On v_i-products of automata

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In this paper we introduce a family of compositions and investigate it from the point of view of isomorphic completeness. Using results concerning well-known types of compositions, we give necessary and sufficient conditions for a system of automata to be isomorphically complete with respect to these products.

By an *automaton* we mean a finite automaton without output. For any nonvoid set X let us denote by X^* the free monoid generated by X. Furthermore, denote by X^+ the free semigroup generated by X. Considering an automaton $\mathbf{A}=(X, A, \delta)$, the transition function δ can be extended to $A\times X^*\to A$ in the following way: $\delta(a, \lambda)=a$ and $\delta(a, p)=\delta(\delta(a, p'), x)$ for any $a\in A, p=p'x\in X^*$, where λ denotes the empty word of X^*. Further on we shall use the notation $ap_{\mathbf{A}}$ for $\delta(a, p)$. If there is no danger of confusion then we omit the index $\mathbf{A}$ in $ap_{\mathbf{A}}$. Let M be an arbitrary nonvoid set. Denote by $P(M)$ the set of all subsets of M.

Let $\mathbf{A}_t=(X_t, A_t, \delta_t)$ $(t=0, \ldots, n-1)$ be a system of automata. Moreover let X be a finite nonvoid set, φ a mapping of $A_0\times\ldots\times A_{n-1}\times X$ into $X_0\times\ldots\times X_{n-1}$ and γ a mapping of $\{0, \ldots, n-1\}$ into $P(\{0, \ldots, n-1\})$ such that φ can be given in the form

$$\varphi(a_0, \ldots, a_{n-1}, x) = (\varphi_0(a_0, \ldots, a_{n-1}, x), \ldots, \varphi_{n-1}(a_0, \ldots, a_{n-1}, x))$$

where each φ_t $(0\leqq t\leqq n-1)$ is independent of states, which have indices not contained in the set $\gamma(t)$. We say that $\mathbf{A}=\left(X, \prod_{t=0}^{n-1} A_t, \delta\right)$ is a *v_i-product* of $\mathbf{A}_t$ $(t=0, \ldots, n-1)$ with respect to X, φ and γ if $|\gamma(t)|\leqq i$ $(t=0, \ldots, n-1)$ and for any $(a_0, \ldots, a_{n-1})\in \prod_{t=0}^{n-1} A_t$ and $x\in X$

$$\delta((a_0, \ldots, a_{n-1}), x) =$$
$$= (\delta_0(a_0, \varphi_0(a_0, \ldots, a_{n-1}, x)), \ldots, \delta_{n-1}(a_{n-1}, \varphi_{n-1}(a_0, \ldots, a_{n-1}, x))).$$

For this product we use the notation $\prod_{t=0}^{n-1} \mathbf{A}_t(X, \varphi, \gamma)$.

It is clear that the v_0-product is the same as the quasi-direct product. Therefore, we consider the case $i\geqq 1$ only. Furthermore, it is interesting to note that if $n=2, i=1, \gamma(0)=\{1\}, \gamma(1)=\{0\}$ then we obtain the cross product (see [2]) as a special case of the ν_1-product. Finally, observe that the ν_i-product is rearrangable, i.e. changing the order of components of a ν_i-product $\prod_{t=0}^{n-1} \mathbf{A}_t(X, \varphi, \gamma)$ and choosing suitable mappings φ', γ' we get such a ν_i-product which is isomorphic to the original one.

Let Σ be a system of automata. Σ is called *isomorphically complete* with respect to the ν_i-product if any automaton can be embedded isomorphically into a ν_i-product of automata from Σ. Furthermore, Σ is called a *minimal* isomorphically complete system if Σ is isomorphically complete and for arbitrary $\mathbf{A}\in\Sigma$ the system $\Sigma\setminus\{\mathbf{A}\}$ is not isomorphically complete.

For any natural number $n\geqq 1$ denote by $\mathbf{D}_n=(X_n, \{1, \dots, n\}, \delta_n)$ the automaton for which $X_n=\{x_{rs}\colon 1\leqq r, s\leqq n\}$ and

$$\delta_n(t, x_{rs}) = \begin{cases} s & \text{if } t = r, \\ t & \text{otherwise} \end{cases}$$

for any $t\in\{1, \dots, n\}$ and $x_{rs}\in X_n$.

The following theorem holds for the ν_i-products if $i\geqq 1$.

Theorem 1. A system Σ of automata is isomorphically complete with respect to the ν_i-product $(i\geqq 1)$ if and only if for any natural number $n\geqq 1$, there exists an automaton $\mathbf{A}\in\Sigma$ such that $\mathbf{D}_n$ can be embedded isomorphically into a ν_i-product of $\mathbf{A}$ with a single factor.

Proof. Theorem 1 can be proved in a similar way as the corresponding statement for the α_i-products in [4]. The sufficiency follows from Theorem 2 in [4], but it is not difficult to see directly. In order to prove the necessity we show that for any $n\geqq 1$ if $\mathbf{D}_n$ can be embedded isomorphically into a ν_i-product of automata from Σ then there exists an automaton $\mathbf{A}\in\Sigma$ such that $\mathbf{D}_{[\sqrt[i+1]{n}]}$ can be embedded isomorphically into a ν_i-product of $\mathbf{A}$ with a single factor, where $[\sqrt[i+1]{n}]$ denotes the largest integer less than or equal to $\sqrt[i+1]{n}$.

