Epis of some categories of Z-continuous partial algebras

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§ 1. Introductory remarks on the connections with Computer Science

Let $\perp\omega\,\mathrm{Alg}_\Sigma$ denote the category of ω-continuous Σ-algebras with bottom and bottom-preserving ω-continuous homomorphisms between them in the sense e.g. of p. 132 of [13]. The structures $\mathfrak{A}\in\mathrm{Ob}\perp\omega\,\mathrm{Alg}_\Sigma$ are simply algebraic systems in the sense of [6] and the morphisms $h\colon \mathfrak{A}\to\mathfrak{B}$ of $\perp\omega\,\mathrm{Alg}_\Sigma$ are homomorphisms in the sense of [6]. What is special about $\perp\omega\,\mathrm{Alg}_\Sigma$ is that these algebraic systems and homomorphisms have to satisfy certain conditions. The present paper investigates $\perp\omega\,\mathrm{Alg}_\Sigma$ and certain strongly related categories.

Nowadays a very large part of Theoretical Computer Science (TCS) is based on $\perp\omega\,\mathrm{Alg}_\Sigma$ see e.g. [13] or [4] or [8]. We do not give here more references but it is very easy to find them in any recent publication on "Algebraic Semantics of Programming" or in the recent volumes of MFCS or FCT. Just for referential purposes we note that the French school of TCS uses the word "*complete magma*" for an algebraic system $\mathfrak{A}\in\mathrm{Ob}(\perp\omega\,\mathrm{Alg}_\Sigma)$. The importance of $\perp\omega\,\mathrm{Alg}_\Sigma$ for computer science was perhaps first discovered by Dana Scott and his co-workers during their pioneering work a long time ago but of course at that time the tool they found did not have its present polished form. Among others, the fixed point semantics of programming is based mostly on $\perp\omega\,\mathrm{Alg}_\Sigma$ (though this may not be explicit in some of the papers on the subject).

In computer science one has to deal with recursion (or iteration). In $\perp\omega\,\mathrm{Alg}_\Sigma$ recursion is treated as the supremum of an ω-chain where the members of that ω-chain are the finite approximations of the recursion in question.

Since $\perp\omega\,\mathrm{Alg}_\Sigma$ is the foundation for a large part of TCS, we think it is important for TCS — and what is more, it is indispensable for TCS — to investigate the *basic* properties of $\perp\omega\,\mathrm{Alg}_\Sigma$. Such basic questions are to characterize the epimorphisms of $\perp\omega\,\mathrm{Alg}_\Sigma$ and to know e.g. whether or not it is co-well-powered. The present paper investigates these questions. We note that these questions are indeed basic, e.g. in algebraic logic the epimorphism problem is equivalent to the problem of the connections between explicit definitions and implicit definitions in the logic under (algebraic) investigation.

§ 2. Introduction

In [12], when characterizing the epis of POS (Z) — the category of Z-complete posets with bottom and Z-continuous bottom-preserving maps — I just solved the first problem which arised on the way of characterizing the epis in $\perp\mathrm{Alg}_\Sigma(Z)$ — the category of Z-continuous Σ-algebras with bottom and of Z-continuous, bottom-preserving homomorphisms. The present paper solves some other problems which seem to play an important role in solving the main problem.

Instead of going on in an abstract manner, I will give first the *basic definitions.* A *subset system* is a map Z which assigns to each poset A a collection $Z(A)$ of its subsets such that for each monotonic map $f\colon A\to B$, if $X\in Z(A)$, then $f(X):=$ $:=\{f(x)\colon x\in X\}\in Z(B)$.
A poset A is *Z-complete* if every element of $Z(A)$ has a l.u.b. (or $\sup_{\leqq_A}$) in A.

A map $f\colon A\to B$ is *Z-continuous* if it is monotonic and whenever $X\in Z(A)$ and $\sup_{\leqq_A} X$ exists, then $\sup_{\leqq_B} f(X)$ also exists and equals $f(\sup_{\leqq_A} X)$.
Let Σ be a similarity type or signature, i.e. a set of function symbols. For any $\sigma\in\Sigma$, $r(\sigma)$ denotes the arity of σ, which is an *arbitrary ordinal number.*
A *partial Σ-algebra* $\mathfrak{A}$ consists of a set A and of a family $\langle\sigma^A\colon \mathrm{dom}\,\sigma^A\to A\rangle_{\sigma\in\Sigma}$ of partial operations on A, i.e. for each $\sigma\in\Sigma$, $\mathrm{dom}\,\sigma^A\subseteqq A^{r(\sigma)}$. Given two partial Σ-algebras $\mathfrak{A}$ and $\mathfrak{B}$, a *homomorphism* $f\colon\mathfrak{A}\to\mathfrak{B}$ is a map $f\colon A\to B$ with the property that for any $\sigma\in\Sigma$, whenever $\mathbf{a}\in\mathrm{dom}\,\sigma^A$, $f\circ\mathbf{a}\in\mathrm{dom}\,\sigma^B$ and $f(\sigma^A(\mathbf{a}))=$ $=\sigma^B(f\circ\mathbf{a})$.
A partial Σ-algebra $\mathfrak{A}$ is *total,* if for any $\sigma\in\Sigma$, $\mathrm{dom}\,\sigma^A=A^{r(\sigma)}$.

For more about subset systems Z see [1], [9], [7]. For more about the theory of partial Σ-algebras see [2], [11], [10], [3].

The frame category of the present paper will be $\perp ZP\,\mathrm{Alg}_\Sigma$ defined as follows. $\mathfrak{A}\in\mathrm{Ob}\perp ZP\,\mathrm{Alg}_\Sigma$ iff $\mathfrak{A}$ is a partial Σ-algebra, A is partially ordered by $\leqq_A$ with least element and all the operations of $\mathfrak{A}$ are monotonic. $f\colon\mathfrak{A}\to\mathfrak{B}\in\mathrm{Mor}\perp ZP\,\mathrm{Alg}_\Sigma$ iff f is a Z-continuous bottom-preserving homomorphism.

The present paper gives a characterization of the epis in $\perp ZP\,\mathrm{Alg}_\Sigma$, for any Z and for any Σ.

Actually, we are more interested in some full subcategories of $\perp ZP\,\mathrm{Alg}_\Sigma$, which we define below.
$\perp Z\,\mathrm{Alg}_\Sigma$ denotes the full subcategory of $\perp ZP\,\mathrm{Alg}_\Sigma$ defined as $\mathrm{Ob}\perp Z\,\mathrm{Alg}_\Sigma=$ $=\{\mathfrak{A}\in\mathrm{Ob}\perp ZP\,\mathrm{Alg}_\Sigma\colon \mathfrak{A}$ is total$\}$.
$\perp P\,\mathrm{Alg}_\Sigma(Z)$ denotes the full subcategory of $\perp ZP\,\mathrm{Alg}_\Sigma$ with objects $\mathfrak{A}$ which are Z-complete and in which the operations are Z-continuous, i.e. for any $\sigma\in\Sigma$, if $X\in Z(\mathrm{dom}\,\sigma^A)$ and if $\sup_{\leqq_{A^{r(\sigma)}}} X\in\mathrm{dom}\,\sigma^A$, then $\sup_{\leqq_A}\{\sigma^A(\mathbf{x})\colon \mathbf{x}\in X\}=\sigma^A(\sup_{\leqq_{A^{r(\sigma)}}} X)$.
The objects of $\perp P\,\mathrm{Alg}_\Sigma(Z)$ are called *Z-continuous partial Σ-algebras.*
$\perp\mathrm{Alg}_\Sigma(Z)$ is the full subcategory of $\perp P\,\mathrm{Alg}_\Sigma(Z)$ with objects in which all operations are total.

