), the delays appearing in the T -net components do not vary in time (but of course it is allowed that the individual delays of different com-

ponents vary within $[v_{\min}, v_{\max}]$. In this case the distribution of any subsequent clock signal will occur in exactly the same manner as in case of the first signal, provided that each cell that receives a signal offspring, generated by a signal T_{k+1} following T_k , has already re-entered T -state l . This is, however, obviously satisfied under the above assumption, if only the input of a successor signal T_{k+1} , given as a T -input $\langle t_{k+1}^0 \rangle$ to the origin cell, takes place *after* the k -th re-enabling of the origin cell by the event $\langle r_k^0 \rangle$. To get from the event $\langle t_k^0 \rangle$ to $\langle r_k^0 \rangle$ will take the minimum duration $D_{\min} := v_{\max} + s_{\max} + \bar{v}_{\max}$ (see Fig. 5), so that the above conditions are satisfied if

$$t_{k+1}^0 \cong t_k^0 + D_{\min} \quad (k = 1, 2, 3, \dots) \quad (9)$$

In this case, by each clock signal T_k the k -th deduction step in the calculus is put into execution which can be seen from the following facts.

Upon occurrence of a clock pulse $\langle t_2^0 \rangle$, in correspondence with a second clock signal T_2 , all neighbours of a cell α will have executed their first state transition, including re-entry of T -state l , and be in a welldefined state, keeping ready for a new causal T -input. That means, even though not every cell of (\hat{Z}, T) has executed a state transition at the time of input of T_2 , this actually has happened *locally* (within one neighbourhood) wherever a clock pulse event caused by T_2 occurs.

The proof of the theorem relied exclusively on *local arguments* (in each case only neighbouring cells have been considered), so that the hypotheses of the theorem have only to be valid locally, i.e. the statements proven in the theorem for a single clock signal input are implied in the same manner for a second signal input etc. Therefore, from statement (b) of the theorem, for any k the clock pulse events of neighbouring cells differ in time at most by v_{\max} ($k=1, 2, 3, \dots$). Under these assumptions, hence, the conditions of definition 3 are satisfied, and we get the

Corollary. Provided that the component delays appearing in the T -net are *time-invariant*, it follows that, for given \hat{Z} , the set of all cells in structure (\hat{Z}, T) works locally synchronous.

But that means: With each clock signal, (\hat{Z}, T) puts into execution one deduction step in the corresponding simultaneous calculus, in a locally synchronous performance. Thereby, it is no obstacle that, as was shown above, at certain times the region in Z^2 being reached by clock signal offsprings can possibly be of "genus" greater than zero: As far as at time t clock pulse events, generated by T_k but not T_{k+1} , have occurred, and state transitions initiated by this are executed, there is present the partial result of the k -th deduction step in the simultaneous calculus that arises from an initial pattern.

Thus it is proved: The (sufficient) condition of synchrony is not necessary; an adequate execution of cellular calculi in the sense of John von Neumann is possible by way of locally synchronous cellular automata.

6. T -synchrony

The execution of a simultaneous cellular calculus in a structure (\hat{Z}, T) is by no means attained through synchrony of the local substitutions, referring to a global or discrete time scale as introduced in the second section. This seems to make impossible the view of a global state of a locally synchronous cellular automaton: Compared with the global time t referring to the origin, (\hat{Z}, T) does not perform the com-

putation of a successor "configuration" at once — precisely: between two subsequent instants of the discrete time scale \tilde{t} — but instead does this successively from the inner to the outer region of Z^2 . However, concurrently with the foregoing computation, the computation of another successor can be started after short delay $D \cong D_{\min}$ as given by (9), before the first computation is completed.

Thus at no time (of \tilde{t}) a global state, as the total result of a deduction step, appears in a structure (\tilde{Z}, T) . Note that the definition of a configuration, as well as of a global transformation, was based on a concept of synchrony, allowing the simultaneous consideration of all cells of a cellular automaton at "global" instants, more precisely, at simultaneously considered local instants; see the discussion of these matters in the second section.