If $n=1$ then the statement is obvious. Now let $n>1$ and assume that $\mathbf{D}_n$ can be embedded isomorphically into a ν_i-product $\mathbf{B}=\prod_{t=0}^{k} \mathbf{A}_t(X_n, \varphi, \gamma)$ of automata $\mathbf{A}_t=(X_t', A_t, \delta_t)\in\Sigma$ $(t=0, \dots, k)$. Let us denote by μ such an isomorphism and for any $t\in\{1, \dots, n\}$ denote by $(a_{t0}, \dots, a_{tk})$ the image of t under μ. We distinguish two cases depending on the sets $\gamma(t)$ $(t=0, \dots, k)$. If $\gamma(t)=\emptyset$ for all $t\in\{0, \dots, k\}$ then $\mathbf{B}$ is a quasi-direct product. Since the quasi-direct product can be considered as a special α_{i+1}-product we have that $\mathbf{D}_n$ can be embedded isomorphically into an α_{i+1}-product $\prod_{t=1}^{k} \mathbf{A}_t(X_n, \varphi)$ of automata from Σ. From this, by the proof of Theorem 2 in [4], it follows that there exists an automaton $\mathbf{A}\in\Sigma$ such that $\mathbf{D}_{[\sqrt[i+1]{n}]}$ can be embedded isomorphically into an α_{i+1}-product of $\mathbf{A}$ with a single factor. Since an α_{i+1}-product with a single factor is a ν_i-product with a single factor we have proved the statement for this case.

Now assume that $\gamma(t) \neq \emptyset$ for some $t \in \{0, \ldots, k\}$. By the rearrangability of ν_i-products, without loss of generality we may suppose that $\gamma(0) \neq \emptyset$. We show that $\mathbf{D}_n$ can be embedded isomorphically into a ν_{i+1}-product of automata from $\{\mathbf{A}_0, \ldots, \mathbf{A}_k\}$ with at most $i+1$ factors. If $k \leqq i$ then we are ready. Assume that $k > i$. We may suppose that there exist natural numbers $r \neq s$ $(1 \leqq r, s \leqq n)$ such that $a_{r0} \neq a_{s0}$ since otherwise $\mathbf{D}_n$ can be embedded isomorphically into a ν_i-product of automata from $\{\mathbf{A}_0, \ldots, \mathbf{A}_k\}$ with k factors. Let $\gamma(0) = \{j_1, \ldots, j_w\}$. By the definition of the ν_i-product, we have that $w \leqq i$ and

$$\varphi_0(a_0, \ldots, a_k, x) = \varphi_0(a_{j_1}, \ldots, a_{j_w}, x) \quad \text{for any} \quad (a_0, \ldots, a_k) \in \prod_{t=0}^{k} A_t \quad \text{and} \quad x \in X_n.$$

We prove that the elements $(a_{t0}, a_{tj_1}, \ldots, a_{tj_w})$ $(t=1, \ldots, n)$ are pairwise different. Indeed, assume that $a_{u0} = a_{v0}$ and $a_{ut} = a_{vt}$ $(t = j_1, \ldots, j_w)$ for some $u \neq v$ $(1 \leqq u, v \leqq n)$. Then $\varphi_0(a_{uj_1}, \ldots, a_{uj_w}, x) = \varphi_0(a_{vj_1}, \ldots, a_{vj_w}, x)$ for any $x \in X_n$. Therefore, in the ν_i-product $\mathbf{B}$ the automaton $\mathbf{A}_0$ obtains the same input signal in the states a_{u0} and a_{v0} for any $x \in X_n$. Since μ is isomorphism, $u \neq v$ and $a_{u0} = a_{v0}$, thus the automaton $\mathbf{A}_0$ goes from the state a_{u0} into the state a_{t0} and from the state a_{v0} it goes into the state a_{v0} for any x_{ut} $(t=1, \ldots, n)$. This implies $a_{v0} = a_{t0}$ $(t=1, \ldots, n)$ which contradicts our assumption $a_{r0} \neq a_{s0}$. Therefore, we have that the elements $(a_{t0}, a_{tj_1}, \ldots, a_{tj_w})$ $(t=1, \ldots, n)$ are pairwise different. Now take the following ν_{i+1}-product $\mathbf{C} = \mathbf{A}_0 \times \mathbf{A}_{j_1} \times \ldots \times \mathbf{A}_{j_w}(X_n, \psi, \bar{\gamma})$ where for any $t \in \{0, \ldots, w\}$ $\bar{\gamma}(t) = \{0, 1, \ldots, w\}$ and