$\perp P\,\mathrm{Alg}_{\Sigma,Z}$ is the full subcategory of $\perp P\,\mathrm{Alg}_\Sigma(Z)$ in which the objects are such that the domains of the operations are Z-complete.

In §3 we define the closure operator CL_Σ (see Definition 6) and, in Theorem 1, we prove that a morphism $f\colon\mathfrak{A}\to\mathfrak{B}\in\mathrm{Mor}\perp ZP\,\mathrm{Alg}_\Sigma$ is an epi iff $\mathrm{CL}_\Sigma(f(A))=B$.

This is a characterization of epis in $\perp ZP\,\mathrm{Alg}_\Sigma$. In Theorem 2 we extend this characterization to many other categories. At the end of §3 we show the connection of CL_Σ with CL of [12].

In §4 we use the above characterization of epis to show co-well-poweredness, assuming that the subset system Z is bounded. This assumption cannot be omitted.

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§ 3. Characterization of epis

Throughout the paper, let a signature Σ and a subset system Z be fixed.

Throughout this section, let $\mathfrak{A} \in \mathrm{Ob} \perp ZP\,\mathrm{Alg}_\Sigma$, $X \subseteq A$ and $a, b, c, d \in A$.

Definition 0. $\mathrm{cl}(X)$ is the least subset Y of A such that $X \subseteq Y$ and whenever $V \in Z(Y)$, then $\sup_{\leqq_A} V \in Y$.

Definition 1. We define a to be *X-greater or equal than b* $\left(a \xrightarrow{X} b\right)$ iff there is an ordinal α such that a is α, X-greater than b $\left(a \xrightarrow{\alpha, X} b\right)$ and the latter is defined as follows: $a \xrightarrow{0, X} b$ iff $b \leqq_A x \leqq_A a$ for some $x \in X$. Let $\alpha > 0$. Then $a \xrightarrow{\alpha, X} b$ iff there is a term-function symbol t of type Σ such that $b \leqq_A t^{\mathfrak{A}}(\mathbf{b})$ and $a \,{}_A\!\geqq t^{\mathfrak{A}}(\mathbf{a})$ for some $\mathbf{b}, \mathbf{a} \in \mathrm{dom}\, t^{\mathfrak{A}}$ and for any $i < r(t)$, $\mathbf{b}(i) \in \mathrm{cl}(Y^i)$ for some $Y^i \subseteq A$ and for each $y \in Y^i$ there is an ordinal $\alpha_y < \alpha$ with $\mathbf{a}(i) \xrightarrow{\alpha_y, X} y$.

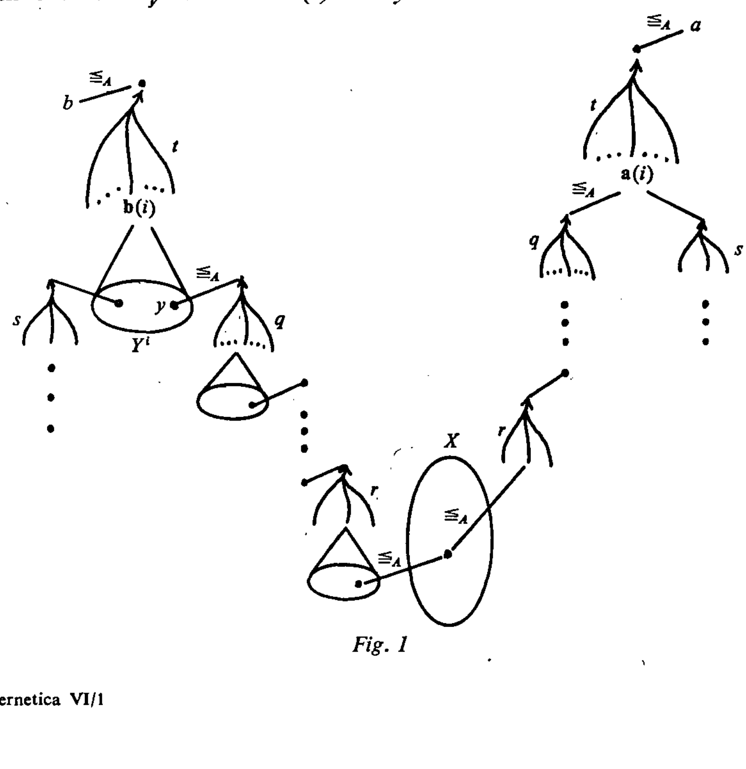

Fig. 1

Remark 2. Comparing this definition with Definition 1 of [12], notice that $a \overset{\alpha, X}{\longmapsto} b$ implies $a \xrightarrow{\alpha, X} b$ (just take for t the identity). If the operations of $\mathfrak{A}$ have all empty domains, then $a \xrightarrow{X} b$ iff ($a \overset{\alpha, X}{\longmapsto} b$ for some ordinal α).

Lemma 3. Suppose $a \xrightarrow{X} b$ and let $f, g\colon \mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor}\,\bot ZP\,\mathrm{Alg}_\Sigma$. Then $f \upharpoonright X = g \upharpoonright X$ implies $f(a)\ {}_B{\geqq}\ g(b)$.

Proof. $a \xrightarrow{X} b$ means $a \xrightarrow{\alpha, X} b$ for some ordinal α.

If $\alpha = 0$ then there is an $x \in X$ with $b \leqq_A x \leqq_A a$. Hence by the monotonity of f and g we have $g(b) \leqq_B g(x) = f(x) \leqq_B f(a)$.

Suppose $\alpha > 0$. Then $b \leqq_A t^{\mathfrak{A}}(\mathbf{b})$ and $t^{\mathfrak{A}}(\mathbf{a}) \leqq_A a$ for some termfunction t of type Σ and some $\mathbf{b}, \mathbf{a} \in \mathrm{dom}\, t^{\mathfrak{A}}$, and for any $i < r(t)$, $\mathbf{b}(i) \in \mathrm{cl}\,(Y^i)$ for some $Y^i \subseteqq A$ and for each $y \in Y^i$ there is an ordinal $\alpha_y < \alpha$ with $\mathbf{a}(i) \xrightarrow{\alpha_y, X} y$. By the induction hypothesis, for any $i < r(t)$ and for any $y \in Y^i$ we have $g(y) \leqq_B f(\mathbf{a}(i))$. But, since by the Z-continuity of g we have $g(\mathbf{b}(i)) \in \mathrm{cl}\,(g(Y^i))$, also $g(\mathbf{b}(i)) \leqq_B f(\mathbf{a}(i))$ must hold for any $i < r(t)$. By the monotonity of the operations we get then $g(b) \leqq_B g(t^{\mathfrak{A}}(\mathbf{b})) = t^{\mathfrak{B}}(g \circ \mathbf{b}) \leqq_B t^{\mathfrak{B}}(f \circ \mathbf{a}) = f(t^{\mathfrak{A}}(\mathbf{a})) \leqq_B f(a)$. □

Corollary 4. $a \xrightarrow{X} b$ implies $a\ {}_A{\geqq}\ b$.

Corollary 5. Let $f, g\colon \mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor}\,\bot ZP\,\mathrm{Alg}_\Sigma$ with $f \upharpoonright X = g \upharpoonright X$. Then $a \xrightarrow{X} a$ implies $f(a) = g(a)$.

Definition 6. $\mathrm{CL}_\Sigma(X) := \{a \in A\colon a \xrightarrow{X} a\}$.

Corollary 7. If for an $f\colon \mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor}\,\bot ZP\,\mathrm{Alg}_\Sigma$ we have $\mathrm{CL}_\Sigma(f(A)) = B$, then f is an epi!

Now we are going to prove the converse of Corollary 7.