But, as was shown by the main result, a simultaneous cellular calculus is actually put into execution in structure (\tilde{Z}, T) . Furthermore, under the above assumption of time-invariant component delays in the T -net, the execution of a next deduction step can, at any given case, be initialized after a delay nearly as small as in the case of an ideal synchronous cellular automaton which is seen from the following.

Consider a cellular automaton \tilde{Z} working synchronously, in a hypothetical instance, by clock pulses occurring instantaneously at each cell whenever a clock signal is given to the automaton. (This could be visualized as a (\tilde{Z}, T) -structure with T -inputs occurring synchronously at each cell.) In this case, the minimum signal distance would be $\varepsilon + s_{\max}$, see (2).

In a locally synchronous cellular automaton (\tilde{Z}, T) , the minimum time distance of clock signals given to the origin cell is D_{\min} , as seen from (9). As $D_{\min} = v_{\max} + \bar{v}_{\max} + s_{\max}$, we have, by $\varepsilon' := v_{\max} + \bar{v}_{\max}$, a computation speed of the (\tilde{Z}, T) automaton which relates to the above case of instantaneous signal transmission! (Of course, in a finite run of a cellular automaton, corresponding to a sequence of a certain number of deduction steps in the calculus, special considerations have to be made of the time by which the total result of such a computation can be brought out, referring to \tilde{t} . The discussion of this is deferred to [9]; at the moment we are considering infinite runs.)

To attain a concept of a "configuration of a locally synchronous cellular automaton", we now introduce a *new concept of synchrony* which is based on local observation times: Corresponding to each clock signal T_k , we consider local time scales \tilde{t}_k^α for each individual cell α in a locally synchronous cellular automaton: $\tilde{t}_k^\alpha = (\mathbf{R}, \cong, \tilde{t}_k^\alpha)$, where $\tilde{t}_k^\alpha \in \tilde{t}$ ($k=1, 2, \dots$). Thus for each cell α , with each causal T -input $\langle \tilde{t}_k^\alpha \rangle$ a new observation time is beginning (" $\langle \tilde{t}_k^\alpha \rangle$ brings along the time"; compared with \tilde{t} , the starting points of the local scales of neighbouring cells differ, at any given time, at most by the maximum component delay, v_{\max}). Then, for each k , each cell α is in a well-defined state at time \tilde{t}_k^α (see section 4), before the k -th clock signal, represented by the k -th clock pulse $\langle \tilde{t}_k^\alpha \rangle$, $\tilde{t}_k^\alpha < \tilde{t}_k^\alpha$, appears at α .

In definition 3, the synchrony of clock pulse events was defined by coincidence of the times of their occurrences, with respect to \tilde{t} . (This could as well be applied to define the synchrony of causal T -inputs.) Here, we define the "synchrony" of events of causal T -inputs occurring in a T -net by covering the starting points of the local time scales that correspond to the same clock signal:

Definition 4. In a (\tilde{Z}, T) -structure, two events $\langle \tilde{t}_k^\alpha \rangle$ and $\langle \tilde{t}_l^\beta \rangle$, where $\tilde{t}_k^\alpha, \tilde{t}_l^\beta \in \tilde{t}$, are said to be *T-synchronous* iff $k=l$. To denote this we shall write $\langle \tilde{t}_k^\alpha \rangle$ *T-syn* $\langle \tilde{t}_l^\beta \rangle$.