$$\psi_t(b_0, \ldots, b_w, x) = \begin{cases} \varphi_0(a_{r0}, \ldots, a_{rk}, x) & \text{if } t=0 \text{ and there exists } 1 \leqq r \leqq n \\ \quad \text{such that } b_0 = a_{r0}, \; b_s = a_{rj_s} \; (s=1, \ldots, w), \\ \varphi_{j_t}(a_{r0}, \ldots, a_{rk}, x) & \text{if } t \neq 0 \text{ and there exists } 1 \leqq r \leqq n \\ \quad \text{such that } b_0 = a_{r0}, \; b_s = a_{rj_s} \; (s=1, \ldots, w), \\ \text{otherwise arbitrary input signal from } X_0' \text{ if} \\ \quad t=0 \text{ and from } X_{j_t}' \text{ if } t \neq 0, \end{cases}$$

for all $(b_0, \ldots, b_w) \in A_0 \times A_{j_1} \times \ldots \times A_{j_w}$ and $x \in X_n$. It is not difficult to see that the correspondence $\mu' \colon t \to (a_{t0}, a_{tj_1}, \ldots, a_{tj_w})$ $(t=1, \ldots, n)$ is an isomorphism of $\mathbf{D}_n$ into $\mathbf{C}$. Therefore, we have that $\mathbf{D}_n$ can be embedded isomorphically into a ν_{i+1}-product of automata from $\{\mathbf{A}_0, \ldots, \mathbf{A}_k\}$ with at most $i+1$ factors. But a ν_{i+1}-product with at most $i+1$ factors is an α_{i+1}-product and thus, in a similar way as in the first case, we obtain that $\mathbf{D}_{[\sqrt[i+1]{n}]}$ can be embedded isomorphically into a ν_i-product of $\mathbf{A}_t$ with a single factor for some $0 \leqq t \leqq k$. This ends the proof of Theorem 1.

Observe that $\mathbf{D}_m$ can be embedded isomorphically into a ν_0-product of $\mathbf{D}_n$ with a single factor for any $m \leqq n$. Using this fact, by Theorem 1, we get the following

Corollary. There exists no system of automata which is isomorphically complete with respect to the ν_i-product $(i \geqq 1)$ and minimal.

In [1] F. Gécseg has introduced the concepts of the generalized α_i-product and the simulation and characterized the isomorphically and homomorphically complete systems with respect to them. Further on we shall introduce the concept of the generalized ν_i-product and investigate the isomorphically complete systems with respect to this product and the simulation.

We say that an automaton $\mathbf{A}=(X, A, \delta)$ *isomorphically simulates* $\mathbf{B}=(Y, B, \delta')$ if there exist one-to-one mappings $\mu: B\to A$ and $\tau: Y\to X^+$ such that $\mu(\delta'(b, y))=\delta(\mu(b), \tau(y))$ for any $b\in B$ and $y\in Y$. The following obvious observation holds for the isomorphic simulation.

Lemma 1. If **A** can be simulated isomorphically by **B** and **B** can be simulated isomorphically by **C** then **C** isomorphically simulates **A**.

Let $\mathbf{A}_t=(X_t, A_t, \delta_t)$ $(t=0, \ldots, n-1)$ be a system of automata. Moreover let X be a finite nonvoid set, φ a mapping of $A_0\times\ldots\times A_{n-1}\times X$ into $X_0^+\times\ldots\times X_{n-1}^+$ and γ a mapping of $\{0, \ldots, n-1\}$ into $P(\{0, \ldots, n-1\})$ such that φ can be given in the form

$$\varphi(a_0, \ldots, a_{n-1}, x)=(\varphi_0(a_0, \ldots, a_{n-1}, x), \ldots, \varphi_{n-1}(a_0, \ldots, a_{n-1}, x))$$

where each $\varphi_t (0\leqq t\leqq n-1)$ is independent of states, which have indices not contained in the set $\gamma(t)$. We say that $\mathbf{A}=\left(X, \prod_{t=0}^{n-1} A_t, \delta\right)$ is a *generalized ν_i-product* of $\mathbf{A}_t$ $(t=0,\ldots, n-1)$ with respect to X, φ and γ if $|\gamma(t)|\leqq i$ $(t=0, \ldots, n-1)$ and for any $(a_0, \ldots, a_{n-1})\in\prod_{t=0}^{n-1} A_t$ and $x\in X$ $\delta((a_0, \ldots, a_{n-1}), x)=(\delta_0(a_0, \varphi_0(a_0, \ldots, a_{n-1}, x)), \ldots$ $\ldots, \delta_{n-1}(a_{n-1}, \varphi_{n-1}(a_0, \ldots, a_{n-1}, x)))$.

A system Σ of automata is called *isomorphically S-complete* with respect to the generalized ν_i-product if any automaton can be simulated isomorphically by a generalized ν_i-product of automata from Σ.

Observe that in the definitions of the simulation and the generalized ν_i-product all input words are different from the empty word. Therefore, further on, by an input word we mean a nonempty word. Also the following notation will be used. If k, s are integers and t is a natural number then $k+s \pmod t$ denotes the least nonnegative residue of $k+s$ modulo t. Furthermore, for any $n\geqq 1$ denote by $\mathbf{T}_n=(T_n, \{0, \ldots, n-1\}, \delta_n)$ the automaton for which T_n is the set of all transformations of $\{0, \ldots, n-1\}$ and $\delta_n(k, t)=t(k)$ for any $k\in\{0, \ldots, n-1\}$ and $t\in T_n$.

