Linear deterministic attributed transformations

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Introduction

This paper is based on and continues our earlier work [2] in the subject. Our point of view is close to that of the authors' of [3] inasmuch as we, too, translate an attribute grammar (or transformation) into a system of recursive definitions. Our aim was to define attributed transformations as homomorphisms between suitable algebras that can be constructed from well-known ones in a natural way. Rational algebraic theories (cf. [13]) and magmoids (cf. [1]) turned out to be the most appropriate for this purpose. Two questions may arise in connection with our new definition.

1. Why do we use these complex many sorted algebras if our aim is to map T_Σ, the free Σ-algebra, into a certain attributed structure? It would be enough to define an appropriate Σ-algebra on this structure.

Beyond the notational convenience and elegance of proofs there is one more reason. Investigating one specific attributed transformation it is generally easier to deal with Σ-algebras only. However, if we investigate e.g. the composition properties of these transformations (tree transformations here), the process of "translating" into a Σ-algebra becomes rather tedious and affected. In this case the main advantage is that we can get rid of the alphabet Σ.

2. Wouldn't it be enough to use algebraic theories only instead of magmoids?

It is true that most of the results in [4] concerning top-down tree transformations could be formulated within the framework of projective magmoids, i.e. nondegenerate algebraic theories. An attributed transformation, however, is defined by a homomorphism $h\colon \tilde{T}(\Sigma)\to \mathbf{D}R[k,l]$ (for the precise definitions see later), i.e. a homomorphism between (decomposable) magmoids. One might say that the homomorphism $\mathbf{T}h\colon T(\Sigma)\to \mathbf{TD}R[k,l]$ is already between algebraic theories. This is true, but it turns out that homomorphisms of $T(\Sigma)$ into $\mathbf{TD}R[k,l]$ generally define more complex transformations, called macro transformations (cf. [7]).

For simplicity we assume that the set of possible values is the same for all the attributes. A natural way to generalize our definition could be the introduction of "many sorted" rational theories.

1. Preliminaries

In this section we recall the basic concepts and definitions from [2] concerning attributed transformations.

A magmoid $M=(\{M(p,q)|p,q\geqq 0\}, \cdot, +, 1, 1_0)$ is a special many-sorted algebra whose sorting set consists of all pairs of nonnegative integers. $\cdot$ and $+$ denote binary operations called composition and separated sum (tensor product), respectively. Composition (rather denoted by juxtaposition) maps $M(p,q)\times M(q,r)$ into $M(p,r)$, and separated sum maps $M(p_1,q_1)\times M(p_2,q_2)$ into $M(p_1+p_2, q_1+q_2)$. $1\in M(1,1)$ and $1_0\in M(0,0)$ denote nullary operations. The following axioms must be valid in M.

(i) $(ab)c=a(bc)$ for any composable pairs $\langle a,b\rangle$ and $\langle b,c\rangle$;

(ii) $(a+b)+c=a+(b+c)$;

(iii) $(ab)+(cd)=(a+c)(b+d)$;

(iv) $a1_p=1_qa=a$ if $a\in M(p,q)$ and $1_n=\sum_{i=1}^{n} 1$ for $n\geqq 1$;

(v) $a+1_0=1_0+a=a$.

Due to (i) and (iv) M becomes a category whose objects are the nonnegative integers and the identities are the elements 1_n $(n\geqq 0)$. (For a complex categorical definition of magmoids see [11].) Therefore, $a\in M(p,q)$ is often written as $a\colon p\to q$ if M is understood.

Let $\Theta(p,q)$ denote the set of all mappings of $[p]=\{1,\dots,p\}$ into $[q]$. Defining the composition and separated sum of mappings as it is usual, and taking the identity map of $[n]$ for 1_n we get the magmoid Θ. We denote the unique element of $\Theta(0,q)$ by 0_q $(0_0=1_0)$, and the injection $1\to p$ which picks out i from $[p]$ by π_p^i (or π_i if p is understood). For an arbitrary $\theta\in\Theta(p,q)$, $i\theta$ stands for the image of $i\in[p]$ under θ.

A magmoid M is called projective if it contains a submagmoid Θ_M isomorphic to Θ, and the following holds for every $a,b\in M(p,q)$. If $\pi_i a=\pi_i b$ for each $i\in[p]$, then $a=b$. Generally we shall assume that $\Theta_M=\Theta$, to be able to use the same notations in M as in Θ. It can be proved that for any $a_1,\dots,a_p\colon 1\to q$ there exists a unique $a\colon p\to q$ such that $\pi_i a=a_i$ for each $i\in[p]$. This element will be denoted by $\langle\!\langle a_1,\dots,a_p\rangle\!\rangle$. We shall use $\langle\!\langle$and$\rangle\!\rangle$ (source-tupling) as a derived operation, extending it to the case $a_i\colon p_i\to q$ in the usual way. (In this case $\langle\!\langle a_1,\dots,a_p\rangle\!\rangle\colon \sum_{i=1}^{p} p_i\to q$.) It was pointed out in [1] that every projective magmoid is in fact a non-degenerate algebraic theory and vice versa, depending on whether separated sum, or source tupling and the injections are considered as basic operations.

It is well-known that for every ranked alphabet $\Sigma=\bigcup_{n\geqq 0}\Sigma_n$ there exists a free projective magmoid generated by Σ, which we denote by $T(\Sigma)$. $T(\Sigma)$ has a representation by finite Σ-trees on the variables $X=\{x_1,x_2,\dots\}$ (cf. [1]). Viewing $\sigma\in\Sigma_n$ as $\sigma(x_1,\dots,x_n)\in T(\Sigma)(1,n)$ (which makes Σ a subsystem of $T(\Sigma)$), $T(\Sigma)$ has the property that any ranked alphabet map $h\colon\Sigma\to M$ into a projective magmoid M has a unique homomorphic extension $\bar{h}\colon T(\Sigma)\to M$. In particular, if Σ is the void alphabet, then $T(\Sigma)=\Theta$.

$T(\Sigma)$ has an important subsystem $\tilde{T}(\Sigma)$ defined as follows. $t\in\tilde{T}(\Sigma)(p,q)$ iff the frontier of t, i.e. the sequence of variables appearing at the leaves of t, is exactly

$x_1, \ldots, x_q$. $\tilde{T}(\Sigma)(1,0)=T(\Sigma)(1,0)$ will be denoted by T_Σ. $\tilde{T}(\Sigma)$ is a submagmoid of $T(\Sigma)$, and it is the free magmoid generated by Σ. It has the property that every $t\in T(\Sigma)$ can be uniquely written in the form $\tilde{t}\vartheta$ with $\tilde{t}\in\tilde{T}(\Sigma)$ and $\vartheta\in\Theta$.

Let $\hat{\Theta}$ denote the submagmoid of all injective mappings in Θ. $t\in T(\Sigma)$ is called linear if $\vartheta\in\hat{\Theta}$ by the decomposition $t=\tilde{t}\vartheta$ above. Clearly, the linear elements also form a submagmoid of $T(\Sigma)$, which we denote by $\hat{T}(\Sigma)$.

$\tilde{T}(\Sigma)$ is free in the important subclass of decomposable magmoids, too. A magmoid M is called decomposable if the following two conditions are satisfied:

(i) for every $a\colon p\to q$ $(p\geqq 2, q\geqq 0)$ and $i\in[p]$ there exists exactly one integer q_i and $a_i\colon 1\to q_i$ such that $a=a_1+\ldots+a_p$;

(ii) $M(0,0)=\{1_0\}$.

Any magmoid M can be made decomposable by the application of the functor **D**. **D** operates as follows:

(i) $\mathbf{D}M(1,q)=M(1,q)$ if $q\geqq 0$,
$1=1_M$,
$\mathbf{D}M(0,q)=\mathit{if}\ q=0\ \mathit{then}\ \{\emptyset\}\ \mathit{else}\ \emptyset$,
if $p\geqq 2$, then $\mathbf{D}M(p,q)\subseteq(\bigcup_{r\geqq 0} M(1,r))^p$ such that $\langle a_1, \ldots, a_p\rangle\in\mathbf{D}M(p,q)$ with $a_i\colon 1\to q_i$ iff $\sum_{i=1}^{p} q_i=q$;

(ii) $\langle a_1, \ldots, a_{p_1}\rangle+\langle b_1, \ldots, b_{p_2}\rangle=\langle a_1, \ldots, a_{p_1}, b_1, \ldots, b_{p_2}\rangle$;

(iii) if $a=\langle a_1, \ldots, a_p\rangle\colon p\to q$ with $a_i\colon 1\to q_i$ and $b=\langle b_1, \ldots, b_q\rangle\colon q\to r$, then

$$a.b = \langle a_1(.)_M b^{(1)}, \ldots, a_p(.)_M b^{(p)}\rangle,$$

where $b^{(i)}=\left(\sum_{j=q^{(i)}}^{q^{(i+1)}}\right)_M b_j$ and $q^{(i)}=\sum_{j=1}^{i-1} q_j$ $(i\in[p+1])$;

(iv) if $h\colon M\to M'$ is a homomorphism, then

$$\mathbf{D}h(\langle a_1, \ldots, a_p\rangle) = \langle h(a_1), \ldots, h(a_p)\rangle.$$

There is a natural homomorphism $\zeta\colon \mathbf{D}M\to M$ for which $\zeta(\langle a_1, \ldots, a_p\rangle)= =a_1+\ldots+a_p$.

Any decomposable magmoid M can be made projective by the application of the functor **T** which operates as follows:

(i) $\mathbf{T}M(p,q)=\bigcup(M(p,q')\times\Theta(q',q)\mid q'\geqq 0)$,
$1=\langle 1_M, 1_\Theta\rangle$, $1_0=\langle(1_0)_M, (1_0)_\Theta\rangle$;

(ii) $\langle a_1, \vartheta_1\rangle+\langle a_2, \vartheta_2\rangle=\langle a_1+a_2, \vartheta_1+\vartheta_2\rangle$;

(iii) let $a\colon p\to q'$, $\vartheta\colon q'\to q$, $b=\langle b_1, \ldots, b_q\rangle\colon q\to r$ with $b_i\colon 1\to r_i$ $(i\in[q])$ and $\varphi\colon r'\to r$. φ can be uniquely written in the form $\nleftarrow\varphi_1, \ldots, \varphi_q\nrightarrow$, where for each $i\in[q]\,\varphi_i\colon r_i\to r$. Now $\langle a, \vartheta\rangle.\langle b, \varphi\rangle=\langle a(.)_M\langle b_{1\vartheta}, \ldots, b_{q'\vartheta}\rangle, \nleftarrow\varphi_{1\vartheta}, \ldots, \varphi_{q'\vartheta}\nrightarrow\rangle$;

(iv) if $h\colon M\to M'$ is a homomorphism between decomposable magmoids, then $\mathbf{T}h(\langle a, \vartheta\rangle)=\langle h(a), \vartheta\rangle$.

We shall also use a restriction of **T** denoted by $\hat{\mathbf{T}}$. $\langle a, \vartheta\rangle\in\hat{\mathbf{T}}M$ iff $\vartheta\in\hat{\Theta}$. It is easy to see that $\hat{\mathbf{T}}M$ is a submagmoid of $\mathbf{T}M$, so $\hat{\mathbf{T}}$ is also a functor. It is well-known that $\mathbf{T}(\tilde{T}(\Sigma))\cong T(\Sigma)$ and $\hat{\mathbf{T}}(\tilde{T}(\Sigma))\cong\hat{T}(\Sigma)$.

