On homomorphically α_i-complete systems of automata

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In [2] there is introduced a family of semi-cascade products named α_i-products, where the index i is a nonnegative integer, which denotes the maximal admissible length of feedbacks. By results of F. GÉCSEG (see for example [3]) it can be seen that there exist no finite homomorphically complete systems with respect to the α_0- and α_1-products.

From [1] it follows that every automaton having n states can be represented (in a certain sense) by an α_0-product of automata, such that all components of this product are either two-state reset automata, or special n-state automata, named "standard automata". Using results of [4] we get that these "standard automata" can be embedded state-isomorphically into an α_2-product of two-state automata. Therefore, taking into consideration the fact that an α_0-product of α_2-products is an α_2-product, every automaton can be represented (in a certain sense) by an α_2-product of two-state automata.

In this paper we present a direct proof of this statement. By this result we receive that for every $i \geqq 2$ there exists a finite homomorphically complete system of automata with respect to the α_i-product. For the notions and notations that will not be defined here, we refer to the book [3].

By an *automaton* $\mathbf{A}=(X, A, Y, \delta, \lambda)$ we mean a finite Mealy-type automaton, where X, A and Y are the finite input, state and output sets, respectively; furthermore $\delta\colon A \times X \to A$ denotes the transition and $\lambda\colon A \times X \to Y$ is the output function.

Let $\mathbf{A}_t=(X_t, A_t, Y_t, \delta_t, \lambda_t)$ $(t=1, \dots, n)$ be a system of automata. Moreover, let X and Y be finite nonvoid sets and

$$\varphi\colon A_1 \times \dots \times A_n \times X \to X_1 \times \dots \times X_n, \quad \psi\colon A_1 \times \dots \times A_n \times X \to Y$$

mappings. We say that the automaton $\mathbf{A}=(A, X, Y, \delta, \lambda)$ with $A=A_1 \times \dots \times A_n$, $\lambda((a_1, \dots, a_n), x)=\psi(a_1, \dots, a_n, x)$

$$\delta((a_1, \dots, a_n), x) = \big(\delta_1(a_1, \varphi_1(a_1, \dots, a_n, x)), \dots, \delta_n(a_n, \varphi_n(a_1, \dots, a_n, x))\big),$$

is the *α_i-product* of $\mathbf{A}_t$ $(t=1, \dots, n)$ with respect to X Y, φ, ψ if φ can be given in the form $\varphi(a_1, \dots, a_n, x)=\big(\varphi_1(a_1, \dots, a_n, x), \dots, \varphi_n(a_1, \dots, a_n, x)\big)$, such that $\varphi_j (1 \leqq j \leqq n)$ is independent of states having indices greater than or equal to $j+i$, where i is a fixed nonnegative integer. For this product we shall use the short notation

$\mathbf{A}=\prod_{t=1}^{n} \mathbf{A}_t[X, Y, \varphi, \psi]$. The mappings φ and ψ are called *feedback function* and *output function*, respectively.

Let $\mathbf{A}, \mathbf{B}$ be a pair of automata. We say that $\mathbf{A}$ *can be embedded state-isomorphically* into $\mathbf{B}$ if $\mathbf{B}$ has an A-subautomaton $\mathbf{B}'$, such that $\mathbf{B}'$ is A-isomorphic to $\mathbf{A}$. If $\mathbf{B}$ has an A-subautomaton $\mathbf{B}'$, such that $\mathbf{B}'$ can be mapped A-homomorphically onto $\mathbf{A}$ then it is said that $\mathbf{A}$ *can be strongly covered* (or *can be represented*) by $\mathbf{B}$.

Take a nonnegative integer i. Any system Σ of automata is *homomorphically complete with respect to the α_i-product*, or briefly, Σ is *homomorphically α_i-complete* if every automaton can be strongly covered by an appropriate α_i-product of components from Σ. Moreover, the system Σ is *finite* if it has finite-many elements.

Consider an automaton $\mathbf{A}=(X, A, Y, \delta, \lambda)$ with n states. For an arbitrary positive integer $m \leqq n$ we say that $\mathbf{A}$ is *m-husked* if there exists an arrangement $a_1, \ldots, a_n$ of states in $\mathbf{A}$, such that for $a_l \in A$, $x \in X$, $l<m$ we have $\delta(a_l, x) \in \langle a_1, \ldots, a_{l+1}\rangle$. (Obviously, for $m=1$ this is a formal requirement. Therefore, all automata are 1-husked.)