Lemma 2. If $\mathbf{T}_n$ can be simulated isomorphically by a generalized α_0-product $\prod_{t=0}^{k}\mathbf{A}_t(X, \varphi)$ then $\mathbf{T}_n$ can be simulated isomorphically by $\mathbf{A}_j$ for some $j\in\{0, \ldots, k\}$.

Proof. Lemma 2 follows from the proof of Theorem 1 in [1]. Now we give another proof. Obviously it is enough to prove the statement for the generalized α_0-product of two factors. Indeed, assume that $\mathbf{T}_n$ can be simulated isomorphically by the generalized α_0-product $\mathbf{A}\times\mathbf{B}(X, \varphi)$ under μ and τ. Let us denote by (a_t, b_t) the image of t under μ $(t=0, \ldots, n-1)$. If $a_0=a_t$ for all $t\in\{1, \ldots, n-1\}$ then the elements b_t $(t=0, \ldots, n-1)$ are pairwise different. Now define the mapping τ' in the following way: for any $t_u\in T_n$ $\tau'(t_u)=\varphi_1(a_0, y_1)\ldots\varphi_1(a_0, y_s)$ if $\tau(t_u)=y_1\ldots y_s$. Let us denote by μ' the mapping determined by $\mu'(t)=b_t$ $(t=0, \ldots, n-1)$. It is not difficult to see that **B** isomorphically simulates $\mathbf{T}_n$ under μ' and τ'. Now assume that there exist natural numbers $r\neq s$ $(0\leqq r, s\leqq n-1)$ such that $a_r\neq a_s$. In this case we show that the states a_t $(t=0, \ldots, n-1)$ are pairwise different. Suppose that $a_u=a_v$ for some $u\neq v$ $(0\leqq u, v\leqq n-1)$. Let us denote by t_{ij} the element of T_n for which $t_{ij}(i)=j$ and $t_{ij}(k)=k$ if $k\neq i$ $(k=0, 1, \ldots, n-1)$ for all

i, j $(0 \leqq i, j \leqq n-1)$. Now let $w \in \{0, \ldots, n-1\}$ be arbitrary. Then $t_{uw}(u) = w$ and $t_{uw}(v) = v$. By isomorphic simulation, $(a_u, b_u)\tau(t_{uw}) = (a_w, b_w)$ and $(a_v, b_v)\tau(t_{uw}) = (a_v, b_v)$. Let $\tau(t_{uw}) = y_1 \ldots y_m$. Then $a_u \varphi_0(y_1) \ldots \varphi_0(y_m) = a_w$ and $a_v \varphi_0(y_1) \ldots \varphi_0(y_m) = a_v$. Therefore, by $a_u = a_v$, we obtain $a_w = a_v$. Since w was arbitrary we got that $a_t = a_v$ for all $t \in \{0, \ldots, n-1\}$ which contradicts our assumption $a_r \neq a_s$. Now we have that the states a_t $(t = 0, \ldots, n-1)$ are pairwise different. In this case it is not difficult to see that $\mathbf{A}$ isomorphically simulates $\mathbf{T}_n$ under μ' and τ' where $\mu'(t) = a_t$ $(t = 0, \ldots, n-1)$ and for any $t_u \in T_n$ $\tau'(t_u) = \varphi_0(y_1) \ldots \varphi_0(y_s)$ if $\tau(t_u) = y_1 \ldots y_s$.

Lemma 3. If $\mathbf{T}_n$ can be simulated isomorphically by a generalized ν_1-product $\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi, \gamma)$ then $\mathbf{T}_n$ can be simulated by a generalized ν_1-product $\prod_{t=0}^{r} \mathbf{B}_t(X, \varphi', \gamma')$ where $r \leqq k$, $\mathbf{B}_t \in \{\mathbf{A}_0, \ldots, \mathbf{A}_k\}$ and $\gamma'(t) = \{t-1 \pmod{r+1}\}$ for any $t \in \{0, \ldots, r\}$.

Proof. We proceed by induction on the number of components of the generalized ν_1-product. If $k=0$ then the statement is obvious. Now let $k>0$ and assume that the statement is valid for any l less than k. Moreover, suppose that $\mathbf{T}_n$ can be simulated isomorphically by a generalized ν_1-product $\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi, \gamma)$. Define the binary relation ϱ on the set $\{0, \ldots, k\}$ as follows: $i \varrho j$ if and only if $i=j$ or $\gamma(i) = \{j\}$ or $\gamma(j) = \{i\}$ for any $i, j \in \{0, \ldots, k\}$. Denote by $\hat{\varrho}$ the transitive closure of ϱ. Then $\hat{\varrho}$ is an equivalence relation on $\{0, \ldots, k\}$. Depending on $\hat{\varrho}$, we shall distinguish three cases.