Lemma 8. $a\ {}_A{\geqq}\ b \xrightarrow{\alpha, X} c\ {}_A{\geqq}\ d$ imply $a \xrightarrow{\alpha, X} d$.

Proof. Immediate by Definition 1. □

Lemma 9. $a \xrightarrow{\mathrm{CL}_\Sigma(X)} b$ implies $a \xrightarrow{X} b$.

Proof. Suppose $a \xrightarrow{\alpha, \mathrm{CL}_\Sigma(X)} b$. We prove by transfinit induction on α that $a \xrightarrow{X} b$.

First let $\alpha = 0$. Then there is an $x \in \mathrm{CL}_\Sigma(X)$ such that $b \leqq_A x \leqq_A a$. But since $x \xrightarrow{X} x$, it follows from Lemma 8 that $a \xrightarrow{X} b$.

Now suppose $\alpha > 0$. Then $b \leqq_A t^{\mathfrak{A}}(\mathbf{b})$, $t^{\mathfrak{A}}(\mathbf{a}) \leqq_A a$ for some termfunction t of type Σ and for some $\mathbf{b}, \mathbf{a} \in \mathrm{dom}\,(t^{\mathfrak{A}})$ and for each $i < r(t)$, $\mathbf{b}(i) \in \mathrm{cl}\,(Y^i)$ for some $Y^i \subseteqq A$ and for any $y \in Y^i$ there is an ordinal $\alpha_y < \alpha$ with $\mathbf{a}(i) \xrightarrow{\alpha_y, \mathrm{CL}_\Sigma(X)} y$ and hence by the induction hypothesis there is another ordinal β_y with $\mathbf{a}(i) \xrightarrow{\beta_y, X} y$. Applying Definition 1 we get then $a \xrightarrow{\beta, X} b$ for e.g. $\beta = \Sigma\{(\beta_y + 1)\colon y \in Y^i,\ i < r(t)\}$ (see Fig. 2). □

Corollary 10. The operator $\mathrm{CL}_\Sigma\colon \mathscr{P}(A) \to \mathscr{P}(A)$ which assigns $\mathrm{CL}_\Sigma(X)$ to each $X \subseteqq A$ is a closure operator.

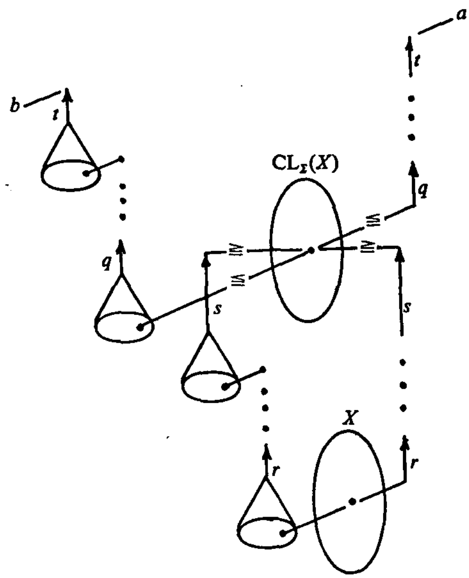

Fig. 2

Proof. 1) It follows immediately from Definition 1 that $a \xrightarrow{X} b$ for some $a, b \in A$ and $X \subseteqq Y \subseteqq A$ imply $a \xrightarrow{Y} b$. Hence $X \subseteqq Y \subseteqq A$ implies $CL_\Sigma(X) \subseteqq CL_\Sigma(Y)$.

2) $X \subseteqq CL_\Sigma(X)$ follows from Definition 1.

3) $CL_\Sigma(CL_\Sigma(X)) \subseteqq CL_\Sigma(X)$ follows from Lemma 9. □

Remark 11. Note that, by Lemma 9, if we suppose that either a or b is in $CL_\Sigma(X)$, then $a \xrightarrow{X} b$ iff $a \;{}_A{\geqq}\; b$.

Lemma 12. The operations of $\mathfrak{A}$ are monotonic w.r.t. the relation "X-greater than or equal to" $\xrightarrow{X}$.

Proof. Let $\sigma \in \Sigma$ be arbitrary and suppose that for any $i < r(\sigma)$, $a_i \xrightarrow{\alpha_i, X} b_i$. Then by Definition 1, $\sigma^{\mathfrak{A}}(a_i \colon i < r(\sigma)) \xrightarrow{\alpha, X} \sigma^{\mathfrak{A}}(b_i \colon i < r(\sigma))$, where e.g. $\alpha := \Sigma \{(\alpha_i + 1) \colon i < r(\sigma)\}$ (just let $t = \sigma$ and $Y^i = \{b_i\}$). □

Corollary 13. $CL_\Sigma(X)$ is closed w.r.t. all operations of $\mathfrak{A}$.

Lemma 14. Let $Y \subseteqq A$. If $a \xrightarrow{X} y$ for every $y \in Y$ then $a \xrightarrow{X} b$ for every $b \in \mathrm{cl}(Y)$.

Proof. Let $b \in \mathrm{cl}(Y)$. For every $y \in Y$ let α_y be such that $a \xrightarrow{\alpha_y, X} y$. Let $\alpha := \Sigma\{(\alpha_y + 1) \colon y \in Y\}$. Then $a \xrightarrow{\alpha, X} b$ by Definition 1 (just take for t the identity termfunction), i.e. $a \xrightarrow{X} b$. □

Corollary 15. $\mathrm{cl}(CL_\Sigma(X)) = CL_\Sigma(X)$.

Remark 16. In general, $CL_\Sigma(X)$ is greater than the least subset $Y \subseteqq A$ such that $X \subseteq Y$ and Y is closed under the operations and $\mathrm{cl}(Y)=Y$. This follows from the fact that $\mathrm{cl} \neq \mathrm{CL}$ in POS which is proved in LEHMANN—PASZTOR [5].

Having arrived at this point we formulate the main result of the present paper.

THEOREM 1. If $f\colon \mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor}\perp ZP\,\mathrm{Alg}_\Sigma$ is an epi then $\mathrm{CL}_\Sigma(f(A))=B$.

Proof. Denote $\mathrm{CL}_\Sigma(f(A))$ by B_0 and suppose that $B-B_0 \neq 0$. We will construct $\varphi, \psi\colon \mathfrak{B} \to \mathfrak{C} \in \mathrm{Mor}\perp ZP\,\mathrm{Alg}_\Sigma$ with $f\cdot\varphi = f\cdot\psi$ but $\varphi \neq \psi$, which contradicts the epiness of f.

Let $\varrho\colon B-B_0 \to B_1$ be a set isomorphism, where B_1 is disjoint from B. Let $C := B \cup B_1$ (the second copower of B with amalgam B_0), $\varphi := \mathrm{id}_B$ and $\psi := \mathrm{id}_{B_0} \cup \varrho$ (the injections), where id_B and id_{B_0} denote the identity maps on B and B_0 respectively. Let $\delta := \varphi \cup \varrho^{-1}$. Then $\delta\colon C \to B$.

Definition 17. We define on C the relation $\leqq_C$ as follows. For any $a, b \in C$

$$a \leqq_C b \quad \text{iff} \quad \begin{cases} \delta(a) \leqq_B \delta(b) & \text{if } a, b \in B \text{ or } a, b \in B_1 \\ \delta(a) \xleftarrow{B_0} \delta(b) & \text{otherwise.} \end{cases}$$

ASSERTION 1. $\leqq_C$ is a partial order on C.

Proof. 1) $\leqq_C$ is reflexive since $\leqq_B$ is reflexive.