If an automaton $\mathbf{A}$ with n states is n-husked then it is said to be *right-husked.* (We note that all $(n-1)$-husked automata with $n>1$ states need necessarily be right-husked.)

The following holds.

Lemma 1. Every m-husked automaton $\mathbf{A}$ having $n>m$ states can be strongly covered by a suitable α_0-product $\mathbf{M}=\prod_{t=1}^{2} \mathbf{A}_t[X, Y, \varphi, \psi]$ whose components satisfy the following conditions:

(i) $\mathbf{A}_1$ has $n-m$ states;

(ii) $\mathbf{A}_2$ is an $(m+1)$-husked automaton the number of states of which is equal to n.

Proof. Take an m-husked automaton $\mathbf{A}=(X, A, Y, \delta, \lambda)$ with $n>m$ number of states and let $a_1, \ldots, a_n$ be an arrangement of states in $\mathbf{A}$, such that for $a_l \in A$, $x \in X$, $l<m$ it holds that $\delta(a_l, x) \in \langle a_1, \ldots, a_{l+1}\rangle$. For any triplet $u, v, w \in \langle 1, \ldots, n\rangle$ we introduce the notation

$$a_{(u,v,w)} = \begin{cases} a_u & \text{if} \quad u \notin \langle v, w\rangle, \\ a_v & \text{if} \quad u = w, \\ a_w & \text{if} \quad u = v. \end{cases}$$

Construct the automata $\mathbf{A}_1=(X, B, B\times X, \delta_1, \lambda_1)$ and $\mathbf{A}_2=(B\times X, A, Y, \delta_2, \lambda_2)$ in the following way. $B=\langle m+1, \ldots, n\rangle$, furthermore, for every triplet $v \in B$, $a_l \in A$, $x \in X$

$$\delta_1(v, x) = \begin{cases} v & \text{if} \quad \delta(a_m, x) \in \langle a_1, \ldots, a_m\rangle, \\ w & \text{if} \quad \delta(a_m, x) \notin \langle a_1, \ldots, a_m\rangle \quad \text{and} \quad \delta(a_m, x) = a_w, \end{cases}$$

$$\delta_2(a_l, (v, x)) = \begin{cases} a_{(z,m+1,v)} & \text{if} \quad \delta(a_m, x) \in \langle a_1, \ldots, a_m\rangle \quad \text{and} \quad \delta(a_{(l,m+1,v)}, x) = a_z, \\ a_{(z,m+1,w)} & \text{if} \quad \delta(a_m, x) = a_w \notin \langle a_1, \ldots, a_m\rangle \\ & \qquad\qquad \text{and} \quad \delta(a_{(l,m+1,v)}, x) = a_z, \end{cases}$$

$$\lambda_1(v, x) = (v, x), \quad \lambda_2(a_l, (v, x)) = \lambda(a_{(l,m+1,v)}, x).$$

Define the α_0-product $\mathbf{M}=\sum_{t=1}^{2}\mathbf{A}_t[X, Y, \varphi, \psi]$, where in case of every pair $(v, a_l)\in B\times A$, $x\in X$

$$\varphi(v, a_l, x) = (x, (v, x)),$$

$$\psi(v, a_l, x) = \lambda_2(a_l, (v, x)).$$

By an elementary computation we obtain that the mapping μ: $B\times A\to A$ with $\mu(v, a_l)=a_{(l,m+1,v)}$ is an A-homomorphism of $\mathbf{M}$ onto $\mathbf{A}$. By the definitions of $\mathbf{A}_1$ and $\mathbf{A}_2$ this completes the proof of Lemma 1.

The following statement is trivial.

Lemma 2. Let $\langle\mathbf{A}_1, ..., \mathbf{A}_n\rangle$ and $\langle\mathbf{B}_1, ..., \mathbf{B}_m\rangle$ be arbitrary finite systems of automata. If any automaton $\mathbf{A}$ can be strongly covered by an α_0-product of components from $\langle\mathbf{A}_1, ..., \mathbf{A}_n\rangle$, moreover, an element $\mathbf{A}_t$ of $\langle\mathbf{A}_1, ..., \mathbf{A}_n\rangle$ can be strongly covered by an α_0-product of components from $\langle\mathbf{B}_1, ..., \mathbf{B}_m\rangle$ then $\mathbf{A}$ can be strongly covered by an α_0-product of components from $\langle\mathbf{A}_1, ..., \mathbf{A}_{t-1}, \mathbf{B}_1, ..., \mathbf{B}_m, \mathbf{A}_{t+1}, ..., A_n\rangle$.