First assume that the partition induced by $\hat{\varrho}$ has at least two blocks. Let us denote by $\hat{\varrho}(j)$ the block containing j. By the rearrangability of the ν_i-product, we may assume that $\hat{\varrho}(0) = \{0, \ldots, m-1\}$. From this, using the fact that $\bigcup_{s \in \hat{\varrho}(t)} \gamma(s) \subseteq \hat{\varrho}(t)$ holds for any $t \in \{0, \ldots, k-1\}$, we obtain that $\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi, \gamma)$ is isomorphic to a quasi-direct product of two automata $\mathbf{C}_1$ and $\mathbf{C}_2$ where $\mathbf{C}_1$ is a generalized ν_1-product of $\mathbf{A}_0, \ldots, \mathbf{A}_{m-1}$ and $\mathbf{C}_2$ is a generalized ν_1-product of $\mathbf{A}_m, \ldots, \mathbf{A}_k$. Therefore, by Lemma 1, Lemma 2 and our induction hypothesis, we get that the statement is valid.

Now let us suppose that the partition induced by $\hat{\varrho}$ has one block only and there exists an $u \in \{0, \ldots, k\}$ with $u \notin \bigcup_{t=0}^{k} \gamma(t)$. By the rearrangability of ν_i-product, we may suppose that $u=k$. Then observe that $\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi, \gamma)$ is isomorphic to a generalized α_0-product of two automata $\mathbf{C}_1$ and $\mathbf{A}_k$ where $\mathbf{C}_1$ is a generalized ν_1-product of $\mathbf{A}_0, \ldots, \mathbf{A}_{k-1}$. From this, by Lemma 1, Lemma 2 and induction hypothesis, the statement follows.

Finally, assume that the partition induced by $\hat{\varrho}$ has one block only and $\bigcup_{t=0}^{k} \gamma(t) = \{0, \ldots, k\}$. Consider the mapping f determined as follows: for any $t \in \{0, \ldots, k\}$ $f(t) = j$ if and only if $j \in \gamma(t)$. By the definition of $\hat{\varrho}$ and our assumption on $\hat{\varrho}$, it can be seen that f is a cyclic permutation of the set $\{0, \ldots, k\}$. Now rearrange

$\prod_{t=0}^{k} \mathbf{A}_t(X, \varphi, \gamma)$ in the form $\prod_{t=0}^{k} \mathbf{A}_{f_{(0)}^{k-t-1}}(X, \varphi', \gamma')$. Then, by the rearrangability of v_i-product and Lemma 1, we obtain that $\mathbf{T}_n$ can be simulated isomorphically by $\prod_{t=0}^{k} \mathbf{A}_{f_{(0)}^{k-t-1}}(X, \varphi', \gamma')$. On the other hand, it is not difficult to see that $\prod_{t=0}^{k} \mathbf{A}_{f_{(0)}^{k-t-1}}(X, \varphi', \gamma')$ satisfies the condition of our statement. This ends the proof of Lemma 3.

Now we are ready to study the generalized v_1-product. We have

Theorem 2. A system Σ of automata is isomorphically S-complete with respect to the generalized v_1-product if and only if one of the following three conditions is satisfied by Σ:

(1) for any natural number $n>1$ there exists an automaton in Σ having n different states a_t $(t=0, \ldots, n-1)$ and input words q_t $(t=0, \ldots, n-1)$ such that $a_t q_t = a_{t+1 \,(\mathrm{mod}\, n)}$ $(t=0, \ldots, n-1)$,

(2) Σ contains an automaton which has two different states a, b and input words p, q, r such that $ap=br=a$ and $aq=bp=b$,

(3) there exists an automaton in Σ which has two different states a, b and input words p, q, r such that $ap \neq bp$, $apq=bpq=a$ and $ar=b$.

Proof. In order to prove the sufficiency of conditions (1)—(3) we use the following observation.

For any automaton $\mathbf{A}=(X, A, \delta)$, $\mathbf{A}$ can be simulated isomorphically by $\mathbf{T}_n$ with $n \geqq \max(2, |A|)$. Therefore, by Lemma 1, if for any $n \geqq 2$ the automaton $\mathbf{T}_n$ can be simulated isomorphically by a generalized v_1-product of automata from Σ then Σ is isomorphically S-complete with respect to the generalized v_1-product. On the other hand, take the following elements t_1, t_2 and t_3 of T_n

$$t_1(k) = k+1 \pmod n \quad (k=0, \ldots, n-1),$$
$$t_2(0) = 1,\ t_2(1) = 0,\ t_2(k) = k \quad (k=2, \ldots, n-1),$$
$$t_3(0) = t_3(1) = 0 \quad \text{and} \quad t_3(k) = k \quad (k=2, \ldots, n-1).$$

It can be proved (see [3]) that the mappings t_1, t_2, t_3 generate the complete transformation semigroup over the set $\{0, \ldots, n-1\}$. Therefore, the automaton $\mathbf{T}_n$ can be simulated isomorphically by the automaton $\mathbf{T}'_n=(\{t_1, t_2, t_3\}, \{0, \ldots, n-1\}, \delta'_n)$ where $\delta'_n=\delta_n|_{\{0,\ldots,n-1\}\times\{t_1,t_2,t_3\}}$. From this we obtain that if for any $n \geqq 2$ the automaton $\mathbf{T}'_n$ can be simulated isomorphically by a generalized v_1-product of automata from Σ then Σ is isomorphically S-complete with respect to the generalized v_1-product.