2) Suppose $a \leqq_C b \leqq_C c$. Assume $a, c \in B$ or $a, c \in B_1$. Then $\delta(a) \leqq_B \delta(b) \leqq_B \delta(c)$ by Definition 17 and Corollary 4, hence $\delta(a) \leqq_B \delta(c)$ by transitivity of $\leqq_B$, i.e. $a \leqq_C c$ by Definition 17. Assume that one of a and c is in B and the other one is in B_1. Then either $\delta(a) \xleftarrow{B_0} \delta(b)$ or $\delta(b) \xleftarrow{B_0} \delta(c)$, by Definition 17. Then $\delta(a) \xleftarrow{B_0} \delta(c)$ by Lemma 8 and Corollary 4, i.e. $a \leqq_C c$ by Definition 17.

3) Let $a \leqq_C b$ and $b \leqq_C a$ for some $a, b \in C$. If $a, b \in B$ or $a, b \in B_1$ then $a=b$ by antisymmetry of $\leqq_B$ and since δ is one to one on B_1. Suppose one of a, b is in B and the other one is in B_1. Then $\delta(a) \xleftarrow{B_0} \delta(b) \xleftarrow{B_0} \delta(a)$ by Definition 17 and hence $\delta(a)=\delta(b) \in B_0$ by Corollary 4 and Lemma 8. Then $a=\delta(a)=\delta(b)=b$ (contradicting our hypothesis). □

ASSERTION 2. $\delta\colon C \to B$ is monotonic and $\varphi\cdot\delta = \psi\cdot\delta = \mathrm{id}_B$.

Proof. Immediate by Corollary 4 and by the definitions. □

ASSERTION 3. $\varphi, \psi\colon B \to C$ are Z-continuous.

Proof. 1) Clearly φ is monotonic. Let $a, b \in B$ be such that $a \leqq_B b$. If $a \in B_0$ or $b \in B_0$ then $a \xleftarrow{B_0} b$ by Remark 11 and hence $\psi(a) \leqq_C \psi(b)$. If $a, b \in B - B_0$ then $\psi(a) \leqq_C \psi(b)$ by $\psi\cdot\delta = \mathrm{id}_B$ and Definition 17. Thus ψ is monotonic.

2) Let $Y \in Z(B)$ and assume that $b := \sup_{\leqq_B} Y$ exists. By Definition 17, $y \leqq_C b$ for any $y \in Y$, i.e. $b = \varphi(b)$ is an upper bound of $Y = \varphi(Y)$ in C. Now let $c \in C$ be another upper bound of Y. If $c \in B$ then $b \leqq_C c$ by Definition 17 and since $b = \sup_{\leqq_B} Y$. Suppose $c \in B_1$. Then $\delta(c) \xrightarrow{B_0} \delta(y) = y$ for every $y \in Y$, by Definition 17. Thus $\delta(c) \xrightarrow{B_0} b$ by Lemma 14, i.e. $b \leqq_C c$. Thus $b = \sup_{\leqq_C} Y$.

Since ψ is monotonic, $\psi(b)$ is an upper bound of $\psi(Y)$. Let $c \in C$ be another upper bound of $\psi(Y)$. Suppose $c \in B_0 \cup B_1 = \psi(B)$. Then $\delta(c)$ is an upper bound of Y in B since δ is monotonic and $\psi \cdot \delta = \mathrm{id}_B$, therefore $b \leqq_B \delta(c)$ by $b = \sup_{\leqq_B} Y$. Then $\psi(b) \leqq_C \psi\delta c = c$ by monotonity of ψ. Suppose $c \in B - B_0$. Then $c \xrightarrow{B_0} y$ for every $y \in Y$ by Definition 17 and Remark 11, therefore $c \xrightarrow{B_0} b$ by Lemma 14, i.e. $c = \delta(c) \xrightarrow{B_0} \delta\psi b$, hence $\psi(b) \leqq_C c$ by Definition 17. Thus $\psi(b) = \sup_{\leqq_C} \psi(Y)$.

Now we take on C the structure inherited from $\mathfrak{B}$, i.e. for any $\sigma \in \Sigma$, $\sigma^C := \sigma^B \cup \psi \circ \sigma^B$. Since by Corollary 13 B_0 is closed under the operations of $\mathfrak{B}$, σ^C is a partial operation on C.

Remarks 18. 1) If $\mathfrak{B}$ is a *total* Σ-algebra and if Σ contains at most unary operation symbols, then $\mathfrak{C} := (C, \sigma^C)_{\sigma \in \Sigma}$ is also a *total* Σ-algebra.

2) $\mathfrak{C}$ is the second copower of $\mathfrak{B}$ with amalgam B_0 in the category of all partial Σ-algebras.

3) $\delta \colon \mathfrak{C} \to \mathfrak{B}$ is a homomorphism. □

By its definition and by Lemma 12, σ^C is monotonic. Let $\mathfrak{C} := (C, \sigma^C)_{\sigma \in \Sigma}$ with partial order $\leqq_C$. Then $\mathfrak{C} \in \mathrm{Ob}\perp ZP\,\mathrm{Alg}_\Sigma$. Clearly, $\varphi, \psi \colon \mathfrak{B} \to \mathfrak{C}$ are homomorphisms, therefore $\varphi, \psi \in \mathrm{Mor}\perp ZP\,\mathrm{Alg}_\Sigma$, by Assertion 3. By $B - B_0 \neq 0$ we have $\varphi \neq \psi$ and by $f(B) \subseteqq B_0$ we have $f \cdot \varphi = f \cdot \psi$. Thus f is not an epi. □

To prove Theorem 2, we shall need Lemma 19.

Lemma 19. Let P be any poset. Then conditions (i) and (ii) below are *equivalent*.

(i) P is directed.

(ii) For any $X \subseteqq P$ either X is cofinal in P or $P - X$ is cofinal in P.

Proof. Suppose that $X \subseteqq P$ is such that neither X nor $P - X$ is cofinal in P. Then there are $x, p \in P$ such that $x \leqq a$ implies $a \in X$ and $p \leqq a$ implies $a \notin X$. Then $\{x, p\}$ cannot have an upper bound.

Suppose that $\{x, p\}$ does not have an upper bound. Then neither $\{a \in P \colon a \geqq x\}$ nor $\{a \in P \colon a \ngeqq x\}$ is cofinal in P. □

Notation. $Z \subseteqq \Delta$ denotes the fact that X is directed for any poset P and $X \in Z(P)$.

Theorem 2. 1) For any Z and for any type Σ we have $f \colon \mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor}\perp ZP\,\mathrm{Alg}_\Sigma$ is an epi iff $\mathrm{CL}_\Sigma(f(A)) = B$.

2) Suppose that $Z \subseteqq \Delta$. Then a)—c) below hold.

a) For any type Σ,
$f \colon \mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor}\perp P\,\mathrm{Alg}_\Sigma(Z)$ is an epi iff $\mathrm{CL}_\Sigma(f(A)) = B$.

b) For any type Σ,
$f \colon \mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor}\perp P\,\mathrm{Alg}_{\Sigma, Z}$ is an epi iff $\mathrm{CL}_\Sigma(f(A)) = B$.

c) If the type Σ contains only 0- or 1-ary operation symbols then
$f \colon \mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor}\perp \mathrm{Alg}_\Sigma(Z)$ is an epi iff $\mathrm{CL}_\Sigma(f(A)) = B$.

Proof. A) By Corollary 7, if for $f \colon \mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor}\perp ZP\,\mathrm{Alg}_\Sigma$ we have $\mathrm{CL}_\Sigma(f(A)) = B$ then f is an epi. Further on, for any category $\mathscr{C}$ and any subcategory $\mathscr{B}$ of $\mathscr{C}$, if $f \in \mathrm{Mor}\,\mathscr{B}$ is an epi in $\mathscr{C}$, then it is an epi also in $\mathscr{B}$.