Let M be a magmoid and k an arbitrary natural number. k-dil M denotes the magmoid for which $(k\text{-dil}\, M)(p,q)=M(kp,kq)$, $1=(1_k)_M$, $1_0=(1_0)_M$ and the further operations are performed in it just as in M. Clearly, the operator k-dil can also be extended to a functor. Let η_k denote the inclusion function: k-dil $M \to M$. η_k is not a homomorphism, it is only a so called k-morphism. To avoid ambiguity, $M = k$-dil Θ will be the only exeption when we distinguish Θ_M from Θ, using the unique embedding $\iota_k: \Theta \to k$-dil Θ.

Rational algebraic theories were introduced in [13]. To remain in circles of magmoids we define this concept by means of projective magmoids, thus excluding the trivial degenerate rational theory. A rational theory R is a projective magmoid equipped with a new unary operation $^\dagger: R(p, p+q) \to R(p, q)$, called iteration. The carrier sets and the operations are required to satisfy the following conditions:

(i) for each $p, q \geqq 0$, $R(p, q)$ is partially ordered with minimal element $\perp_{p,q}$ ($\perp_p$ if q is understood);

(ii) separated sum and composition are monotonic, and the latter is left strict, i.e. $\perp_{p,q} a = \perp_{p,r}$ for $a: q \to r$;

(iii) let $a: p \to p+q$, and construct the sequence $(a_i: p \to q \mid i \geqq 0)$ as follows

$$a_0 = \perp_{p,q},\ a_{i+1} = a \langle a_i, 1_q \rangle \quad \text{for} \quad i \geqq 0.$$

Then $\bigcup_{i \geqq 0} a_i$ exists and equals $a^\dagger$;

(iv) composition is both left and right continuous.

Since rational theories are ordered algebras, a homomorphism between them is required to preserve the ordering, too. It was shown in [13] that for every ranked alphabet Σ the free rational theory generated by Σ exists. This theory $R(\Sigma)$ has a representation by infinite $\Sigma_\perp$-trees on X, where $\Sigma_\perp = \Sigma \cup \{\perp\}$ and $\perp$ is a new symbol with rank 0. Reg (Σ) will denote the rational theory of all regular forests of finite Σ-trees on X.

Definition 1.1. Let R be a rational theory, $k \geqq 1$, $l \geqq 0$ integers. Define $R[k, l] = (\{R[k, l](p, q) \mid p, q \geqq 0\}, \cdot, +, 1, 1_0)$ to be the following structure. (We do not use the subscript M to indicate the magmoid in which the operations are performed if only one M is reasonable from the context.)

(i) $R[k, l](p, q) = R(kp+lq, kq+lp)$,
$1 = (1_{k+l})_R$, $1_0 = (1_0)_R$;

(ii) if $a \in R[k, l](p_1, q_1)$, $b \in R[k, l](p_2, q_2)$, then

$$a+b = \langle \mu_{lq_1}^{kp_1} + \mu_{lq_2}^{kp_2}, \nu_{lq_1}^{kp_1} + \nu_{lq_2}^{kp_2} \rangle \cdot (a+b) \cdot \langle \mu_{lp_1}^{kq_1} + \mu_{lp_2}^{kq_2}, \nu_{lp_1}^{kq_1} + \nu_{lp_2}^{kq_2} \rangle^{-1},$$

where μ_m^n (μ_n if m is understood) $= 1_n + 0_m$, ν_m^n (ν_m if n is understood) $= 0_n + 1_m$;

(iii) if $a \in R[k, l](p, q)$, $b \in R[k, l](q, r)$, then $a \cdot b = \langle \mu_{kp}, \nu_{lr} \rangle \cdot \langle a\vartheta, b\varphi \rangle^\dagger$, where $\vartheta = \nu_{kq}^{kp+lq} + \nu_{lp}^{(k+l)r}$, $\varphi = 0_{kp} + \langle \nu_{kr}^{(k+l)q}, \mu_{kq+(k+l)r}^{lq} \rangle + 0_{lp}$ (see also Fig. 1).

In [2] we proved that $R[k, l]$ is a magmoid. Let $\xi: R \to R'$ be a homomorphism between rational theories. Clearly, ξ defines a homomorphism $\xi[k, l]: R[k, l] \to R'[k, l]$, and so the operator $[k, l]$ becomes a functor.

Definition 1.2. An attributed transducer (a-transducer) is a 6-tuple $\mathbf{A} = (\Sigma, R, k, l, h, S)$, where

(i) Σ is a finite ranked alphabet, $S \notin \Sigma$;

(ii) R is a rational theory, $k \geqq 1$, $l \geqq 0$ are integers;

(iii) $h: \Sigma_S \to \mathbf{D}R[k, l]$ is a ranked alphabet map, where $\Sigma_S = \Sigma \cup \{S\}$ with S

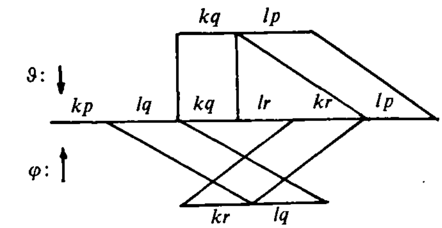


Fig. 1

having rank 1. $h(S)$ is required to be a synthesizer, i.e. $h(S)=a+0_l$ for some $a\in R(k+l, k)$.

Extend h to a homomorphism $h: \tilde{T}(\Sigma)\to \mathbf{D}R[k, l]$. $\tau_{\mathbf{A}}: T_\Sigma \to R(1, 0)$, the transformation defined by $\mathbf{A}$ is the following function. $\tau_{\mathbf{A}}(t)=a$, where $\pi_k^1 h(S(t))= =a+0_l$.

Let Δ be a ranked alphabet, and consider the homomorphism $\varepsilon_\Delta: R(\Delta)\to \mathrm{Reg}\,(\Delta)$ for which $\varepsilon_\Delta(\delta)=\{\delta(x_1, \ldots, x_n)\}$ if $\delta\in\Delta_n$. Let Θ_Δ denote the congruence relation induced by ε_Δ. For simplicity we shall identify each $t\in T(\Delta)$ with its class $[t]\Theta_\Delta$.

Definition 1.3. A deterministic attributed tree transducer (a-tree transducer) from Σ into Δ is an a-transducer $\mathbf{A}=(\Sigma, R(\Delta)/\Theta_\Delta, k, l, h, S)$. In this case we consider $\tau_{\mathbf{A}}\subseteq T_\Sigma\times T_\Delta$ as a relation

$$\tau_{\mathbf{A}} = \{\langle t, u\rangle \mid \pi_k^1(h(S(t))) = u+0_l \quad \text{and} \quad u\in T_\Delta\}.$$

Further on a deterministic a-tree transducer from Σ into Δ will rather be denoted by the 6-tuple $(\Sigma, \Delta, k, l, h, S)$.

$\mathbf{A}=(\Sigma, \Delta, k, l, h, S)$ is called total if $h(\sigma)\in T(\Delta)$ for each $\sigma\in\Sigma_S$. Determinism, totality and linearity of tree transducers will be denoted by d, t and l, respectively. Since k-dil $T(\Delta)$ is a submagmoid of $R(\Delta)[k, 0]$, every dta-tree transducer with s-attributes only ($l=0$) is in fact a dt-top-down tree transducer and vice versa.

Let $t\in R(\Delta)[k, l](p, q)$. It is convenient to consider t as the image of a tree $u\in\tilde{T}(\Sigma)(p, q)$ under a suitable homomorphism $h: \tilde{T}(\Sigma)\to R(\Delta)[k, l]$. To underline the attributed feature of t we introduce the following notations

$$\underline{t}(r, i) = \pi_{k(r-1)+i}t \quad \text{if} \quad r\in[p], \quad i\in[k];$$

$$\bar{t}(j, m) = \pi_{kp+l(j-1)+m}t \quad \text{if} \quad j\in[q], \quad m\in[l];$$

$$x(j, i) = x_{k(j-1)+i} \quad \text{if} \quad j\in[q], \quad i\in[k];$$

$$y(r, m) = x_{kq+l(r-1)+m} \quad \text{if} \quad r\in[p], \quad m\in[l].$$

The intuitive meaning of these items is the following.

$\underline{t}(r, i)$: the value of the i-th synthesized attribute (s-attribute) of the r-th root;
$\bar{t}(j, m)$: the value of the m-th inherited attribute (i-attribute) of the j-th leaf;
$x(j, i)$: reference to the i-th s-attribute of the j-th leaf;
$y(r, m)$: reference to the m-th i-attribute of the r-th root.

Naturally, the roots and leaves above belong to u that we never mention explicitely. If $p=1$, then $\underline{t}(i)$ and $y(m)$ stand for $\underline{t}(1, i)$ and $y(1, m)$, respectively. We shall use these notations for $a \in R[k, l](p, q)$, too, after defining the concept of dependence on a variable in an arbitrary rational theory.

2. The composition of a dtla-tree transformation and an arbitrary a-transformation

Linear top-down tree transducers can be defined in two different ways. The original definition in [5] requires all the rules of the transducer be linear in the sense that no variable occurs more than once on the right-hand side of a rule. If we represent the transducer by a k-morphism of magmoids, say h: $T(\Sigma) \to k$-dil $T(\Delta)$ (the transducer is taken dt for simplicity), then $h(\sigma)$ $(\sigma \in \Sigma)$ resumes all the rules above in which σ appears on the left-hand side. However, the meaning of the variables in $h(\sigma)$ differs from that of the variables occuring in the rules. Therefore, if we require for all $\sigma \in \Sigma$ $h(\sigma)$ not contain two different occurences of the same variable, which is the second way to define linearity, the transducer need not be linear in the original sense, and vice versa.

Unfortunately, the original definition cannot be carried out in the case of a-tree transducers (even if the transducer is described by a set of rules as in [9]), but the second one can be adopted quite naturally.

Definition 2.1. $t \in R(\Delta)[k, l](1, q)$ is called linear if $t \in \hat{T}(\Delta_\perp)$. $t \in \mathbf{D}R(\Delta)[k, l](p, q)$ is linear if $t = 1_0$, or $t = t_1 + \ldots + t_p$ and each t_i: $1 \to q_i$ $(i \in [p])$ is linear. $\mathbf{A} = (\Sigma, \Delta, k, l, h, S)$ is linear if $h(\sigma)$ has a linear representant for every $\sigma \in \Sigma_S$.

Let $L_\perp(\Delta)[k, l]$ denote the system of all linear elements in $\mathbf{D}R(\Delta)[k, l]$, and $L(\Delta)[k, l]$ that of all linear and total ones.

Lemma 2.2. $L_\perp(\Delta)[k, l]$ and $L(\Delta)[k, l]$ are submagmoids of $\mathbf{D}R(\Delta)[k, l]$.

Proof. It is enough to prove the lemma for $L(\Delta)[k, l]$. Indeed, let φ: $R(\Delta_\perp) \to R(\Delta)$ be the homomorphism extending the identity map $\Delta_\perp \to \Delta \cup \{\perp\}$. If $L(\Delta_\perp)[k, l]$ is a submagmoid, then so is $L_\perp(\Delta)[k, l]$, which is the image of it under the embedding $\mathbf{D}\varphi[k, l]$.