Using Lemma 1 and Lemma 2 by an induction we get the following

Lemma 3. Every automaton $\mathbf{A}$ can be strongly covered by an α_0-product of right-husked automata having not more states than $\mathbf{A}$.

Lemma 4. Every right-husked automaton can be embedded state-isomorphically into and α_2-product of two-state automata.

Proof. Let $\mathbf{A}=(X, A, Y, \delta, \lambda)$ be an arbitrary right-husked automaton and take an arrangement $a_1, ..., a_n$ of its states with $\delta(a_t, x)\in\langle a_1, ..., a_{t+1}\rangle$ $(t=1, ..., n-1$, $x\in X)$. Consider the automaton $\mathbf{B}=(\langle u, v\rangle, \langle 0, 1\rangle, \langle z\rangle, \delta_{\mathbf{B}}, \lambda_{\mathbf{B}})$ where $\delta_{\mathbf{B}}(0, u)= =\delta_{\mathbf{B}}(1, v)=0$, $\delta_{\mathbf{B}}(0, v)=\delta_{\mathbf{B}}(1, u)=1$ and $\lambda_{\mathbf{B}}(j, x)=z$ for any $j\in\langle 0, 1\rangle, x\in\langle u, v\rangle$. Construct the α_2-product $\mathbf{C}=(X, C, Y, \delta_{\mathbf{C}}, \lambda_{\mathbf{C}})=\prod_{t=1}^{n}\mathbf{B}_t[X, Y, \varphi, \psi]$ with $\mathbf{B}_1= =...=\mathbf{B}_n=\mathbf{B}$ as follows. For any $1\leqq s\leqq n$, $(d_1, ..., d_n)\in\prod_{t=1}^{n}B_t$ and $x\in X$

$$\varphi_s(d_1, ..., d_n, x)=\begin{cases} v & \text{if } d_j=1,\ \delta(a_{n-j+1}, x)=a_{n-s+1} \text{ for some} \\ & \quad j\in\langle 1, ..., s-1, s+1\rangle\cap\langle 1, ..., n\rangle \text{ or} \\ & \quad d_s=1,\ \delta(a_{n-s+1}, x)\neq a_{n-s+1}, \\ u & \text{otherwise,}\end{cases}$$

$$\psi(d_1, ..., d_n, x)=\begin{cases}\lambda(a_{n-j+1}, x) & \text{if } d_j=1 \text{ for some } 1\leqq j\leqq n \\ & \qquad\text{and } \sum_{t=1}^{n} d_t=1, \\ \text{arbitrary fixed element of } Y \text{ otherwise.}\end{cases}$$

Denote C' the set of all elements $(d_1, ..., d_n)\in\prod_{t=1}^{n}B_t$ for which $\sum_{t=1}^{n}d_t=1$. It is clear that $\mathbf{C}'=(X, C', Y, \delta_{\mathbf{C}|C'\times X}, \lambda_{\mathbf{C}|C'\times X})$ is an A-subautomaton of $\mathbf{C}$. Now consider the mapping $\nu: (d_1, \ldots, d_n) \to a_{\sum_{t=1}^{n} d_t \cdot (n-t+1)}$ $((d_1, \ldots, d_n) \in C')$. It can be seen that ν is an A-isomorphism of $\mathbf{C}'$ onto $\mathbf{A}$. This ends the proof of Lemma 4.

It is evident that any α_0-product of α_2-products also is an α_2-product. Therefore, by Lemma 3 and Lemma 4, the following result is shown.

Theorem. Every automaton can be strongly covered by an α_2-product of two-state automata.

We know, by definition, that for every $i > 2$ the concept of α_i-product is a generalization of α_2-product. (In [4] it is shown that this generalization is proper.) Thus, the above Theorem and our remark about α_0-product and α_1-product jointly imply the following result.

Corollary. For every nonnegative integer i there exists a finite homomorphically α_i-complete system if and only if $i \geqq 2$.

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