B) Now 1) follows from Theorem 1. Suppose that $Z \subseteq \Delta$. To prove 2) we shall use the construction in Theorem 1, i.e. we shall use $\varphi, \psi, \mathfrak{B}, \delta$, and $\mathfrak{C}$.

ASSERTION 4. If $\mathfrak{B}$ is Z-complete then so is $\mathfrak{C}$.

Proof. Let $X \in Z(C)$. Then $\delta(X) \in Z(B)$ since δ is monotonic, hence $b := \sup_{\leqq_B} \delta(X)$ exists. Then $\varphi(b) = \sup_{\leqq_C} \varphi\delta X$ and $\psi(b) = \sup_{\leqq_C} \psi\delta X$ by Assertion 3. Suppose that $B \cap X$ is cofinal in X. Then $\varphi\delta(B \cap X) = B \cap X$ is cofinal in $\varphi\delta X$, hence $\varphi(b) = \sup_{\leqq_C} (B \cap X) = \sup_{\leqq_C} X$. If $B \cap X$ is not cofinal in X then $B_1 \cap X$ is cofinal in X by $Z \subseteq \Delta$ and Lemma 19. Then, similarly as before, $\sup_{\leqq_C} X = \psi(b)$. □

ASSERTION 5. If $\mathfrak{B}$ is Z-continuous then so is $\mathfrak{C}$ and if $\mathfrak{B} \in \mathrm{Ob}\perp P\,\mathrm{Alg}_{\Sigma,Z}$ then $\mathfrak{C} \in \mathrm{Ob}\perp P\,\mathrm{Alg}_{\Sigma,Z}$.

Proof. $\mathfrak{C}$ is Z-complete by Assertion 4. Let $\sigma \in \Sigma$ and $X \in Z(\mathrm{dom}\,\sigma^C)$. Let us denote $B^{r(\sigma)}$, $C^{r(\sigma)}$, $\varphi^{r(\sigma)}$, $\psi^{r(\sigma)}$ and $\delta^{r(\sigma)}$ by $\bar{B}$, $\bar{C}$, $\bar{\varphi}$, $\bar{\psi}$ and $\bar{\delta}$ respectively. By the definition of $\mathfrak{C}$ we have $\mathrm{dom}\,\sigma^C \subseteq \bar{\varphi}(\bar{B}) \cup \bar{\psi}(\bar{B})$. Therefore either $X_\varphi := X \cap \bar{\varphi}(\bar{B})$ or $X_\psi := X \cap \bar{\psi}(\bar{B})$ is cofinal in X, by Lemma 19.

Suppose X_φ is cofinal in X. Then $\sigma^C(X_\varphi)$ is cofinal in $\sigma^C(X)$ by monotonity of σ^C, hence $\sup_{\leqq_C} X = \sup_{\leqq_C} X_\varphi$ and $\sup_{\leqq_C} \sigma^C(X) = \sup_{\leqq_C} \sigma^C(X_\varphi)$. By $X_\varphi \subseteq \bar{\varphi}(\bar{B})$ and $\varphi \cdot \delta = \mathrm{id}_B$ we have $X_\varphi = \bar{\varphi}\bar{\delta}X_\varphi$. Since $\delta\colon \mathfrak{C} \to \mathfrak{B}$ is a monotonic homomorphism, we have $\bar{\delta}(X_\varphi) \in Z(\mathrm{dom}\,\sigma^B)$ and thus $\sigma^B(\bar{\delta}X_\varphi) \in Z(B)$. Now, since φ is a homomorphism, we have $\sigma^C(\bar{\varphi}\bar{\delta}X_\varphi) = \varphi\sigma^B(\bar{\delta}X_\varphi)$, and then $\sup_{\leqq_C} \varphi\sigma^B(\bar{\delta}X_\varphi) = \varphi \sup_{\leqq_B} \sigma^B(\bar{\delta}X_\varphi)$ by Z-continuity of φ. By Z-completeness of $\mathfrak{B}$ and $\mathfrak{C}$ and by Z-continuity of φ we have that $\bar{\varphi}(\sup_{\leqq_B} Y) = \sup_{\leqq_C} \bar{\varphi}(Y)$ for any $Y \in Z(\bar{B})$ (because $\varphi[(\sup_{\leqq_B} Y)(i)] = \varphi \sup_{\leqq_B} \{\mathbf{y}(i) \colon \mathbf{y} \in Y\} = \sup_{\leqq_C} \{\varphi\mathbf{y}(i) \colon \mathbf{y} \in Y\} = (\sup_{\leqq_C} \bar{\varphi}Y)(i)$ for any $i < r(\sigma)$). Thus $\bar{\varphi} \sup_{\leqq_B} \bar{\delta}X_\varphi = \sup_{\leqq_C} X_\varphi$ and $\sup_{\leqq_B} \bar{\delta}(X_\varphi) = \bar{\delta}(\sup_{\leqq_C} X_\varphi)$ by $\varphi \cdot \delta = \mathrm{id}_B$, therefore $\sup_{\leqq_C} X_\varphi \in \mathrm{dom}\,\sigma^C$ iff $\sup_{\leqq_B} \bar{\delta}(X_\varphi) \in \mathrm{dom}\,\sigma^B$ since φ and δ are homomorphisms. Suppose $\sup_{\leqq_B} \bar{\delta}(X_\varphi) \in \mathrm{dom}\,\sigma^B$. Then $\sup_{\leqq_B} \sigma^B(\bar{\delta}X_\varphi) = \sigma^B(\sup_{\leqq_B} \bar{\delta}X_\varphi)$ by Z-continuity of $\mathfrak{B}$ and $\varphi\sigma^B(\sup_{\leqq_B} \bar{\delta}X_\varphi) = \sigma^C(\bar{\varphi} \sup_{\leqq_B} \bar{\delta}X_\varphi)$ since φ is a homomorphism.

Summing up:

$$\sup_{\leqq_C} \sigma^C(X) = \sup_{\leqq_C} \sigma^C(X_\varphi) = \sup_{\leqq_C} \sigma^C(\bar{\varphi}\bar{\delta}X_\varphi) = \sup_{\leqq_C} \varphi\sigma^B(\bar{\delta}X_\varphi) = \varphi \sup_{\leqq_B} \sigma^B(\bar{\delta}X_\varphi) =$$

$$= \varphi\sigma^B(\sup_{\leqq_B} \bar{\delta}X_\varphi) = \sigma^C(\bar{\varphi} \sup_{\leqq_B} \bar{\delta}X) = \sigma^C(\sup_{\leqq_C} \bar{\varphi}\bar{\delta}X_\varphi) = \sigma^C(\sup_{\leqq_C} X_\varphi) = \sigma^C(\sup_{\leqq_C} X).$$

If X_ψ is cofinal in X then the proof is the same as above, only φ has to be replaced by ψ everywhere. □

Now 2) follows from Assertion 5, Remarks 18 and from the proof of Theorem 1. □

In part 2 of [12] we defined the closure operator CL on posets. Now we are going to prove that in some cases CL_Σ on Z-continuous Σ-algebras equals CL.

Lemma 20. Let $Z \subseteqq \Delta$ and let Σ be finitary. Then for any $\mathfrak{A} \in \mathrm{Ob}\,\bot\mathrm{Alg}_\Sigma(Z)$, if $X \subseteqq A$ is a subalgebra of $\mathfrak{A}$, then for any $\sigma \in \Sigma$ $a_i \stackrel{\alpha_i, X}{\longmapsto} b_i$, $i < r(\sigma)$ implies $a := \sigma^A(a_i\colon i<r(\sigma)) \stackrel{\alpha, X}{\longmapsto} \sigma^A(b_i\colon i<r(\sigma)) =: b$ for $\alpha := \sup\limits_{i<r(\sigma)} \alpha_i$.