Let $t \in R(\Delta)[k, l](1, q)$ be arbitrary, and construct the directed graph G_t as follows. The nodes of G_t are

$$\{\mathbf{rs}(i), \mathbf{ri}(m), \mathbf{ls}(j, i), \mathbf{li}(j, m) | i \in [k], m \in [l], j \in [q]\}$$

(**s**, **i**, **r** and **l** suggest synthesized, inherited, root and leaf, respectively). There is an arc from $\mathbf{rs}(i)$ to $\mathbf{ls}(j, i')(\mathbf{ri}(m))$ iff $\underline{t}(i)$ contains an occurence of $x(j, i')(y(m)$, resp.). Similarly, there is an arc from $\mathbf{li}(j, m)$ to $\mathbf{ls}(j', i)(\mathbf{ri}(m'))$ iff $\bar{t}(j, m)$ contains an occurence of $x(j', i)(y(m')$, resp.). G_t has no more arcs than those listed above. G_t is called a dependency graph. Unfortunately, the direction of the arcs is just the opposite of the direction used in most of works concerning attribute grammars (e.g. [8], [9], [12]). However, this direction is more natural from the point of view that t: $k + lq \to kq + l \in R(\Delta)$, where the arrow leads from the "components" to the "variables".

Clearly, $t \in L(\Delta)[k, l]$ iff
(i) $t \in T(\Delta)$;
(ii) there is at most one arc entering each of the nodes

$$\{\mathbf{ls}(j, i),\ \mathbf{ri}(m) | i \in [k],\ m \in [l],\ j \in [q]\}$$

in G_t (i.e. G_t is a forest).

Let $q_0 \geqq 0$ and for each $0 \leqq s \leqq q_0$, $t_s \in L(\Delta)[k, l](1, q_s)$ with $\sum_{s=1}^{q_0} q_s = q$. It suffices to prove that $t = t_0 \cdot \sum_{s=1}^{q_0} t_s \in L(\Delta)[k, l](1, q)$. Construct the graphs G_{t_s} for each $0 \leqq s \leqq q_0$, marking the nodes of G_{t_s} with a subscript s. For each $i \in [k]$, $m \in [l]$, $j \in [q_0]$ identify the node $\mathbf{ls}_0(j, i)$ with $\mathbf{rs}_j(i)$ and $\mathbf{li}_0(j, m)$ with $\mathbf{ri}_j(m)$ to get the graph G. This graph fully describes the dependence relation of the attributes while performing the composition $t_0 \cdot \sum_{s=1}^{q_0} t_s$. Therefore it is easy to see that $G_t = G^+ \setminus N_{\text{in}}$, where G^+ denotes the transitive closure of G and

$$N_{\text{in}} = \{\mathbf{ls}_0(j, i)(\equiv \mathbf{rs}_j(i)),\ \mathbf{li}_0(j, m)(\equiv \mathbf{ri}_j(m)) | i \in [k],\ m \in [l],\ j \in [q_0]\}.$$

Let us remark that, by construction, there is at most one arc entering each node of G, moreover, no arc enters the nodes

$$\{\mathbf{rs}_0(i),\ \mathbf{li}_s(j, m) | i \in [k],\ m \in [l],\ s \in [q_0],\ j \in [q_s]\}.$$

This implies that the connected subgraphs starting from these nodes are trees, so t is finite and G_t is a forest, which was to be proved.

Observe that the connected subgraphs starting from the nodes of N_{in} might be circles. This means that circularity might appear if we want to achieve the result of the composition by computing the value of all the concerning attributes, but this "inside" circularity does not affect the value of the important attributes.

Now we generalize the notion of "dependence on a variable" to projective magmoids.

Lemma 2.3. Let M be a projective magmoid with $M(1, 0) \neq \emptyset$. For any $a \in M(p, q)$ let $a = a'\vartheta$, where $a'\colon p \to q'$, $\vartheta \in \hat{\Theta}(q', q)$ and q' is minimal. The image of ϑ, $\operatorname{Im}(\vartheta)$ is then uniquely determined.

Proof. Suppose the decompositions $a = a_1'\vartheta_1 = a_2'\vartheta_2$ both satisfy the conditions of the lemma and $\operatorname{Im}(\vartheta_1) \neq \operatorname{Im}(\vartheta_2)$, e.g. $i \in \operatorname{Im}(\vartheta_1)$ but $i \notin \operatorname{Im}(\vartheta_2)$. Let $\perp \in M(1, 0)$, and consider the element $\varrho = 1_{i-1} + \perp + 0_1 + 1_{q-i}\colon q \to q$. Since $i \notin \operatorname{Im}(\vartheta_2)$, we have $\vartheta_2\varrho = \vartheta_2$, thus, $a = a_1'\vartheta_1\varrho$. Observe that $\vartheta_1\varrho = (1_{j-1} + \perp + 0_1 + 1_{q'-j})\vartheta_1$, where $i = j\vartheta_1$. On the other hand

$$1_{j-1} + \perp + 0_1 + 1_{q'-j} = (1_{j-1} + \perp + 1_{q'-j})(1_{j-1} + 0_1 + 1_{q'-j}),$$

that is

$$a_1'\vartheta_1\varrho = (a_1'(1_{j-1} + \perp + 1_{q'-j}))(1_{j-1} + 0_1 + 1_{q'-j})\vartheta_1.$$

This is a contradiction, since q' was supposed to be minimal.

We shall say that $a: p\to q$ depends on x_i ($i\in[q]$) if $i\in\mathrm{Im}(\vartheta)$ by the decomposition $a=a'\vartheta$ above.

Let R be a rational theory, and extend the homomorphism ζ: $\mathbf{D}R[k,l]\to R[k,l]$ to a mapping $\hat{\zeta}$: $\hat{\mathbf{T}}\mathbf{D}R[k,l]\to R[k,l]$ as follows. For $\vartheta\in\hat{\Theta}(p,q)$ let

$$\hat{\zeta}(\vartheta)=\langle\!\langle \eta_k(\iota_k(\vartheta))+0_{lp},\ 0_{kq}+\varphi\rangle\!\rangle,$$

where φ: $lq\to lp$ satisfies

$$\pi_{(j-1)+m}\varphi = \mathit{if}\quad j\in\mathrm{Im}(\vartheta)\quad \mathit{and}\quad j=i\vartheta\quad \mathit{then}\quad \pi_{l(i-1)+m}\quad \mathit{else}\quad \bot_1$$

for each $j\in[q]$ and $m\in[l]$. Now, for any $a: n\to p$ and $\vartheta: p\to q$ let $\hat{\zeta}(\langle a,\vartheta\rangle)=\zeta(a).\hat{\zeta}(\vartheta)$.

Intuitively, $\hat{\zeta}(\langle a,\vartheta\rangle)$ can be obtained as follows (for simplicity let $R=R(\Delta)$ and a: $1\to p$). Starting from $\zeta(a)$, any reference to an **s**-attribute of a leaf (say the i-th) must be pointed to the corresponding **s**-attribute of the $i\vartheta$-th leaf. References to **i**-attributes of the root remain unaltered (though the corresponding variable indices may be shifted), but the values of the **i**-attributes of the leaves must also be rearranged according to ϑ. The value of all the **i**-attributes of a "fictive" leaf is set to $\bot$.

The following example shows that, contrary to our expectations, $\hat{\zeta}$ is not a homomorphism.

Let $R=R(\Delta)$ with $\Delta=\Delta_1=\{\delta\}, k=l=1$. Consider the elements $a=\delta(y(1))$: $1\to 0$ and $b=\langle\bot,\delta(y(1))\rangle$: $1\to 1$ of $\mathbf{D}R(\Delta)[1,1]$. Then

$$\hat{\zeta}(\langle a,0_1\rangle.\langle b,1\rangle)=\hat{\zeta}(\langle a,0_1\rangle)=\langle\delta(y(1)),\bot\rangle,$$

but $\hat{\zeta}(\langle a,0_1\rangle).\hat{\zeta}(\langle b,1\rangle)=\langle\delta(y(1)),\delta(\bot)\rangle$.

However, it must be noticed that the only difference is between the values of the **i**-attrbutes of the "fictive" leaf.

Let $\hat{R}[k,l]\subseteq R[k,l]$ be the following system. $a\in\hat{R}[k,l](p,q)$ iff there exists a system $I=\{I(r)\subseteq[q]\,|\,r\in[p]\}$ of pairwise disjoint subsets of $[q]$ for which the following two conditions are satisfied:

(i) if $\underline{a}(r,i)$ depends on $x(j,i')$ $r\in[p]$, $j\in[q]$, $i,i'\in[k]$), then $j\in I(r)$, moreover, if $\underline{a}(r,i)$ depends on $y(r',m)$ ($r'\in[p], m\in[l]$), then $r=r'$;

(ii) if $\bar{a}(j,m)$ depends on $y(r,m')$, then $j\in I(r)$, moreover, if $\bar{a}(j,m)$ depends on $x(j',i)$, then for each $r\in[p]$ we have: $j\in I(r)$ iff $j'\in I(r)$.

For a fixed $a\in\hat{R}[k,l](p,q)$ there might be several systems I satisfying (i) and (ii) above. There exists, however, a minimal one I_a, in which for every $r\in[p]$, $I_a(r)$ is the least subset of $[q]$ satisfying the following two conditions:

(i) if $\underline{a}(r,i)$ depends on $x(j,i')$, then $j\in I_a(r)$ i

(ii) if $\bar{a}(j,m)$ depends on $x(j',i)$ for some; $j\in I_a(r)$, then $j'\in I_a(r)$, too.

Define the binary relation Ψ on $\hat{R}[k,l]$ as follows. For every a,b: $p\to q$, $a\Psi b$ iff

(i) $I_a=I_b$;

(ii) $\underline{a}(r,i)=\underline{b}(r,i)$ for each $r\in[p], i\in[k]$;

(iii) $\bar{a}(j,m)\neq\bar{b}(j,m)$ implies that $j\notin I_a(r)$ for any $r\in[p]$. We shall see that $\hat{R}[k,l]$ is a submagmoid of $R[k,l]$ and Ψ is a congruence relation. It could also be proved that $\hat{\zeta}\Psi$: $\hat{\mathbf{T}}\mathbf{D}R[k,l]\to\hat{R}[k,l]/\Psi$ is already a homomorphism.

Let us start with two easy observations.

Proposition 2.4. 1. For any appropriate $a \in \mathbf{D}R[k,l]$ and $\vartheta \in \hat{\Theta}$, $\zeta(\langle a, \vartheta \rangle) \in \hat{R}[k,l]$.

2. If $\xi: R \to R'$ is a homomorphism and $a\Psi b$ holds in $\hat{R}[k,l]$, then $\xi[k,l](a)\Psi\xi[k,l](b)$ holds in $\hat{R}'[k,l]$.

The first statement is trivial, while the second follows from the fact that the components of $\xi[k,l](a)$ and $\xi[k,l](b)$ depend on at most the same variables as the corresponding components of a and b do.

Let $k' \geqq 1, l' \geqq 0$, and for each $q \geqq 0$ define the bijections ϱ_q and ϱ'_q as follows

$$\varrho_q = 1_{kk'} + \langle 0_{ll'} + \langle \sum_{j=1}^{q} \mu_{kl'}^{k'l}, \sum_{j=1}^{q} \nu_{kl'}^{k'l} \rangle, \mu_{ll'} \rangle,$$

$$\varrho'_q = \langle \sum_{j=1}^{q} \mu_{ll'}^{kk'} + 1_{k'l+kl'}, \sum_{j=1}^{q} \nu_{ll'}^{kk'} + 0_{k'l+kl'} \rangle^{-1}.$$

See also Fig. 2.