First suppose that Σ satisfies (1). Then it is not difficult to see that for any automaton $\mathbf{A}$ there exists an automaton $\mathbf{B} \in \Sigma$ such that $\mathbf{A}$ can be simulated isomorphically by a generalized v_1-product of $\mathbf{B}$ with a single factor.

Now assume that Σ satisfies (2) by $\mathbf{A} \in \Sigma$. Let $n \geqq 5$ be arbitrary and take the generalized v_1-product $\mathbf{A}^n(X, \varphi, \gamma)$ where

$$X = \{u_i\colon 1 \leqq i < n\} \cup$$
$$\cup \{v_i\colon 0 \leqq i < n\} \cup \{x_i\colon 1 < i < n\} \cup \{y_i\colon 1 \leqq i < n-1\} \cup \{v, x, y, z, w\}$$

and the mappings γ and φ are defined in the following way: for any $t \in \{0, \ldots, n-1\}$

$$\gamma(t) = t-1 (\text{mod } n),$$

$$\varphi_t(a, u_i) = p, \quad \varphi_t(b, u_i) = \begin{cases} q & \text{if} \quad t = i, \\ p & \text{otherwise} \quad (i = 1, \ldots, n-1), \end{cases}$$

$$\varphi_t(a, v_i) = \begin{cases} r & \text{if} \quad t = i, \\ p & \text{otherwise}, \end{cases} \quad \varphi_t(b, v_i) = \begin{cases} r & \text{if} \quad 0 < t < i, \\ p & \text{otherwise} \quad (i = 0, \ldots, n-1), \end{cases}$$

$$\varphi_t(a, x_i) = p, \quad \varphi_t(b, x_i) = \begin{cases} r & \text{if} \quad i \leqq t \leqq n-1, \\ p & \text{otherwise} \quad (i = 2, \ldots, n-1), \end{cases}$$

$$\varphi_0(a, y_i) = p, \qquad \varphi_0(b, y_i) = q,$$

$$\varphi_t(a, y_i) = p, \quad \varphi_t(b, y_i) = \begin{cases} r & \text{if} \quad 1 \leqq t < i, \quad i \neq 2, \\ p & \text{otherwise} \quad (i = 1, \ldots, n-2 \quad \text{and} \quad t \geqq 1) \end{cases}$$

$$\varphi_t(a, v) = p, \quad \varphi_t(b, v) = \begin{cases} r & \text{if} \quad 1 \leqq t \leqq n-2, \\ p & \text{otherwise}, \end{cases}$$

$$\varphi_0(a, x) = p, \quad \varphi_0(b, x) = r, \quad \varphi_t(a, x) = \varphi_t(b, x) = p \quad (t \geqq 1),$$

$$\varphi_0(a, z) = p, \quad \varphi_0(b, z) = r, \quad \varphi_1(a, z) = r, \quad \varphi_1(b, z) = p,$$

$$\varphi_2(a, z) = \varphi_2(b, z) = p, \quad \varphi_t(a, z) = p, \quad \varphi_t(b, z) = r \quad (t > 2),$$

$$\varphi_0(a, w) = q, \quad \varphi_0(b, w) = p, \quad \varphi_t(a, w) = p, \quad \varphi_t(b, w) = r \quad (t \geqq 1),$$

$$\varphi_0(a, y) = q, \quad \varphi_0(b, y) = \varphi_t(a, y) = \varphi_t(b, y) = p \quad (t \geqq 1).$$

Take the mappings

$$\mu: \begin{array}{l} 0 \quad \to (b, a, \ldots, a), \\ \vdots \\ n-1 \to (a, a, \ldots, b), \end{array}$$

$$\tau: \begin{array}{l} t_1 \to q_1 \ldots q_{n-1}, \\ t_2 \to u_3 \ldots u_{n-1} y_1 z u_1 \ldots u_{n-1} y x_2 u_3 \ldots u_{n-1} v_0 x_3 u_2 \ldots u_{n-1} y x_2, \\ t_3 \to u_3 \ldots u_{n-1} y_1 z u_1 \ldots u_{n-1} w, \end{array}$$

where

$$q_1 = u_1 \ldots u_{n-2} v_{n-1} u_1 \ldots u_{n-1} v y,$$

$$q_2 = u_1 \ldots u_{n-3} v_{n-2} v_0 u_1 \ldots u_{n-2} x_{n-1} y_{n-2} u_{n-1} y,$$

$$q_3 = u_1 \ldots u_{n-4} v_{n-3} v_0 x_{n-1} x u_{n-1} u_1 \ldots u_{n-3} x_{n-2} u_{n-1} y_{n-3} x_{n-1} u_{n-2} u_{n-1} y,$$

$$q_i = u_1 \ldots u_{n-i-1} v_{n-i} v_0 x_{n-i+2} u_{n-i+3} \ldots u_{n-1} x x_{n-i+3} u_{n-i+2} \ldots u_{n-1}$$
$$u_1 \ldots u_{n-i} x_{n-i+1} u_{n-i+2} \ldots u_{n-1} y_{n-i} x_{n-i+2} u_{n-i+1} \ldots u_{n-1} y$$

if $4 \leqq i < n-1$ and

$$q_{n-1} = v_1 x_2 u_4 \ldots u_{n-1} x x_4 u_3 \ldots u_{n-1} v_0 x_3 u_2 \ldots u_{n-1} y x_2.$$