Proof. If $\alpha = 0$ then for each $i < r(\sigma)$, $b_i \in \mathrm{cl}(Y^i)$ for some $Y^i \subseteqq A$ and for each $y \in Y^i$ there is an $x_y \in X$ such that $y \leqq x_y \leqq a_i$. Let $Y := \{\sigma^A(y)\colon y \in \bigtimes\limits_{i<r(\sigma)} Y_i\}$.

By NELSON [9], for $Z \subseteqq \Delta$ and finitary Σ, $\sigma^A(\mathrm{cl}(Y_1), \ldots, \mathrm{cl}(Y_n)) = \mathrm{cl}(\sigma^A(Y_1, \ldots, Y_n))$, hence $b \in \mathrm{cl}(Y)$ and by the monotonity of σ for any $y \in Y$, since $y = \sigma^A(\mathbf{y})$ for some $\mathbf{y} \in \bigtimes\limits_{i<r(\sigma)} Y_i$, there is an $x_y := \sigma^A(x_{\mathbf{y}(i)}\colon i<r(\sigma)) \in X$ (X is closed w.r.t. σ), such that $y \leqq_A x_y \leqq_A a$. Hence $a \stackrel{0, X}{\longmapsto} b$.

Let $\alpha > 0$ and suppose that whenever $\sup\limits_{i<r(\sigma)} \alpha_i < \alpha$ the statement holds. Then for any $i < r(\sigma)$ $b_i \in \mathrm{cl}(Y_i)$ for some $Y_i \subseteqq A$ and for any $y^i \in Y_i$ there is a $b_{y^i} {}_A\!\geqq y^i$ and an ordinal $\beta_{y^i} < \alpha_i$ such that $a_i \stackrel{\beta_{y^i}, X}{\longmapsto} b_{y^i}$. Let $Y := \{\sigma^A(y)\colon y \in \bigtimes\limits_{i<r(\sigma)} Y_i\}$. Then for any $y \in Y$, $y = \sigma^A(\mathbf{y})$ for some $\mathbf{y} \in \bigtimes\limits_{i<r(\sigma)} Y_i$ and by the monotonity of σ $y \leqq_A \sigma^A(b_{\mathbf{y}(i)}\colon i<r(\sigma)) =: b_y$. By the induction hypothesis, $a \stackrel{\beta_y, X}{\longmapsto} b_y$, where $\beta_y := \sup\limits_{i<r(\sigma)} \beta_{\mathbf{y}(i)}$. Since by the assumption $b \in \mathrm{cl}(Y)$ and since for any $y \in Y$, $\beta_y < \alpha$ (because $r(\sigma) \in \omega$), $a \stackrel{\alpha, X}{\longmapsto} b$. □

Corollary 21. If $Z \subseteqq \Delta$ and Σ is finitary, then for any $f\colon \mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor}\,\bot\mathrm{Alg}_\Sigma(Z)$, $\mathrm{CL}\,(f(A))$ is a subalgebra of $\mathfrak{B}$.

Lemma 22. Let $Z \subseteqq \Delta$ and let Σ be finitary. Then for any $\mathfrak{A} \in \mathrm{Ob}\,\bot\mathrm{Alg}_\Sigma(Z)$, for any $a, b \in A$ and for any subalgebra X of $\mathfrak{A}$, $a \stackrel{X}{\longrightarrow} b$ implies $a \stackrel{\alpha, X}{\longmapsto} b$ for some ordinal α.

Proof. Suppose $a \stackrel{0, X}{\longrightarrow} b$. Then $b \leqq_A x \leqq_A a$ for some $x \in X$, which implies $a \stackrel{0, X}{\longmapsto} b$.

Let $a \stackrel{\alpha, X}{\longrightarrow} b$ and suppose that for any $\beta < \alpha$, $a \stackrel{\beta, X}{\longrightarrow} b$ already implies $a \stackrel{\gamma, X}{\longmapsto} b$ for some ordinal γ. Then $a \stackrel{\alpha, X}{\longrightarrow} b$ means $b \leqq_A t^{\mathfrak{A}}(\mathbf{b})$ and $t^{\mathfrak{A}}(\mathbf{a}) \leqq_A a$ for some term-function symbol t and some $\mathbf{a}, \mathbf{b} \in A^{r(t)}$ and for any $i < r(t)$, $\mathbf{b}(i) \in \mathrm{cl}(Y^i)$ for some $Y^i \subseteqq A$ and for each $y \in Y^i$ there is an ordinal $\alpha_y < \alpha$ with $\mathbf{a}(i) \stackrel{\alpha_y, X}{\longrightarrow} y$. By the induction hypothesis for each $i < r(t)$ and for each $y \in Y^i$, $\mathbf{a}(i) \stackrel{\beta_y, X}{\longmapsto} y$, for some ordinal β_y.

Let $Y := \{t^{\mathfrak{A}}(y)\colon y \in \bigtimes\limits_{i<r(t)} Y^i\}$. Then $t^{\mathfrak{A}}(\mathbf{b}) \in \mathrm{cl}(Y)$, since by NELSON [9] for finitary Σ and for $Z \subseteqq \Delta$, $t^{\mathfrak{A}}(\mathrm{cl}(Y^0), \ldots, \mathrm{cl}(Y^{r(t)-1})) = \mathrm{cl}(t^{\mathfrak{A}}(Y^0, \ldots, Y^{r(t)-1}))$. For any $y \in Y$, $y = t^{\mathfrak{A}}(\mathbf{y})$ for some $\mathbf{y} \in \bigtimes\limits_{i<r(t)} Y^i$, hence by Lemma 20, $t^{\mathfrak{A}}(\mathbf{a}) \stackrel{\beta_y, X}{\longmapsto} y$,

where $\beta_y = \sup_{i<r(t)} \beta_{y(i)}$. Then $t^{\mathfrak{A}}(\mathbf{a}) \overset{\beta, X}{\longmapsto} t^{\mathfrak{A}}(\mathbf{b})$ for some β greater than each β_y, $y \in Y$. By Lemmas 3 and 4 in part 2 of [12], $a \overset{\beta+1, X}{\longmapsto} b$. □

Corollary 23. If $Z \subseteqq \Delta$ and Σ is finitary, then for any f: $\mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor} \perp \mathrm{Alg}_\Sigma(Z)$ we have $\mathrm{CL}_\Sigma(f(A)) \subseteqq \mathrm{CL}(f(A))$.

Corollary 24. If $Z \subseteqq \Delta$ and Σ contains at most unary operations, then 1) and 2) below hold.

1) f: $\mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor} \perp \mathrm{Alg}_\Sigma(Z)$ is an epi iff $\mathrm{CL}(f(A)) = B$.

2) $\perp \mathrm{Alg}_\Sigma(Z)$ is co-well-powered.

Proof. 1) By Corollary 23 and Corollary 21, $\mathrm{CL}_\Sigma(f(A)) = \mathrm{CL}(f(A))$. By Theorem 2, f: $\mathfrak{A} \to \mathfrak{B} \in \mathrm{Mor} \perp \mathrm{Alg}_\Sigma(Z)$ is an epi iff $\mathrm{CL}_\Sigma(f(A)) = B$.

2) In Corollary 2 of part 4 in [12] I proved $\mathrm{CL}(X) \subseteqq \{\sup_{\leqq_A} S\colon S \subseteqq X\}$ for any $A \in \mathrm{POS}(Z)$ (Z arbitrary) and $X \subseteqq A$. □

§ 4. Co-well-poweredness

Suppose that Z is bounded, i.e. there is a cardinal $\delta(Z)$ such that for any poset A, if $X \in Z(A)$ then $|X| < \delta(Z)$.