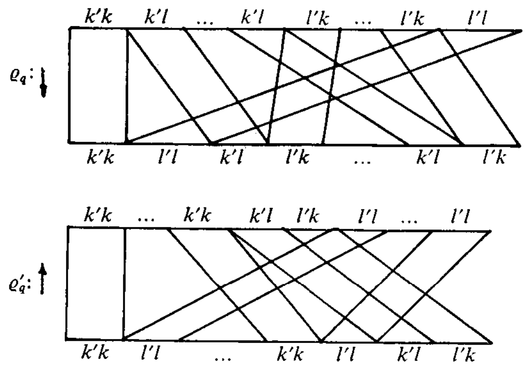


Fig. 2

Definition 2.5. $a \in R[k'k+l'l, k'l+l'k](1,q)$ is called $[k,l]$-linear if $\varrho_q a \varrho'_q \in \hat{R}[k',l'](k+lq, kq+l)$. Generally, $a \in \mathbf{D}R[k'k+l'l, k'l+l'k]$ is $[k,l]$-linear if $a=1_0$, or $a=a_1+\ldots+a_p$ and for each $i \in [p]$, a_i: $1 \to q_i$ is $[k,l]$-linear.

Let $a, b \in \mathbf{D}R[k'k+l'l, k'l+l'k]$ be $[k,l]$-linear elements, $a=\sum_{i=1}^{p} a_i$, $b=\sum_{i=1}^{p} b_i$ with a_i, b_i: $1 \to q_i$ $(i \in [p])$. Define $a\Phi b$ iff $\varrho_q a_i \varrho'_q \Psi \varrho_q b_i \varrho'_q$ for each $i \in [p]$.

Lemma 2.6. The $[k,l]$-linear elements form a (decomposable) submagmoid of $\mathbf{D}R[k'k+l'l, k'l+l'k]$ and Φ is a congruence relation on it.

The proof of this lemma will be given in the Appendix because of the great amount of computation it needs. The submagmoid of $[k,l]$-linear elements will be denoted by $L[k,l]\mathbf{D}R[k'k+l'l, k'l+l'k]$. Taking $k=1$ and $l=0$ in the lemma we get that $\hat{R}[k',l']$ is a submagmoid of $R[k',l']$ and Ψ is a congruence relation, as we stated it before.

Let h: $\tilde{T}(\Delta)\to R[k', l']$ be a homomorphism, and define the mapping $\hat{h}[k, l]$: $L(\Delta)[k, l]\to L[k, l]\mathbf{D}R[k'k+l'l, k'l+l'k]$ as follows (the notation $\hat{h}[k, l]$ is somewhat abusing here):

(i) for $t\in L(\Delta)[k, l](1, q)(=\hat{\mathbf{T}}\tilde{T}(\Delta)(k+lq, kq+l))$
$\hat{h}[k, l](t)=\varrho_q^{-1}\hat{\zeta}(\hat{\mathbf{T}}h(t))\varrho_q'^{-1}$;

(ii) for $t=t_1+\ldots+t_p\in L(\Delta)[k, l](p, q)$ (t_i: $1\to q_i$)
$\hat{h}[k, l](t)=\hat{h}[k, l](t_1)+\ldots+\hat{h}[k, l](t_p)$.

Lemma 2.7. $\hat{h}[k, l]\Phi$: $L(\Delta)[k, l]\to L[k, l]\mathbf{D}R[k'k+l'l, k'l+l'k]/\Phi$ is a homomorphism.

This lemma, too, will be proved in the Appendix.

Now we are ready to prove the main result of this section.

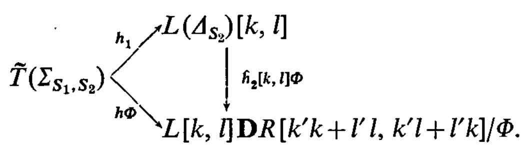
Theorem 2.8. Let $\mathbf{A}_1=(\Sigma, \Delta, k, l, h_1, S_1)$ be a dtla-tree transducer, $\mathbf{A}_2=(\Delta, R, k', l', h_2, S_2)$ an arbitrary a-transducer. Then $\tau_{\mathbf{A}_1}\circ\tau_{\mathbf{A}_2}$ is also an a-transformation.

Proof. By Lemma 2.2, h_1 is in fact a homomorphism of $\tilde{T}(\Sigma_{S_1})$ into $L(\Delta)[k, l]$. Let $\Sigma_{S_1,S_2}=\Sigma\cup\{S_1, S_2\}$, where S_1 and S_2 both have rank 1, and extend h_1 to a homomorphism of $\tilde{T}(\Sigma_{S_1,S_2})$ into $L(\Delta_{S_2})[k, l]$ by $h_1(S_2)=S_2\pi_k^1+\sum_{i=1}^{k+l-1}\delta_0+0_l$. δ_0 is an arbitrary element of Δ_0. (We can suppose that δ_0 exists, because $\Delta_0=\emptyset$ would imply $\tau_{\mathbf{A}_1}=\tau_{\mathbf{A}_1}\circ\tau_{\mathbf{A}_2}=\emptyset$.) Let S be a new symbol, and define the ranked alphabet map h: $\Sigma_S\to\mathbf{D}R[k'k+l'l, k'l+l'k]$ as follows:

(i) $h(\sigma)=\hat{h}_2[k, l](h_1(\sigma))$ if $\sigma\in\Sigma_{S_1,S_2}$;

(ii) $h(S)=h(S_2)h(S_1)$.

We claim that the transducer $\mathbf{A}=(\Sigma, R, k'k+l'l, k'l+l'k, h, S)$ satisfies $\tau_{\mathbf{A}}=\tau_{\mathbf{A}_1}\circ\tau_{\mathbf{A}_2}$. Viewing h as a homomorphism of $\tilde{T}(\Sigma_{S_1,S_2})$ into $L[k, l]\mathbf{D}R[k'k+l'l, k'l+l'k]$ the following diagram commutes:



Now, for any $t\in T_\Sigma$

$$h(S(t)) = h(S_2)h(S_1)h(t)\Phi\hat{h}_2[k, l]\left(\left(S_2\pi_k^1+\sum_{i=1}^{k+l-1}\delta_0+0_l\right).h_1(S_1(t))\right) =$$

$$= \hat{h}_2[k, l]\left(S_2\pi_k^1 h_1(S_1(t))+\sum_{i=1}^{k-1}\delta_0\right) = \varrho_0^{-1}\hat{\zeta}\left(\hat{\mathbf{T}}h_2\left(\langle S_2(\tau_{\mathbf{A}_1}(t)), 0_l\rangle+\sum_{i=1}^{k-1}\langle\delta_0, 0_0\rangle\right)\right)\varrho_0'^{-1} =$$

$$= \hat{\zeta}(\langle h_2(S_2(\tau_{\mathbf{A}_1}(t))), 0_l\rangle)+\sum_{i=1}^{k-1}\hat{\zeta}(\langle h_2(\delta_0), 0_0\rangle).$$

By the definition of Φ,

$$\pi_{k'k+l'l}^1 h(S(t)) = \pi_{k'k+l'l}^1(\hat{\zeta}(\langle h_2(S_2(\tau_{\mathbf{A}_1}(t))), 0_l\rangle)+$$

$$+\sum_{i=1}^{k-1}\hat{\zeta}(\langle h_2(\delta_0), 0_0\rangle)) = 0_{k'l}+(\pi_k^1.h_2(S_2(\tau_{\mathbf{A}_1}(t))))+0_{l'(k-1)} = \tau_{\mathbf{A}_2}(\tau_{\mathbf{A}_1}(t))+0_{k'l+l'k}$$

which was to be proved.

REMARK. The intuitive meaning of the above construction is the following. The attributes of the transducer **A** can be devided into four classes. These are **s**−**s**, **i**−**s**, **s**−**i** and **i**−**i** containing $k'k$, $k'l$, $l'k$ and $l'l$ attributes, respectively. To interpret the value of the four kinds of attributes let $t \in T_\Sigma$ and α a node in t. Suppose that the value of the i-th **s**-attribute (m-th **i**-attribute) of α under h_1 appears as a subtree below the node β_i (γ_m, resp.) in $S_2(\tau_{A_1}(t))$. The following table describes the value of all the attributes of α under the composite transformation.

Attribute	Index	Class	Type	Related node in $S_2(\tau_{A_1}(t))$	Value
$\langle i, i'\rangle$	$k'(i-1)+i'$	**s**−**s**	synthesized	β_i	$\mathbf{s}(\beta_i, i')$
$\langle m, i'\rangle$	$k'k+l'l+k'(m-1)+i'$	**i**−**s**	inherited	γ_m	$\mathbf{s}(\gamma_m, i')$
$\langle i, m'\rangle$	$k'k+l'l+k'l+l'(i-1)+m'$	**s**−**i**	inherited	β_i	$\mathbf{i}(\beta_i, m')$
$\langle m, m'\rangle$	$k'k+l'(m-1)+m'$	**i**−**i**	synthesized	γ_m	$\mathbf{i}(\gamma_m, m')$

In the last column, e.g. $\mathbf{s}(\beta_i, i')$ denotes the value of the i'-th **s**-attribute of β_i under h_2. If "related node in $S_2(\tau_{A_1}(t))$" does not exist, then the value of the corresponding attribute is undefined or unimportant (see the congruence Φ). It is rather surprising that the attributes of class **i**−**i** can be computed in synthesized way.

Theorem 2.9. The class of all dtla-tree transformations is closed under composition.

Proof. Let $\mathbf{A}_2$ be a dtla-tree transformation from Δ into Γ in Theorem 2.8. Then the composite transducer **A** is obviously dl, but in general not total. Let us remark, however, that for each $\sigma \in \Sigma_S$ there exists a total representant in $[h(\sigma)]\Phi$. For example, it is enough to replace the $\perp$ components of $h(\sigma)$ (which in fact correspond to the values of the **i**-attributes of the fictive leaves of $h_1(\sigma)$ under h_2) by an arbitrary $\gamma_0 \in \Gamma_0$. Clearly, this modification does not change the transformation τ_A, so we are through.

Example 2.10. Let $k=l=k'=l'=1$, $\Sigma_0=\Delta_0=\{\bar{a}\}$, $\Sigma_1=\{a\}$, $\Delta_1=\{f, g\}$, $\Sigma= =\Sigma_0\cup\Sigma_1$, $\Delta=\Delta_0\cup\Delta_1$ and $\Gamma=\Delta$. Define h_1 and h_2 as follows

$$h_1(S_1) = h_2(S_2) = \langle x(1),\ \bar{a}\rangle;$$

$$h_1(a) = h_2(f) = \langle f(x(1)),\ g(y(1))\rangle;$$

$$h_1(\bar{a}) = h_2(\bar{a}) = y(1);$$

$$h_2(g) = \langle g(x(1)),\ f(y(1))\rangle.$$

Clearly, $\tau_{A_1} \circ \tau_{A_2} = \{\langle a^n\, \bar{a},\ f^n g^n f^n g^n \bar{a}\rangle \mid n \geqq 0\}$ (parenthesis are omitted for short). Following the construction of Theorem 2.8 we get the transducer $\mathbf{A}=(\Sigma, \Gamma, 2, 2, h, S)$, where

$$h(S) = [\langle x(1),\ \bot,\ x(2),\ \bar{a}\rangle\langle x(1),\ \bot,\ x(2),\ y(2)\rangle]\Theta_\Gamma =$$

$$= [\langle x(1),\ \bot,\ x(2),\ \bar{a}\rangle]\,\Theta_\Gamma \equiv \langle x(1),\ \bar{a},\ x(2),\ \bar{a}\rangle(\Phi);$$

$$h(a) = \langle f(x(1)),\ f(x(2)),\ g(y(1)),\ g(y(2))\rangle;$$

$$h(a) = \langle y(1),\ y(2)\rangle.$$

It is easy to check that, indeed, $\tau_{\mathbf{A}} = \tau_{\mathbf{A}_1} \circ \tau_{\mathbf{A}_2}$.