Now we show that $\mathbf{T}'_n$ can be simulated isomorphically by $\mathbf{A}^n(X, \varphi, \gamma)$ under μ and τ. The validity of the equations $\mu(\delta'_n(j, t_l))=\delta_{\mathbf{A}^n}(\mu(j), \tau(t_l))$ $(l=2, 3)$ $(j=0, \ldots, n-1)$ can be checked by a simple computation.

Introduce the following notation

$$u_{jt}^{(i)} = \begin{cases} b & \text{if} \quad j = t, \quad j \leqq n-i-1 \quad \text{or} \quad t = 1, \quad j > n-i-1 \\ & \qquad \text{or} \quad t > n-i-1, \quad t > j, \\ a & \text{otherwise,} \end{cases}$$

$1 \leqq i < n-2$, $0 \leqq t \leqq n-1$ and $0 \leqq j \leqq n-1$. It can be proved by induction on i that $\mu(j)q_1 \ldots q_i = (u_{j0}^{(i)}, \ldots, u_{jn-1}^{(i)})$ for any $j \in \{0, \ldots, n-1\}$ and $1 \leqq i < n-2$. On the other hand $(u_{j0}^{(n-3)}, \ldots, u_{jn-1}^{(n-3)})q_{n-2}q_{n-1} = \mu(j+1 \,(\mathrm{mod}\, n))$ for any $j \in \{0, \ldots, n-1\}$. Therefore, $\mu(\delta'_n(j, t_1)) = \mu(j+1 \,(\mathrm{mod}\, n)) = (u_{j0}^{(n-3)}, \ldots, u_{jn-1}^{(n-3)})q_{n-2}q_{n-1} = \mu(j)q_1 \ldots q_{n-1} = \delta_{\mathbf{A}^n}(\mu(j), \tau(t_1))$ for any $j \in \{0, \ldots, n-1\}$. This ends the proof of the sufficiency of condition (2).

Now suppose that Σ satisfies (3) by $\mathbf{A} \in \Sigma$. Then there exist states $a \neq b$ of $\mathbf{A}$ and input words p, q, r such that $ap \neq bp$, $apq = bpq = a$ and $ar = b$. Observe that it is enough to prove the sufficiency of (3) for the case $a \notin \{ap, bp\}$. Indeed, assume that $a \in \{ap, bp\}$. We distinguish two cases. If $b \in \{ap, bp\}$ then p is a permutation of the set $\{a, b\}$ and thus the automaton $\mathbf{A}$ has the property required in (2). If $b \notin \{ap, bp\}$ then introducing the notations $a'=b$, $b'=a$, $p'=p$, $q'=qr$, $r'=pq$ we obtain that $a' \neq b'$, $a'p' \neq b'p'$, $a'p'q'=b'p'q'=a'$, $a'r'=b'$ and $a' \notin \{a'p', b'p'\}$. Therefore, without loss of generality we may assume that $a \notin \{ap, bp\}$. Now let $n \geqq 6$ be arbitrary and take the generalized ν_1-product $\mathbf{A}^n(X, \varphi, \gamma)$ where $X = \{x_1, \ldots, x_8\}$ and the mappings γ, φ are defined in the following way: for any $t \in \{0, \ldots, n-1\}$

$$\gamma(t) = \{t-1 \,(\mathrm{mod}\, n)\}$$

$$\varphi_t(a, x_1) = pq, \quad \varphi_t(b, x_1) = r,$$

$$\varphi_t(a, x_2) = \begin{cases} p & \text{if} \quad t = 1, \\ pqp & \text{otherwise,} \end{cases} \quad \varphi_t(b, x_2) = \begin{cases} p & \text{if} \quad t = 2, \\ rp & \text{otherwise,} \end{cases}$$

$$\varphi_t(ap, x_3) = q, \quad \varphi_t(bp, x_3) = qr,$$

$$\varphi_t(a, x_4) = p, \quad \varphi_t(b, x_4) = \begin{cases} pq & \text{if} \quad t = 1, \\ p & \text{otherwise,} \end{cases}$$

$$\varphi_t(a, x_5) = \begin{cases} qp & \text{if} \quad b \neq ap, \\ p & \text{if} \quad b = ap, \end{cases} \quad \varphi_t(ap, x_5) = q, \quad \varphi_t(bp, x_5) = \begin{cases} r & \text{if} \quad t = 1, \\ qr & \text{if} \quad t \neq 1, \end{cases}$$