In what follows our aim is to prove that for such Z-s those categories for which we have proved [f: $\mathfrak{A} \to \mathfrak{B}$ epi $\Leftrightarrow \mathrm{CL}_\Sigma(f(A)) = B$] (see Theorem 2) are co-well-powered.

Let $\delta(\Sigma)$ denote the ordinal dimension of the type Σ, i.e. the least regular ordinal δ such that $|\delta| < |r(\sigma)|$ for any $\delta \in \Sigma$.

Denote by $\delta := \delta(\Sigma, Z)$ the least regular ordinal greater than $\max\{\delta(Z), \delta(\Sigma)\}$.

Notice that for any poset A, if $a \in A$ and $Y \subseteqq A$, then $a \in \mathrm{cl}(Y)$ implies that there is an $Y' \subseteqq Y$ with $|Y'| < \delta(Z)$ and $a \in \mathrm{cl}(Y')$. In the following we will suppose immodiately $|Y| < \delta(Z)$ when writing $a \in \mathrm{cl}(Y)$.

In the following let $\mathfrak{A} \in \mathrm{Ob} \perp ZP\, \mathrm{Alg}_\Sigma$, $X \subseteqq A$ and $a, b, c, d \in A$.

Lemma 25. Suppose that Z is bounded by $\delta(Z)$ and let $\delta(\Sigma, Z)$ be as above. Then $a \xrightarrow{X} b$ implies $a \xrightarrow{\beta, X} b$ for some $\beta < \delta(\Sigma, Z)$.

Proof. Let $a \xrightarrow{X} b$. Then $a \xrightarrow{\alpha, X} b$ for some α.

If $\alpha = 0$ then the statement is true by $\delta > 0$.

Let $\alpha > 0$ and suppose that for every $\beta < \alpha$ the statement holds. $a \xrightarrow{\alpha, X} b$ means that $b \leqq_A t^{\mathfrak{A}}(\mathbf{b})$, $t^{\mathfrak{A}}(\mathbf{a}) \leqq_A a$ for some termfunction symbol t and some $\mathbf{b}, \mathbf{a} \in \mathrm{dom}\, t^{\mathfrak{A}}$ and that for any $i < r(t)$, $\mathbf{b}(i) \in \mathrm{cl}(Y^i)$ for some $Y^i \subseteqq A$, card $(Y^i) < \delta(Z)$, and for any $y \in Y^i$ there is an ordinal $\alpha_y < \alpha$ such that $\mathbf{a}(i) \xrightarrow{\alpha_y, X} y$. By the induction hypothesis for any $i < r(t)$ and for any $y \in Y^i$ there is an ordinal $\beta_y < \delta(\Sigma, Z)$ with $\mathbf{a}(i) \xrightarrow{\beta_y, X} y$. Let $\beta := \Sigma\{(\beta_y + 1)\colon y \in Y^i,\ i < r(t)\}$. By the definition of $\delta(\Sigma, Z)$ we have $\beta < \delta(\Sigma, Z)$ and by Definition 1, $a \xrightarrow{\beta, X} b$. □

Definition 26. For every $a, b \in A$ such that $a \xrightarrow{X} b$ we define $R_{a,b}$ as follows. Let α be the least ordinal for which $a \xrightarrow{\alpha, X} b$.

If $\alpha=0$ then $a\xrightarrow{0,X}b$ means that there is an $x\in X$ with $b\leqq_A x\leqq_A a$. Let us fix one $x\in X$ with this property. Then $R_{a,b}:=\langle\{x\},0\rangle$.

If $\alpha>0$ then $a\xrightarrow{\alpha,X}b$ means that $b\leqq_A t^{\mathfrak{A}}(\mathbf{b})$ and $t^{\mathfrak{A}}(\mathbf{a})\leqq a$ for some termfunction symbol t and some $\mathbf{b},\mathbf{a}\in A^{r(t)}$ and that for each $i<r(t)$, $\mathbf{b}(i)\in\mathrm{cl}(Y^i)$ for some $Y^i\subseteqq A$ and for any $y\in Y^i$ there is an ordinal $\alpha_y<\alpha$ with $\mathbf{a}(i)\xrightarrow{\alpha_y,X}y$. Let us fix t, $\mathbf{a}$, $\mathbf{b}$ and the Y^i-s for any $i<r(t)$. Then

$$R_{a,b}:=\langle t,\langle\{R_{\mathbf{a}(i),y}\colon y\in Y^i\}\rangle_{i<r(t)},\alpha\rangle.$$

Lemma 27. If $a\xrightarrow{X}b$, $c\xrightarrow{X}d$, and $R_{a,b}=R_{c,d}$ then $d\leqq_A a$ (and $b\leqq_A c$).

Proof. By transfinit induction on α of $R_{a,b}$.

Let $R_{a,b}=R_{c,d}=\langle\{x\},0\rangle$. Then $b\leqq_A x\leqq_A a$ and $d\leqq_A x\leqq_A c$, hence $d\leqq_A x\leqq_A a$, i.e. $d\leqq_A a$.

Let $R_{a,b}=R_{c,d}=\langle t,\langle\{R_{\mathbf{a}(i),y}\colon y\in Y^i\}\rangle_{i<r(t)},\alpha\rangle$ where $\alpha>0$. Then $b\leqq_A t^{\mathfrak{A}}(\mathbf{b})$, $t^{\mathfrak{A}}(\mathbf{a})\leqq_A a$, $d\leqq_A t^{\mathfrak{A}}(\mathbf{d})$ and $t^{\mathfrak{A}}(\mathbf{c})\leqq_A c$ for some $\mathbf{a},\mathbf{b},\mathbf{c}$ and $\mathbf{d}\in A^{r(t)}$ and for any $i<r(t)$, $\mathbf{b}(i)\in\mathrm{cl}(Y^i)$ and $\mathbf{d}(i)\in\mathrm{cl}(Z^i)$ for some $Y^i,Z^i\subseteqq A$ and for any $y\in Y^i$ there is a $z\in Z^i$ such that $R_{\mathbf{a}(i),y}=R_{\mathbf{c}(i),z}$ (and since $R_{a,b}=R_{c,d}$ of course for any $z\in Z^i$ there is a $y\in Y^i$ with $R_{\mathbf{c}(i),z}=R_{\mathbf{a}(i),y}$). By the induction hypothesis this implies $z\leqq_A\mathbf{a}(i)$ for any $z\in Z^i$. Since $\mathbf{d}(i)\in\mathrm{cl}(Z^i)$, $\mathbf{d}(i)\leqq_A\mathbf{a}(i)$. Then by the monotonity of t, $d\leqq_A t^{\mathfrak{A}}(\mathbf{d})\leqq_A t^{\mathfrak{A}}(\mathbf{a})\leqq_A a$, i.e. $d\leqq_A a$. □

Let Term (Σ) denote the class of all termfunction symbols of type Σ. It is easy to show that Term (Σ) is a set. Let $\gamma(X,Z)$ be the least regular ordinal greater than $(\max\{|X|,|\mathrm{Term}(\Sigma)|,\delta(\Sigma,Z)\})^{\delta(\Sigma,Z)}$.

Let H_0 be the set of all $R_{a,b}$-s of form $\langle\{x\},0\rangle$. Then $|H_0|=|X|<\gamma(X,Z)$.

Let $0<\alpha<\delta(\Sigma,Z)$. Then we define H_α to be the set of all $R_{a,b}$-s of form $\langle t,\langle\{R_{\mathbf{a}(i),y}\colon y\in Y^i\}\rangle_{i<r(t)},\alpha\rangle$. Then $|H_\alpha|<|\mathrm{Term}(\Sigma)|\cdot|(\bigcup\{H_\beta\colon\beta<\alpha\})^{\delta(Z)\cdot\delta(\Sigma)}|<$ $<\gamma(X,Z)$.