Let $v(\mathbf{A})$ denote the minimal value of the natural number K for which $\mathbf{A}$, a da-tree transducer, is K visit (for the definition of visits see e.g. [8], [12]). The complexity of $\mathbf{A}$, $c(\mathbf{A})$ is defined implicitely by the equation $v(\mathbf{A}) = \left\lceil \frac{c(\mathbf{A})}{2} \right\rceil + 1$. Now, let $\mathbf{A}_1$ and $\mathbf{A}_2$ be dtla-tree transducers with $c(\mathbf{A}_i) = c_i$ $(i = 1, 2)$, and construct the transducer $\mathbf{A}$ defining $\tau_{\mathbf{A}_1} \circ \tau_{\mathbf{A}_2}$. It can be proved that $c(\mathbf{A}) \leqq c_1 c_2 + c_1 + c_2$ and this is the best possible upper bound.

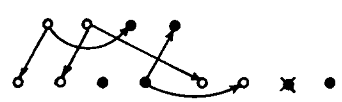
3. The composition of a dla-tree transformation and an arbitrary a-transformation

Let $\mathbf{A} = (\Sigma, \Delta, k, l, h, S)$ be a dla-tree transducer. We define a homomorphism $\mathrm{Ch}\colon \tilde{T}(\Sigma_S) \to L_\bot(\emptyset)[k+l, k+l]$ (called the trace of h) having the property that for every $t \in T_\Sigma$ and $i \in [k]$

$$\pi^i_{k+l}\,\mathrm{Ch}(S(t)) = \mathit{if}\quad \pi^i_k h(S(t)) = \bot_{1,l} \quad \mathit{then} \quad \bot_{1,k+l} \quad \mathit{else} \quad \pi^i_{k+l}.$$

(Obviously, $\mathrm{Ch}(S)$ is not a synthesizer here.)

Instead of presenting a formal description we illustrate Ch via an example. Let $\sigma \in (\Sigma_S)_n$ and t the linear representant of $h(\sigma)$ having the fewest nodes. Construct the graph G_t as in Lemma 2.2. For example, let $k = l = n = 2$. On Fig. 3 s- and i-attributes are represented by ∘-s and •-s, respectively, in the order from left to right. The mark × indicates a $\bot$-valued attribute.

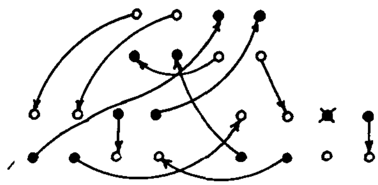


$\{\mathbf{rs}(i), \mathbf{ri}(m) \mid i, m \in [2]\}$

$\{\mathbf{ls}(j, i), \mathbf{li}(j, m) \mid j, i, m \in [2]\}$

ig. 3

Associate to each s-attribute a new i-attribute and to each i-attribute a new s-one. On Fig. 4 the nodes denoting these new attributes are placed below the corresponding old ones. The predicate, whether the value of an attribute a under h is $\bot$ or not will be expressed by: the value of a under Ch is $\bot$ or the same as the value of the associated new attribute a'. This can be achieved by checking the value of all the attributes a depends on, tracing them one after the other in an arbitrary order. In our example $\mathrm{Ch}(\sigma)$ can be represented by the graph of Fig. 4.



$\{\mathbf{rs}(i), \mathbf{ri}(m) | i, m \in [2]\}$
$\{\mathbf{ri}'(i), \mathbf{rs}'(m) | i, m \in [2]\}$
$\{\mathbf{ls}(j, i), \mathbf{li}(j, m) | j, i, m \in [2]\}$
$\{\mathbf{li}'(j, i), \mathbf{ls}'(j, m) | j, i, m \in [2]\}$

Fig. 4

To get the required result we only have to order the (old and new) attributes so that the i-th s-attribute and the i-th i-attribute ($i \in [k]$) should be the i-th old s-attribute and its associated new i-attribute, respectively.

There is a natural embedding $\otimes$: $R[k, l] \times R[k', l'] \to R[k+k', l+l']$ defined as follows. $a \otimes b = \varphi(a+b)\psi$, where $a, b\colon p \to q$ and

$$\varphi = 1_{kp} + \langle\!\langle \nu_{k'p}^{lq}, \mu_{k'p}^{lq} \rangle\!\rangle + 1_{l'q},$$

$$\psi^{-1} = 1_{kq} + \langle\!\langle \nu_{k'q}^{lp}, \mu_{k'q}^{lp} \rangle\!\rangle + 1_{l'p}.$$

It is easy to check that $\otimes$ is indeed a $1-1$ homomorphism.

Lemma 3.1. Let $\mathbf{A}_1 = (\Sigma, \Delta, k, l, h_1, S)$ be a dla-tree transducer, $\mathbf{A}_2 = (\Sigma, R, k', l', h_2, S)$ an arbitrary a-transducer. $\tau_{\mathbf{A}_2} | D\tau_{\mathbf{A}_1}$ (the restriction of $\tau_{\mathbf{A}_2}$ to the domain of $\tau_{\mathbf{A}_1}$) is an a-transformation.

Proof. Let $\mathbf{A}_S = (\Sigma_S, R, k+l+k', k+l+l', h_S, S')$ be the following a-transducer:

(i) $h_S(\sigma) = \mathrm{Ch}_1(\sigma) \otimes h_2(\sigma)$ for $\sigma \in \Sigma_S$,

(ii) $h_S(S') = \langle\!\langle \pi_1, \perp_{k+l+k'-1}, \pi_{k+l+1}, \perp_{k+l+l'-1} \rangle\!\rangle$. Observe that for every $t \in T_\Sigma$

$$\tau_{\mathbf{A}_S}(S(t)) = \textit{if } t \in D\tau_{\mathbf{A}_1} \textit{ then } \tau_{\mathbf{A}_2}(t) \textit{ else } \perp_{1,0}.$$

Now, consider the a-transducer $\mathbf{A} = (\Sigma, R, k+l+k', k+l+l', h, S)$, where $h(\sigma) = h_S(\sigma)$ for $\sigma \in \Sigma$ and $h(S) = h_S(S')h_S(S)$. Clearly, $\tau_{\mathbf{A}} = \tau_{\mathbf{A}_2} | D\tau_{\mathbf{A}_1}$.

Remark 3.2. If $\mathbf{A}_2$ is a dla-tree transducer, then so is $\mathbf{A}$.

Theorem 3.3. Let $\mathbf{A}_1 = (\Sigma, \Delta, k, l, h_1, S_1)$ be a dla-tree transducer, $\mathbf{A}_2 = (\Delta, R, k', l', h_2, S_2)$ an arbitrary a-transducer. Then $\tau_{\mathbf{A}_1} \circ \tau_{\mathbf{A}_2}$ is also an a-transformation.

Proof. Choosing any linear representant of $h_1(\sigma)$ ($\sigma \in \Sigma_{S_1}$) we get a homomorphism $h_1'\colon \tilde{T}(\Sigma_{S_1}) \to L_\perp(\Delta)[k, l]$. Since $L_\perp(\Delta)[k, l] \cong L(\Delta_\perp)[k, l]$, we can use h_1' to define the dtla-tree transducer $\mathbf{A}_1' = (\Sigma, \Delta_\perp, k, l, h_1', S_1)$. Extend h_2 to $\Delta_\perp$ by $h_2(\perp) = \perp_{k', l'}$. Theorem 2.8 implies that $\tau_{\mathbf{A}_1'} \circ \tau_{\mathbf{A}_2} = \tau_{\mathbf{A}}$ for an appropriate a-transducer $\mathbf{A}$. Clearly, $\tau_{\mathbf{A}} | D\tau_{\mathbf{A}_1} = \tau_{\mathbf{A}_1} \circ \tau_{\mathbf{A}_2}$, so the statement of the theorem follows from Lemma 3.1.

Corollary 3.4. The class of all dla-tree transformations is closed under composition.

Proof. If $\mathbf{A}_2$ is a dla-tree transducer in Theorem 3.3, then so is the composite transducer $\mathbf{A}$. Thus, the corollary follows from Remark 3.2.

4. dla-tree to string transformations

Let T be a (string) alphabet and let CF(T) denote the rational theory of all context free languages over $T\cup X$ (cf. [13]). CF(T) is in fact the "front" theory of Reg(Σ), supposing $\Sigma_0=T$. $L=\langle L_1,\dots,L_p\rangle\in \mathrm{CF}(T)(p,q)$ is called linear deterministic if each L_i $(i\in[p])$ contains at most one string and no variable occurs more than once in L.

Definition 4.1. A dla-tree to string transducer from Σ into T^* (T is finite) is an α-transducer $(\Sigma, \mathrm{CF}(T), k, l, h, S)$, where $h(\sigma)$ is linear deterministic for each $\sigma\in\Sigma_S$.

By Lemma 2.2 the linear deterministic elements form a submagmoid of CF$(T)[k,l]$ which will be denoted by LCF$(T)[k,l]$. If **A** is a dla-tree to string transducer from Σ into T^*, we consider $\tau_{\mathbf{A}}$ as a relation, $\tau_{\mathbf{A}}\subseteq T_\Sigma\times T^*$.

Theorem 4.2. Let $\mathbf{A}=(\Sigma, \mathrm{CF}(T), k, l, h, S)$ be a dla-tree to string transducer. Then $\tau_{\mathbf{A}}=(\varphi, K, \psi)$, where K is a regular forest, φ is a relabeling tree homomorphism (injective on K) and ψ is a dtl-top-down tree to string transformation. (Recall from [4] that the transformation defined by the bimorphism (φ, K, ψ) is $\{\langle\varphi(t),\psi(t)\rangle | t\in K\}$.)

Proof. Let ξ: CF$(T)\to$CF$(\emptyset)$ denote the homomorphism defined by the unique homomorphism of T^* into $\emptyset^*=\{\lambda\}$. Let $A_0\subseteq \mathrm{LCF}(\emptyset)[k,l](1,0)$ such that $a\in A_0$ iff $\pi_k^1 a=\{\lambda\}$ and $\pi_k^i a\subseteq\{\lambda\}$ for each $i\in[k]$. Since LCF$(\emptyset)[k,l]$ is a finite magmoid, the pair $(h\circ\xi[k,l], A_0)$ can be considered a deterministic finite state bottom-up tree automaton working on T_{Σ_S}, where $A=\mathrm{LCF}(\emptyset)[k,l](1,0)$ is the set of states, $(h\circ\xi[k,l](\sigma)|\sigma\in\Sigma_S)$ describes the transitions and A_0 is the set of final states. Let **Q** denote the relabeling transducer defined by this automaton. **Q** marks each node of a tree $t\in T_{\Sigma_S}$ by a new label, which is a pair consisting of the old label $\sigma\in(\Sigma_S)_n$ and a vector of states $\langle a_0,\dots,a_n\rangle$ in which the automaton passes through the node and its sons, respectively, during the recognition (or refuse) of t. Let Σ_S' denote the ranked alphabet of these new labels.

Define $K=\tau_{\mathbf{Q}}(F_S)\subseteq T_{\Sigma_S'}$, where $F_S=\{u\in T_{\Sigma_S} | u=S(t)$ for some $t\in T_\Sigma\}$. Furthermore, let φ: $\tilde T(\Sigma_S')\to T(\Sigma)$ be such that $\varphi(\langle S,\langle a_0,a\rangle\rangle)=x_1$ and $\varphi(\langle\sigma,\langle a_0,\dots,a_n\rangle\rangle)=\sigma$ if $\sigma\in\Sigma_n$. Obviously, K is regular, φ is injective on K and $\varphi(K)=D\tau_{\mathbf{A}}$.