The concept of T -synchrony is well-defined in a locally synchronous cellular automaton since by each clock signal T_k there occurs exactly one clock pulse event at each cell α and, hence, exactly one T -input $\langle t_k^\alpha \rangle$ causal for this. T -synchrony gives an equivalence relation, in a canonical way, dividing the set of causal T -inputs into a countable number of classes, each containing all causal T -inputs generated by the same clock signal. We define special sets t_k called (k -th) *time cuts*, by "cutting through the local times":

$$t_k := \{t_i^\alpha | \langle t_i^\alpha \rangle T\text{-syn} \langle t_k^0 \rangle\}$$

and order them by means of their natural sequence, i.e. along k . This concept makes possible a simultaneous consideration of the local time scales t_k^α , for each k (refer to the remarks given on synchrony, in section 2): While in the above hypothetic case there were considered *synchronous* events of clock pulse inputs at all cells, we consider now, in structure (Z, T) , the case of *T-synchronous* T -inputs at all cells. A time cut t_k contains exactly the instants of all events of k -th causal T -inputs, namely, the starting points of the local times t_k^α . To set up the relation to the global observation time t , we can also express the t_k by the following:

$$t_k = \{t_k^\alpha | t_k^\alpha \in t \ \& \ \exists \alpha \langle t_k^\alpha \rangle\}$$

An element of t_k can, of course, be the starting point of several local time scales $t_k^\alpha, t_k^\beta, \dots$ (with fixed k), in case that several of these T -synchronous events occur synchronously, in the original sense. Each time cut thus contains at most as many instants as the number of cells under consideration because, in locally synchronous working, no cell can get more than one k -th causal T -input ($k=1, 2, \dots$). On the other hand, each causal T -input lies in exactly one time cut.

By use of the concepts of T -synchrony and time cut we now proceed to explain the terms "configuration" and "global state transition of a locally synchronous cellular automaton".

With reference to any given time cut t_k , all cells of such an automaton are T -synchronously in well-defined states, and thus it makes sense to speak of a global state, or a configuration, of the locally synchronous cellular automaton "at time cut t_k ". The relation between the continuous time scale t and a discrete time scale to take as a basis for the consideration of global steps is now to be worked out. The times of input of clock signals T_1, T_2, \dots at the origin 0 were denoted by t_1^0, t_2^0, \dots , and from (9) it is, for all k , $|t_{k+1}^0 - t_k^0| \cong D_{\min} = \varepsilon' + s_{\max}$. So we could say: *Locally*, i.e. at each individual cell α , time passes by in discrete steps $t_1^\alpha \rightarrow t_2^\alpha \rightarrow \dots$; each time a step $t_k^\alpha \rightarrow t_{k+1}^\alpha$ has elapsed (by occurrence of $\langle t_{k+1}^\alpha \rangle$), α has once changed its state.

For a locally synchronous cellular automaton, we now define a discrete time scale the "instants" of which are *time cuts*:

$$\vec{t} = (\{t_k\}_{k \in \mathbb{N}}, \cong, t_1)$$

In section 2, we introduced a continuous time scale to explain locally synchronous working; now we have returned to a discrete time concept, by means of the equivalence relation of T -synchrony. While, in the synchronous case, saying that between each two subsequent instants t_1^0, t_2^0 of \vec{t} there takes place a global state transition of Z leading from a configuration c to a successor configuration c' , we could say that,

in a locally synchronous cellular automaton, between each two subsequent time cuts t_k, t_{k+1} of \vec{t} there takes place a global state transition of (\vec{Z}, T) from c to a c' .

We conclude this discussion by a remark added to clarify the above ideas. The concept of T -synchrony might seem to be a bit rough and only of technical value to re-open the view of a global state. But think of an imaginary observer gliding across the cellular plane, at the same bounded speed as is actually given by the component delays appearing in the T -net, along his course. Then, at any region of the plane, he would get the impression of a *static* global state from the fact that, during his observation, all cells passed are T -synchronously in well-defined states, corresponding to the so far obtained result of the execution of a deduction step in the simultaneous cellular calculus.

Abstract

It is shown that, for a cellular automaton in the sense of John von Neumann, the assumption of synchronous state transitions of all cells is not necessary but can be weakened to an assumption of locally synchronous working. An organization scheme is described which achieves locally synchronous performance in any cellular automaton of the von Neumann type. A special concept of synchrony is introduced which makes possible the consideration of configurations and global state transitions of locally synchronous cellular automata.

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