By Lemma 25, if $a\xrightarrow{X}b$, then there is an ordinal $\beta<\delta(\Sigma,Z)$ such that $R_{a,b}\in H_\beta$. By the definition of $\gamma(X,Z)$, $|\bigcup\{H_\beta\colon\beta<\delta(\Sigma,Z)\}|<\gamma(X,Z)$.

By Lemma 26, we know that for any $a,b\in\mathrm{CL}_\Sigma(X)$, if $R_{a,a}=R_{b,b}$ then $a=b$. Hence we can immediately see that we have proved

Corollary 28. $|\mathrm{CL}_\Sigma(X)|<\gamma(X,Z)$.

Corollary 29. 1) Let Z be bounded. Then for any similarity type Σ, $\perp ZP\,\mathrm{Alg}_\Sigma$ is co-well-powered.

2) Suppose that Z is bounded and $Z\subseteqq\Delta$. Then for any type Σ, $\perp P\,\mathrm{Alg}_\Sigma(Z)$ and also $\perp P\,\mathrm{Alg}_{\Sigma,Z}$ are co-well-powered. If Σ contains only 0- or 1-ary operation symbols then $\perp\mathrm{Alg}_\Sigma(Z)$ is co-well-powered. □

Next we prove that in Corollary 29 the condition that Z is bounded cannot be omitted.

Proposition 30. Let Σ be a signature with at least one $f\in\Sigma$ such that $r(f)>0$. Then there is a subset system $Z\subseteqq\Delta$ such that *both* $\perp ZP\,\mathrm{Alg}_\Sigma$ and $\perp Z\,\mathrm{Alg}_\Sigma$ are not co-well-powered.

Proof. For simplicity we assume $\Sigma=\{f\}$ with $r(f)=1$. It is obvious how to extend the present proof for the general case (in the formulation of the present Proposition).

We define Z as follows. For every poset $\langle A, \leqq_A\rangle$ let $Z(\langle A, \leqq_A\rangle):=\{Y\colon Y\subseteq A$ and $(\exists\alpha\in\mathrm{Ord})\langle\alpha, \in\rangle\cong\langle Y, \leqq_A\rangle$. Clearly, Z is a subset system and $Z\subseteq\Delta$.

ω denotes the set of natural numbers and Id_S denotes the identity function on S, for any set S.

Let $\mathfrak{A}:=\langle\omega, \perp_A, \leqq_A, f^A\rangle$ such that $\perp_A=0$, $\leqq_A=\{0\}\times\omega\cup\mathrm{Id}_\omega$ and $f^A=\mathrm{Id}_\omega$. Then $\mathfrak{A}\in\mathrm{Od}\perp Z\,\mathrm{Alg}_\Sigma$.

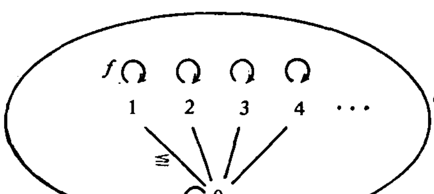

Fig. 3

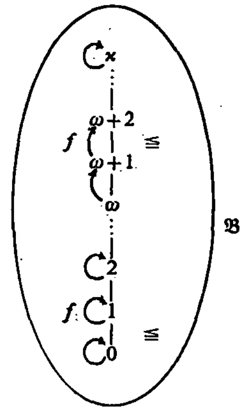

Fig. 4

Let $\varkappa\in\mathrm{Ord}$ be arbitrary but such that $\varkappa\geqq\omega$. Let $\mathfrak{B}:=\langle\varkappa+1, \perp_B, \leqq_B, f^B\rangle$ such that $\perp_B=0$, $\leqq_B=\in\cap(B\times B)\cup\mathrm{Id}_B$ and $f^B=f^A\cup\{\langle\alpha, \alpha+1\rangle\colon \omega\leqq\alpha+1\in B\}\cup\bigcup\{\langle\varkappa, \varkappa\rangle\}$. E.g. if $\omega+1\in B$ then $f^B(\omega)=\omega+1$ (see Fig. 4). Now $\mathfrak{B}\in\mathrm{Ob}\perp Z\,\mathrm{Alg}_\Sigma$ since $f^B\colon\langle B, \leqq_B\rangle\to\langle B, \leqq_B\rangle$ is an endomorphism that is $f^B\colon B\to B$ is monotonic.

Let $h:=\mathrm{Id}_\omega$, i.e. $h\colon A\to B$ is the identical embedding of ω into $\varkappa+1$. Then $h\colon\mathfrak{A}\to\mathfrak{B}\in\mathrm{Mor}\perp Z\,\mathrm{Alg}_\Sigma$ since h is a bottom-preserving homomorphism and h is Z-continuous. Since $\perp Z\,\mathrm{Alg}_\Sigma\subseteq\perp ZP\,\mathrm{Alg}_\Sigma$ we have that $h\colon\mathfrak{A}\to\mathfrak{B}\in\mathrm{Mor}\perp ZP\,\mathrm{Alg}_\Sigma$, too.

ASSERTION 6. $h\colon\mathfrak{A}\to\mathfrak{B}$ is an epi in $\perp ZP\,\mathrm{Alg}_\Sigma$ as well as in $\perp Z\,\mathrm{Alg}_\Sigma$.

Proof. Let $X=h(A)$. Then $X=\omega\subseteq B$. Let $\gamma\in B$. Assume $\gamma\in\mathrm{Cl}_\Sigma(X)$. If $\gamma\in X$ then $\gamma+1\in X\subseteq\mathrm{CL}_\Sigma(X)$ obviously. Assume $\gamma\notin X$. Then $\gamma\geqq\omega$, and $f^B(\gamma)=\gamma+1$. Hence $\gamma+1\in\mathrm{CL}_\Sigma(X)$ by Corollary 13. Let $\alpha\in B$ be a limit ordinal and assume $\alpha\subseteq\mathrm{CL}_\Sigma(X)$. Then by $\alpha=\sup\limits_{\leqq_B}\alpha$ and $\alpha\in Z(B)$ we conclude $\alpha\in\mathrm{cl}\,(\mathrm{CL}_\Sigma(X))\subseteq\mathrm{CL}_\Sigma(X)$ by Corollary 15. Thus by induction we proved $B=\varkappa+1=\mathrm{CL}_\Sigma(X)$. Hence by Lemma 7 we have checked that h is an epi both in $\perp ZP\,\mathrm{Alg}_\Sigma$ and in $\perp Z\,\mathrm{Alg}_\Sigma$. □

By Assertion 6 and the definition of $\mathfrak{B}$ we proved that $\mathfrak{A}$ is such that $(\forall\varkappa\in\mathrm{Ord})\cdot(\exists\mathfrak{B})(\exists\mathfrak{A}\twoheadrightarrow\mathfrak{B})\,|B|\geqq|\varkappa|$, which means that *the epimorphic images of* $\mathfrak{A}$ are not isomorphic to any sub*set* of $\mathrm{Ob}\perp ZP\,\mathrm{Alg}_\Sigma$. Thus $\perp ZP\,\mathrm{Alg}_\Sigma$ is not co-well-powered.

Since h is an epi in $\perp Z\,\mathrm{Alg}_\Sigma$ as well, we have that $\perp Z\,\mathrm{Alg}_\Sigma$ is not co-well-powered either. □

Problem 31. Is $\perp ZP\,\mathrm{Alg}_\Sigma$ co-well-powered for some *un*bounded Z? More precisely, is it true that for all Σ there is some unbounded Z such that $\perp ZP\,\mathrm{Alg}_\Sigma$ is co-well-powered?

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