We describe ψ as a homomorphism of $\tilde T(\Sigma_S')$ into LCF$(T)[k+l,0]$, i.e. ψ will be a $(k+l)$-state dl-top-down tree to string transformation. To avoid ambiguity we shall use the variables $Z=\{z_1,z_2,\dots\}$ $(Z\cap T=\emptyset)$ instead of X in the definition of ψ. Let \$ be a distinguished symbol in T and # a new symbol not in T. Take an arbitrary $a\in A$. If $a=\langle u_1,\dots,u_k\rangle$, then let $\bar a=\bar u_1\#\dots\#\bar u_k$, where $\bar u_i=$ *if* $u_i=\emptyset$ *then* \$ *else* u_i $(i\in[k])$. Let n denote the number of all the #-s and $y(m)$-s $(m\in[l])$ — called separating symbols — in $\bar a$. Clearly $n\leq k+l$. Define the mapping η_a: $((T\cup Z)^*)^{k+l}\to \mathrm{LCF}(T\cup Z)[k,l](1,0)$ as follows. If $w=\langle w_1,\dots,w_{k+l}\rangle$, then $\eta_a(w)=\langle v_1,\dots,v_k\rangle$, where for each $i\in[k]$

(i) if $\bar u_i=\$$, then $v_i=\emptyset$;
(ii) if $\bar u_i=\lambda$, then $v_i=w_{j+1}$, where the # preceding $\bar u_i$ is the j-th separating symbol in $\bar a$.

(iii) if $\bar{u}_i = y(i_j) \dots y(i_{j+n_i})$, where $y(i_j)$ is the j-th separating symbol in a (from left to right), then $v_i = w_j y(i_j) w_{j+1} \dots y(i_{j+n_i}) w_{j+n_i+1}$.

Taking the inverse of some element under η_a we shall assume that the unnecessary components of $\eta_a^{-1}(\langle v_1, \dots, v_k \rangle) = \langle w_1, \dots, w_{k+l} \rangle$ (which do not take part in (ii) and (iii)) are set to \$.

Now, if $\langle \sigma, \langle a_0, \dots, a_n \rangle \rangle \in (\Sigma'_S)_n$, then let

$$\psi(\langle \sigma, \langle a_0, \dots, a_n \rangle \rangle) = \eta_{a_0}^{-1} \left(h(\sigma) \cdot \sum_{i=1}^{n} \eta_{a_i}(\langle z_{(k+l)(i-1)+1}, \dots, z_{(k+l)i} \rangle) \right).$$

ψ is obviously linear, so it is enough to prove that for every $t \in K$ with $\text{root}(t) = \langle S, \langle a_0, a \rangle \rangle$ $\psi(t) = \eta_{a_0}^{-1}(h(\varphi(t)))$. This follows from the following induction.

If $\langle \sigma, \langle a_0, \dots, a_n \rangle \rangle \in (\Sigma'_S)_n$ $(n \geqq 0)$, $t_i \in T_{\Sigma'_S}$ with $\text{root}(t_i) = \langle -, \langle a_i, \dots \rangle \rangle$ and $\psi(t_i) = \eta_{a_i}^{-1}(h(\varphi(t_i)))$, then for $t = \langle \sigma, \langle a_0, \dots, a_n \rangle \rangle (t_1, \dots, t_n)$ we have $\psi(t) = \eta_{a_0}^{-1}(h(\varphi(t)))$. Really,

$$\psi(t) = \psi(\langle \sigma, \langle a_0, \dots, a_n \rangle \rangle) \cdot \sum_{i=1}^{n} \psi(t_i) =$$

$$= \psi(\langle \sigma, \langle a_0, \dots, a_n \rangle \rangle)[z_{(k+l)(i-1)+j} \leftarrow \pi_j \psi(t_i) | i \in [n], j \in [k+l]] =$$

$$= \eta_{a_0}^{-1} \left(h(\sigma) \cdot \sum_{i=1}^{n} \eta_{a_i}(\eta_{a_i}^{-1}(h(\varphi(t_i)))) \right) = \eta_{a_0}^{-1}(h(\varphi(t))).$$

Corollary 4.3. The surface sets of dla-tree to string transformations are the same as that of dtl-top-down tree to string transformations.

This class of languages was investigated e.g. in [10].

5. Problems

The existence of the trace homomorphism Ch described in section 3 raises the following problem. Given any regular forest $F \subseteq T_\Sigma$, is it possible to find a homomorphism $\xi \colon \tilde{T}(\Sigma) \to L_\perp(\emptyset)[k, l]$ such that for any $t \in T_\Sigma$, $\pi_1 \xi(t) = if\ t \in F\ then\ \pi_1\ else\ \perp_1$? The answer is positive if Σ is a unary alphabet $(\Sigma = \Sigma_0 \cup \Sigma_1)$, although a negative answer is more likely in the general case. It is also open whether it is possible to define deterministic finite state bottom-up or look-ahead tree transformations (cf. [6]) by attributed tree transducers. However, it can be shown that the classes of deterministic attributed and macro tree transformations coincide in the monadic case (i.e. if both the domain and range alphabets are unary). The proof of this result will be given in a forthcoming paper.

Appendix

To prove Lemmas 2.6 and 2.7 we need a preliminary observation.

An infinite tree $t \in R(\Delta)(p, q)$ is called local if it is determined by the sequence $\beta \in (\Delta \cup X_q)^p$ of its roots and a "successor" function χ, which for every $\delta \in \Delta_n$ $(n \geqq 0)$ specifies the sequence of labels of the sons of any node in t labeled by δ (i.e. $\chi(\delta) \in (\Delta \cup X_q)^n$). In this case we write $t = (\beta, \chi)$.

Let Ω and Λ be finite ranked alphabets, $T\in\tilde{T}(\Omega)(p,q)$ an ideal (i.e. $\pi_i T\neq x_j$ for any $i\in[p]$, $j\in[q]$) such that any two distinct nodes of T have different labels. This allows us to identify a node of T by its label. Let nds (T) denote the set of nodes (labels) of T, and for $\omega\in\mathrm{nds}(T), \omega\in\Omega_n$, let $\langle\omega(0),\ldots,\omega(n)\rangle$ denote the sequence of nodes obtained by enumerating the father of ω followed by the sons of ω. Take $\omega(0)=i$ if ω is the root of $\pi_i T$. Furthermore, let $\varkappa\colon \tilde{T}(\Omega)\to\mathbf{D}R(\Lambda)[k,l]$ be a homomorphism such that

$$\varkappa(\omega)=\lessdot \underline{a}(1,\omega),\ldots,\underline{a}(k,\omega),\bar{a}(1,\omega),\ldots,\bar{a}(ln,\omega)\gtrdot,$$

where $n\geqq 0, \omega\in\Omega_n$ and $\{\underline{a}(i,\omega),\bar{a}(j,\omega)|i\in[k], j\in[ln]\}\subseteqq\Lambda_{kn+l}$.

It is routine to check that $\zeta(\varkappa(T))=(\beta,\chi)\in R(\Lambda)[k,l](p,q)$ is the following local tree:

(i) $$\beta=\langle A(1),\ldots,A(kp),\ B(1),\ldots,B(lq)\rangle, \tag{1}$$

where $\forall r\in[p], \forall i\in[k]$

$$A\big(k(r-1)+i\big)=\underline{a}(i,\omega)\quad\text{if}\quad r=\omega(0),$$

and $\forall j\in[q], \forall m\in[l]$

$$B\big(l(j-1)+m\big)=\bar{a}\big(l(s-1)+m,\omega\big)\quad\text{if}\quad x_j=\omega(s)$$

for some $\omega\in\Omega_n, s\in[n]$;

(ii) if $\omega\in\mathrm{nds}(T), \omega\in\Omega_n$, then $\forall i\in[k], \forall j\in[ln]$

$$\chi\big(\underline{a}(i,\omega)\big)=\chi\big(\bar{a}(j,\omega)\big)=\langle A(1),\ldots,A(kn),B(1),\ldots,B(l)\rangle,$$

where $\forall s\in[n], \forall i\in[k]$

$$A\big(k(s-1)+i\big)=\mathit{Case}\ \omega(s)\quad\mathit{of}\quad\omega'(\in\Omega)\colon\underline{a}(i,\omega');\ x_j\colon x(j,i);$$

and $\forall m\in[l]$

$$B(m)=\mathit{Case}\ \omega(0)\quad\mathit{of}\quad\omega'(\in\Omega)\colon\bar{a}\big(l(s-1)+m,\omega'\big),\quad\text{where}\quad\omega'(s)=\omega;$$
$$r(\in[p])\colon y(r,m).$$

Moreover, if $\vartheta\in\hat{\Theta}(q,q')$, then $\zeta(\hat{\mathbf{T}}\varkappa(T\vartheta))=(\beta',\chi')$ is the following:

(i) $$\beta'\doteq\langle A(1),\ldots,A(kp),B'(1),\ldots,B'(lq')\rangle,$$

where $A(i)$ and $B(j)$ $(i\in[kp], j\in[lq])$ are as in (1), and $\forall j'\in[q'], \forall m\in[l]$

$$B'\big(l(j'-1)+m\big)=\mathit{if}\quad j'\in\mathrm{Im}(\vartheta)\quad\mathit{and}\quad j'=j\vartheta\quad\mathit{then}\quad B\big(l(j-1)+m\big)\quad\mathit{else}\quad\bot;$$

(ii) for each

$$A\in\{\underline{a}(i,\omega),\bar{a}(j,\omega)|\omega\in\mathrm{nds}(T),\ \omega\in\Omega_n,\ i\in[k],\ j\in[ln]\}$$
$$\chi'(A)=\chi(A)\big[x(j,i)\leftarrow x(j',i)|i\in[k],\ j\in[lq],\ j\vartheta=j'\big].$$

Proof of Lemma 2.6. It is enough to prove the following two statements:

1. if $q_0 \geqq 0$ and for each $0 \leqq s \leqq q_0$, $w_s \in L(k, l)\mathbf{D}R[k'k+l'l, k'l+l'k](1, q_s)$ with $\sum_{s=1}^{q_0} q_s = q$, then

$$w_0 \cdot \sum_{s=1}^{q_0} w_s \in L[k, l]\mathbf{D}R[k'k+l'l, k'l+l'k](1, q);$$

2. if $w_s \Phi w_s'$ for every $0 \leqq s \leqq q_0$, then

$$w = w_0 \cdot \sum_{s=1}^{q_0} w_s \Phi w_0' \cdot \sum_{s=1}^{q_0} w_s' = w'.$$

Let $\varrho(w_s)$ stand for $\varrho_{q_s} w_s \varrho_{q_s}'$, and let

$$I_{\varrho(w_s)} = \{\underline{I}_s(1), \ldots, \underline{I}_s(k), \quad \bar{I}_s(1), \ldots, \bar{I}_s(lq_s)\}$$

with $\|\underline{I}_s(i)\| = \underline{n}_s(i)$, $\|\bar{I}_s(j)\| = \bar{n}_s(j)$ and $kq_s + l - \|\cup I_{\varrho(w_s)}\| = n_s$ $(i \in [k],\ j \in [lq_s])$. Choose injective mappings $\underline{\vartheta}_s(i)$, $\bar{\vartheta}_s(j)$ and ϑ_s, which map $[\underline{n}_s(i)]$, $[\bar{n}_s(j)]$ and $[n_s]$ into $[kq_s+l]$ such that $\operatorname{Im}(\underline{\vartheta}_s(i)) = \underline{I}_s(i)$, $\operatorname{Im}(\bar{\vartheta}_s(j)) = \bar{I}_s(j)$ and $\operatorname{Im}(\vartheta_s) = [kq_s+l] \setminus \cup I_{\varrho(w_s)}$. Let Ω consist of the symbols $\{\underline{T}_s(i), \bar{T}_s(j) | 0 \leqq s \leqq q_0,\ i \in [k],\ j \in [lq_s]\}$, where $\underline{T}_s(i) \in \Omega_{\underline{n}_s(i)}$ and $\bar{T}_s(j) \in \Omega_{\bar{n}_s(j)}$. Define Λ as the least ranked alphabet satisfying the following conditions:

(i) for every $\omega \in \Omega_n$ $(n \geqq 0)$

$$\{\underline{a}(i', \omega), \bar{a}(j', \omega) | i' \in [k'], j' \in [l'n]\} \subseteq \Lambda_{k', n+l'};$$

(ii) for each $0 \leqq s \leqq q_0$

$$\{\bar{a}(j', s), \bar{a}'(j', s) | j' \in [l'n_s]\} \subseteq \Lambda_{k'n_s}.$$

Construct local trees W_s, W_s' and $W_s^{\perp}$ $(0 \leqq s \leqq q_0)$ of $R(\Lambda)[k'k+l'l, k'l+l'k](1, q_s)$ as follows. $W_s = (\beta_s, \chi_s)$ with

(i) $\beta_s = \langle A(1), \ldots, A(k'k), B(1), \ldots, B(l'l), C(1, 1), \ldots, C(1, k'l), D(1, 1), \ldots, D(1, l'k), \ldots, C(q_s, 1), \ldots, C(q_s, k'l), D(q_s, 1), \ldots, D(q_s, l'k)\rangle$, where $\forall i \in [k]$, $\forall i' \in [k']$

$$A(k'(i-1)+i') = \underline{a}(i', \underline{T}_s(i)),$$

$\forall m \in [l]$, $\forall m' \in [l']$

$$B(l'(m-1)+m') = \mathit{Case}\ kq_s+m \quad \mathit{of} \tag{2}$$

$$\begin{array}{l} n\underline{\vartheta}_s(i): \\ n\bar{\vartheta}_s(j): \\ n\vartheta_s: \end{array} \rightarrow \bar{a}(l'(n-1)+m', \leftarrow \begin{array}{l} \underline{T}_s(i)); \\ \bar{T}_s(j)); \\ s); \end{array}$$

$\forall j \in [q_s]$, $\forall m \in [l]$, $\forall i' \in [k']$

$$C(j, k'(m-1)+i') = \underline{a}(i', \bar{T}_s(l(j-1)+m)),$$

and $\forall j' \in [q_s]$, $\forall r \in [k]$, $\forall m' \in [l']$ $D(j', l'(r-1)+m')$ is of the form (2) $[kq_s+m \leftarrow k(j'-1)+r]$, i.e., (2) with kq_s+m replaced by $k(j'-1)+r$;

(iia) $\forall i \in [k]$,
$\forall i' \in [k'], \forall j' \in [l'\underline{n}_s(i)]$

$$\chi_s(\underline{a}(i', \underline{T}_s(i))) = \chi_s(\bar{a}(j', \underline{T}_s(i))) = \langle A(1), \ldots, A(k'\underline{n}_s(i)), B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'], \forall n \in [\underline{n}_s(i)]$

$$A(k'(n-1)+i') = Case\ \ n\underline{\vartheta}_s(i) \quad of \quad k(j'-1)+r(j' \in [q_s], r \in [k]):\ x(j', k(r-1)+i');$$
$$kq_s+m\,(m \in [l]):\ y(k'(m-1)+i'); \tag{3}$$

and $\forall m' \in [l']$ $B(k'\underline{n}_s(i)+m') = y(k'l+l'(i-1)+m')$;

(iib) $\forall j \in [lq_s]$,
$\forall i' \in [k'], \forall j' \in [l'\bar{n}_s(j)]$

$$\chi_s(\underline{a}(i', \overline{T}_s(j))) = \chi_s(\bar{a}(j', \overline{T}_s(j))) = \langle A(1), \ldots, A(k'\bar{n}_s(j)),\ B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'], \forall n \in [\bar{n}_s(j)]$ $A(k'(n-1)+i')$ is of the form (3) $[\underline{\vartheta}_s(i) \leftarrow \bar{\vartheta}_s(j)]$, and $\forall m' \in [l']$ $B(m') = x(j', k'k+l'(m-1)+m')$ if $j = l(j'-1)+m$ for some $j' \in [q_s]$, $m \in [l]$;

(iic) $\forall j' \in [l'n_s]$ $\chi_s(\bar{a}(j', s)) = \langle A(1), \ldots, A(k'n_s)\rangle$, where $\forall i' \in [k'], \forall n \in [n_s]$ $A(k'(n-1)+i')$ is of the form (3) $[\underline{\vartheta}_s(i) \leftarrow \vartheta_s]$.

We get W_s' and $W_s^{\perp}$ from W_s by replacing the symbols $\bar{a}(j', s)$ $(j' \in [l'n_s])$ occuring in it by $\bar{a}'(j', s)$ and $\perp$, respectively.

By construction, for any $0 \leqq s \leqq q_0$ and $i \in [k'k+l'l+(k'l+l'k)q_s]$, $\pi_i W_s$ and $\pi_i W_s'$ depend on all those variables which $\pi_i w_s$ or $\pi_i w_s'$ may depend on. Therefore, if $\pi_i W_s^{(\prime)} = \lambda\varphi$ for some $\lambda \in \Lambda$ and $\varphi \in \hat{\Theta}$, then $\pi_i w_s^{(\prime)} = a\vartheta$, where $\mathrm{Im}\,(\vartheta) \subseteqq \mathrm{Im}\,(\varphi)$. Define the ranked alphabet map $\xi\colon \Lambda \to R$ by $\xi(\lambda) = a\vartheta\varphi^{-1}$, where φ^{-1} is an arbitrary right inverse of φ. ξ is correct, since for every $\lambda \in \Lambda$ there exists exactly one s, i and φ such that $\pi_i W_s^{(\prime)} = \lambda\varphi$. Obviously $\vartheta\varphi^{-1}\varphi = \vartheta$, thus $\xi(W_s) = w_s$ and $\xi(W_s') = w_s'$ hold by the extension of ξ to a homomorphism of $R(\Lambda)$ into R.

Let $\varkappa\colon \tilde{T}(\Omega) \to \mathbf{D}R(\Lambda)[k', l']$ be the homomorphism extending the following ranked alphabet map. For every $\omega \in \Omega_n$

$$\varkappa(\omega) = \nless\underline{a}(1, \omega), \ldots, \underline{a}(k', \omega), \bar{a}(1, \omega), \ldots, \bar{a}(l'n, \omega)\ngtr.$$

Consider the elements

$$T_s = \nless\underline{T}_s(1)\underline{\vartheta}_s(1), \ldots, \underline{T}_s(k)\underline{\vartheta}_s(k), \overline{T}_s(1)\bar{\vartheta}_s(1), \ldots, \overline{T}_s(lq_s)\bar{\vartheta}_s(lq_s)\ngtr$$

of $L(\Omega)[k, l](1, q_s)$, and observe that

$$\hat{\xi}(\hat{\mathbf{T}}\varkappa(T_s)) = \varrho(W_s^{\perp})\Psi\varrho(W_s).$$

To complete the proof it is enough to show that

$$\varrho(W^{(\prime)}) = \varrho(W_0^{(\prime)}) \cdot \sum_{s=1}^{q_0} W_s^{(\prime)}\Psi\hat{\xi}\left[\hat{\mathbf{T}}\varkappa\left(T_0 \cdot \sum_{s=1}^{q_0} T_s\right)\right]. \tag{4}$$

Indeed, (4) shows that both W and W' are $[k, l]$-linear (see Proposition 2.4/1) and $W\Phi W'$. Thus, by Proposition 2.4/2, $w\Phi w'$.

First we compute $T=T_0\cdot\sum_{s=1}^{q_0}T_s$. Following the proof of Lemma 2.2 it is easy to see that $T\in L(\Omega)[k,l](1,q)$ is the following finite Ω-tree. With the notations of our preliminary observation

(i) $\forall i\in[k]$

$$\underline{T}_0(i)(0)=i,$$

$\forall n\in[\underline{n}_0(i)]$

$$\underline{T}_0(i)(n)=\textit{Case}\ \ n\underline{\vartheta}_0(i)\quad\textit{of}\quad k(s-1)+r(s\in[q_0],r\in[k])\colon\ \underline{T}_s(r);$$

$$kq_0+m(m\in[l])\colon\ y(m); \tag{5}$$

$\forall s\in[q_0]$, $\forall m\in[l]$

$$\overline{T}_0(l(s-1)+m)(0)=\textit{Case}\ \ kq_s+m\quad\textit{of}\quad n\underline{\vartheta}_s(i)\colon\ \underline{T}_s(i);\quad n\overline{\vartheta}_s(j)\colon\ \overline{T}_s(j);$$

$$n\vartheta_s\colon\ \overline{T}_0(l(s-1)+m)\notin \operatorname{nds}(T);$$

$\forall j\in[lq_0]$, $\forall n\in[\overline{n}_0(j)]$ $\overline{T}_0(j)(n)$ is of the form (5) $[\underline{\vartheta}_0(i)\leftarrow\overline{\vartheta}_s(j)]$;

(ii) $\forall s\in[q_0]$,
$\forall i\in[k]$

$$\underline{T}_s(i)(0)=\textit{Case}\ \ k(s-i)+i\quad\textit{of}\quad n\underline{\vartheta}_0(r)\colon\ \underline{T}_0(r);\quad n\overline{\vartheta}_0(j)\colon\ \overline{T}_0(j);$$

$$n\vartheta_0\colon\ \underline{T}_s(i)\notin \operatorname{nds}(T);$$

$\forall n\in[\underline{n}_s(i)]$

$$\underline{T}_s(i)(n)=\textit{Case}\ \ n\underline{\vartheta}_s(i)\quad\textit{of}$$

$$k(j'-1)+r(j'\in[q_s],r\in[k])\colon\ x(q^{(s)}+j',r)\left(q^{(s)}=\sum_{p=1}^{s-1}q_p\right); \tag{6}$$

$$kq_s+m(m\in[l])\colon\ \overline{T}_0(l(s-1)+m);$$

$\forall j\in[lq_s]$

$$\overline{T}_s(j)(0)=k+lq^{(s)}+j,$$

$\forall n\in[\overline{n}_s(j)]$ $\overline{T}_s(j)(n)$ is of the form (6) $[\underline{\vartheta}_s(i)\leftarrow\overline{\vartheta}_s(j)]$.

Computing $\hat{\zeta}(\hat{\mathrm{T}}\varkappa(T))$ we get a local tree (β,χ) for which

$$\text{(i)}\ \beta=\langle A(1),\dots,A(k'k),\ C(1,1),\dots,C(1,k'l),\dots,C(q,1),\dots,C(q,k'l),$$

$$D(1,1),\dots,D(1,l'k),\dots,D(q,1),\dots,D(q,l'k),\ B(1),\dots,B(l'l)\rangle, \tag{7}$$

where $\forall i\in[k]$, $\forall i'\in[k']$

$$A(k'(i-1)+i')=\underline{a}(i',\underline{T}_0(i)),$$

$\forall s\in[q_0]$,
$\forall j\in[q_s]$, $m\in[l]$, $\forall i'\in[k']$

$$C(q^{(s)}+j,k'(m-1)+i')=\underline{a}(i',\overline{T}_s(l(j-1)+m)),$$

$\forall i \in [k], \forall m' \in [l']$

$$D(q^{(s)}+j,\, l'(i-1)+m') = Case\ \ k(j-1)+i \quad of \tag{8}$$

$$\begin{array}{l} n\underline{\vartheta}_s(r): \searrow \qquad \nearrow \underline{T}_s(r)); \\ n\bar{\vartheta}_s(j'): \rightarrow \bar{a}(l'(n-1)+m', \leftarrow \bar{T}_s(j')); \\ n\vartheta_s:\ \perp; \end{array}$$

$\forall m \in [l], \forall m' \in [l']$

$$B(l'(m-1)+m') = Case\ \ kq_0+m \quad of \tag{9}$$

$$\begin{array}{l} n\underline{\vartheta}_0(i): \searrow \qquad \nearrow \underline{T}_0(i)); \\ n\bar{\vartheta}_0(j): \rightarrow \bar{a}(l'(n-1)+m', \leftarrow \bar{T}_0(j)); \\ n\vartheta_0:\ \perp; \end{array}$$

(iia) $\forall i \in [k]$,
$\forall i' \in [k'], \forall j' \in [l'\underline{n}_0(i)]$

$$\chi(\underline{a}(i', \underline{T}_0(i))) = \chi(\bar{a}(j', \underline{T}_0(i)) = \langle A(1), \ldots, A(k'\underline{n}_0(i)),\ B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'], \forall n \in [\underline{n}_0(i)]$

$$A(k'(n-1)+i') = Case\ \ n\underline{\vartheta}_0(i) \quad of \quad k(s-1)+r(s \in [q_0], r \in [k]):\ \underline{a}(i', \underline{T}_s(r));$$
$$kq_0+m\ \ (m \in [l]):\ x(kq+m, i'); \tag{10}$$

and $\forall m' \in [l'],\ B(m') = y(i, m')$;

(iib) $\forall j \in [lq_0]$,
$\forall i' \in [k'], \forall j' \in [l'\bar{n}_0(j)]$

$$\chi(\underline{a}(i', \bar{T}_0(j))) = \chi(\bar{a}(j', \bar{T}_0(j))) = \langle A(1), \ldots, A(k'\bar{n}_0(j)),\ B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'], \forall n \in [\bar{n}_0(j)]$ $A(k'(n-1)+i')$ is of the form (10) $[\underline{\vartheta}_0(i) \leftarrow \bar{\vartheta}_0(j)]$, and $\forall m' \in [l']$ if $j = l(s-1)+m$ for some $s \in [q_0], m \in [l]$, then

$$B(m') = Case\ \ kq_s+m \quad of$$

$$\begin{array}{l} n\underline{\vartheta}_s(r): \searrow \qquad \nearrow \underline{T}_s(i)); \\ n\bar{\vartheta}_s(j'): \rightarrow \bar{a}(l'(n-1)+m', \leftarrow \bar{T}_s(j')); \\ n\vartheta_s:\quad \bar{T}_0(j) \notin \mathrm{nds}(T); \end{array}$$

(iic) $\forall s \in [q_0], \forall i \in [k]$,
$\forall i' \in [k'], \forall j' \in [l'\underline{n}_s(i)]$

$$\chi(\underline{a}(i, \underline{T}_s(i))) = \chi(\bar{a}(j', \underline{T}_s(i))) = \langle A(1), \ldots, A(k'\underline{n}_s(i)),\ B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'], \forall n \in [\underline{n}_s(i)]$

$$A(k'(n-1)+i') = Case\ \ n\underline{\vartheta}_s(i) \quad of \quad k(j'-1)+r(j' \in [q_s], r \in [k]):\ x(k(q^{(s)}+j')+r, i');$$
$$kq_s+m(m \in [l]):\ \underline{a}(i', \bar{T}_0(l(s-1)+m)); \tag{11}$$

and $\forall m' \in [l']$

$$B(m') = Case\ k(s-1)+i\ of$$

$$\begin{array}{l} n\underline{\vartheta}_0(r): \searrow \qquad \nearrow \underline{T}_0(r)); \\ n\bar{\vartheta}_0(j): \rightarrow \bar{a}(l'(n-1)+m', \leftarrow \bar{T}_0(j)); \\ n\vartheta_0:\ \underline{T}_s(i) \notin \mathrm{nds}(T); \end{array}$$

(iid) $\forall s \in [q_0]$, $\forall j \in [lq_s]$,
$\forall i' \in [k']$, $\forall j' \in [l'\bar{n}_s(j)]$

$$\chi(\underline{a}(i', \bar{T}_s(j))) = \chi(\bar{a}(j', \bar{T}_s(j))) = \langle A(1), \ldots, A(k'\bar{n}_s(j)),\ B(1), \ldots, B(l')\rangle,$$

where $\forall i' \in [k'], \forall n \in [\bar{n}_s(j)]$ $A(k'(n-1)+i')$ is of the form (11) $[\underline{\vartheta}_s(i) \leftarrow \bar{\vartheta}_s(j)]$, and $\forall m' \in [l'] B(m') = y(k+lq^{(s)}+j-1, m')$.

Now we compute $W = W_0 \cdot \sum_{s=1}^{q_0} W_s$. The result is (β', χ'), the following local tree:

(i) $\beta' = \langle A(1), \ldots, A(k'k),\ B(1), \ldots, B(l'l),\ C(1, 1), \ldots, C(1, k'l),$

$D(1, 1), \ldots, D(1, l'k), \ldots, C(q, 1), \ldots, C(q, k'l),\ D(q, 1), \ldots, D(q, l'k)\rangle$,

where all the symbols in β' are the same as the corresponding ones under (7), exept that in (8) and (9) $\perp$ must be replaced by $\bar{a}(l'(n-1)+m', s)$ and $\bar{a}(l'(n-1)+$ $+m', 0)$, respectively.

(iia) For each symbol occuring in both W and $\zeta(\hat{\mathbf{T}}\varkappa(T))$ we get $\chi'(A)$ from $\chi(A)$ by the following variable transformation:

$$\forall j \in [q], \quad \forall i \in [k], \quad \forall i' \in [k'], \quad \forall m \in [l], \quad \forall m' \in [l']$$

$$x(k(j-1)+i, i') \leftarrow x(j, k'(i-1)+i'), \quad x(kq+m, i') \leftarrow y(k'(m-1)+i'),$$

$$y(i, m') \leftarrow y(l'(i-1)+m'), \quad y(k+l(j-1)+m, m') \leftarrow x(j, k'k+l'(m-1)+m');$$

(iib) $\forall 0 \leqq s \leqq q_0$, $\forall j' \in [l'n_s]$

$$\chi'(\bar{a}(j', s)) = \langle A(1), \ldots, A(k'n_s)\rangle,$$

where $\forall i' \in [k'], \forall n \in [n_s]$ $A(k'(n-1)+i')$ is of the form (9) $[\underline{\vartheta}_0(i) \leftarrow \vartheta_0]$ or (10) $[\underline{\vartheta}_s(i) \leftarrow \vartheta_s]$ depending on $s=0$ or $s \in [q_0]$.

Finally, as it is obvious, we get W' from W by replacing $\bar{a}(j', s)$ by $\bar{a}'(j', s)$ for each $0 \leqq s \leqq q_0$, $j' \in [l'n_s]$.

It is now easy to check that (4) is true, so the lemma is proved.

Proof of Lemma 2.7. Let $q_0 \geqq 0$, and for each $0 \leqq s \leqq q_0$ let

$$t_s = \Lsh \underline{t}_s(1)\underline{\vartheta}_s(1), \ldots, \underline{t}_s(k)\underline{\vartheta}_s(k),\ \bar{\bar{t}}_s(1)\bar{\vartheta}_s(1), \ldots, \bar{\bar{t}}_s(lq_s)\bar{\vartheta}_s(lq_s) \Rsh$$

be an element of $L(\Delta)[k, l](1, q_s)$ with $\underline{t}_s(i) \in \tilde{T}(\Delta)(1, \underline{n}_s(i)), \bar{\bar{t}}_s(j) \in \tilde{T}(\Delta)(1, \bar{n}_s(j))$ $(i \in [k], j \in [lq_s])$. Construct the alphabets Ω and Λ, the homomorphism $\varkappa$: $\tilde{T}(\Omega) \rightarrow$ $\rightarrow \mathbf{D}R(\Lambda)[k', l']$ and the trees $\{T_s | 0 \leqq s \leqq q_0\}$ as in the proof of Lemma 2.6. It is clear that any component of $\hat{h}[k, l](t_s)$ depends on at most the same variables as the corresponding component of $\hat{\varkappa}[k, l](T_s)$ does. Therefore there exists a homomorphism $\xi\colon R(\Lambda)\to R$ such that

$$\xi(\hat{\varkappa}[k, l](T_s)) = \hat{h}[k, l](t_s).$$

Let $\mu\colon \tilde{T}(\Omega)\to\tilde{T}(\Delta)$ be the homomorphism by which $\mu(\underline{T}_s(i))=\underline{\tilde{t}}_s(i)$ and $\mu(\overline{T}_s(j))=\bar{\tilde{t}}_s(j)$ for every $0\leqq s\leqq q_0$, $i\in[k]$, $j\in[lq_s]$. Since $\hat{\mathbf{T}}$ is a functor and $\tilde{T}(\Omega)$ is free, the following diagram commutes:

$$\begin{array}{ccc}
\hat{T}(\Omega) & \xrightarrow{\hat{\mathbf{T}}\varkappa} & \hat{\mathbf{T}}\mathbf{D}R(\Lambda)[k', l'] \\
\big\downarrow{\scriptstyle \hat{\mathbf{T}}\mu} & & \big\downarrow{\scriptstyle \hat{\mathbf{T}}\mathbf{D}\,\xi[k', l']} \\
\hat{T}(\Delta) & \xrightarrow{\hat{\mathbf{T}}h} & \hat{\mathbf{T}}\mathbf{D}R[k', l']
\end{array}$$

This implies that for every $T\in L(\Omega)[k, l](1, q)$

$$\hat{h}[k, l](\hat{\mathbf{T}}\mu(T)) = \xi(\hat{\varkappa}[k, l](T)).$$

Thus, by (4), we get that

$$\hat{h}[k, l](t_0)\cdot\sum_{s=1}^{q_0}\hat{h}[k, l](t_s) = \hat{h}[k, l](\hat{\mathbf{T}}\mu(T_0))\cdot\sum_{s=1}^{q_0}\hat{h}[k, l](\hat{\mathbf{T}}\mu(T_s)) =$$

$$= \xi\left(\hat{\varkappa}[k, l](T_0)\cdot\sum_{s=1}^{q_0}\hat{\varkappa}[k, l](T_s)\right)\Phi\xi\left(\hat{\varkappa}[k, l]\left(T_0\cdot\sum_{s=1}^{q_0}T_s\right)\right) =$$

$$= \hat{h}[k, l]\left(\hat{\mathbf{T}}\mu\left(T_0\cdot\sum_{s=1}^{q_0}T_s\right)\right) = \hat{h}[k, l]\left(t_0\cdot\sum_{s=1}^{q_0}t_s\right),$$

what was to be proved.

Abstract

We define an interesting subclass of deterministic attributed tree transducers. The importance of this subclass lies in its nice closure properties with respect to composition. It is proved that a deterministic and linear attributed tree transformation can be composed by any attributed transformation without leaving the class of attributed transformations. Moreover, the class of linear deterministic attributed tree transformations is closed under composition. Finally we show that the surface sets of linear deterministic attributed tree to string transformations are the same as the surface sets of linear deterministic top-down ones.

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