$$\varphi_t(a, x_6) = p, \quad \varphi_t(b, x_6) = \begin{cases} q & \text{if} \quad t = 2, \\ p & \text{otherwise,} \end{cases}$$

$$\varphi_t(ap, x_6) = \begin{cases} pq & \text{if} \quad b \neq ap, \\ \varphi_t(b, x_6) & \text{otherwise,} \end{cases} \quad \varphi_t(bp, x_6) = \begin{cases} pq & \text{if} \quad b = ap, \\ \varphi_t(b, x_6) & \text{otherwise,} \end{cases}$$

$$\varphi_t(a, x_7) = \begin{cases} p & \text{if } b \neq ap, \quad t = 3, \\ qp & \text{if } b \neq ap, \quad t \neq 3, \\ rp & \text{if } b = ap, \quad t = 3, \\ qrp & \text{if } b = ap, \quad t \neq 3, \end{cases}$$

$$\varphi_t(ap, x_7) = q, \quad \varphi_t(bp, x_7) = \begin{cases} r & \text{if } t = 2, \\ qr & \text{otherwise}, \end{cases}$$

$$\varphi_t(a, x_8) = \begin{cases} p & \text{if } t = 3, \\ pqp & \text{otherwise}, \end{cases} \quad \varphi_t(b, x_8) = \begin{cases} qp & \text{if } t = 3, \\ p & \text{if } t = 4, \\ rp & \text{otherwise}, \end{cases}$$

$$\varphi_t(ap, x_8) = \begin{cases} qrp & \text{if } b \neq ap, \quad t = 4, \\ p & \text{if } b \neq ap, \quad t = 5, \\ \varphi_t(b, x_8) & \text{if } b = ap, \\ \text{an arbitrary input word otherwise}, \end{cases}$$

$$\varphi_t(bp, x_8) = \begin{cases} qrp & \text{if } b = ap, \quad t = 4, \\ p & \text{if } b = ap, \quad t = 5, \\ \varphi_t(b, x_8) & \text{if } b \neq ap, \\ \text{an arbitrary input word otherwise}, \end{cases}$$

and in all other cases φ_t is defined arbitrarily. Take the following mappings

$$\mu\colon \begin{array}{l} 0 \to (b, a, \dots, a) \\ \vdots \\ n-1 \to (a, \dots, a, b) \end{array} \qquad \tau\colon \begin{array}{l} t_1 \to x_1, \\ t_2 \to x_4 x_5 x_6 x_7 x_8 x_3 x_1^{n-4}, \\ t_3 \to x_2 x_3 x_1^{n-2}. \end{array}$$

Distinguishing the cases $b=ap$ and $b\neq ap$ it can be seen easily that $\mu(\delta'_n(j), t_l)=\delta_{A^n}(\mu(j), \tau(t_l))$ for any $j\in\{0, \dots, n-1\}$ and $l\in\{1, 2, 3\}$ which yields the sufficiency of (3).

In order to prove the necessity assume that none of conditions (1)—(3) is satisfied by Σ and Σ is isomorphically S-complete with respect to the generalized ν_1-product. Since Σ does not satisfy (1) there exists a natural number $m>2$ such that Σ does not contain an automaton having the property required in (1) for any $n\geqq m$. Let $n>m^{\binom{m}{2}}$ be an arbitrary fixed natural number. By the assumption on the isomorphic S-completeness of Σ, there exists a generalized ν_1-product $\mathbf{B}=\prod_{t=0}^{k-1} \mathbf{A}_t(X, \varphi, \gamma)$ of automata from Σ such that $\mathbf{T}_n$ can be simulated isomorphically by $\mathbf{B}$ under suitable μ and τ. By Lemma 3, we may suppose that $\gamma(t)=\{t-1 \pmod k\}$ $(t=0, \dots, k-1)$. Let us denote by $(a_{l0}, \dots, a_{lk-1})$ the image of l under μ for any $l\in\{0, \dots, n-1\}$. Consider an arbitrary nonvoid subset $\Gamma=\{j_1, \dots, j_r\}$ of the set $\{0, \dots, k-1\}$. Define a relation π_Γ on $\prod_{t=0}^{k-1} A_t$ in the following way: $(a_0, \dots, a_{k-1})\pi_\Gamma(b_0, \dots, b_{k-1})$ if and only if $a_{j_s-\binom{m}{2}+u \pmod k}=$

$=b_{j_s-\binom{m}{2}+u(\mathrm{mod}\, k)}$ $(u=1, \ldots, \binom{m}{2})$, $(s=1, \ldots, r)$ for any $(a_0, \ldots, a_{k-1}), (b_0, \ldots, b_{k-1}) \in \prod_{t=0}^{k-1} A_t$. It is clear that π_Γ is an equivalence relation on $\prod_{t=0}^{k-1} A_t$. Now let us denote by $\bar{B}$ the set $\{(a_{l0}, \ldots, a_{lk-1})\colon 0 \leqq l \leqq n-1\}$ and let $\bar{\pi}_\Gamma = \pi_\Gamma \cap (\bar{B} \times \bar{B})$.

We shall show that $(a_0, \ldots, a_{k-1}) \bar{\pi}_\Gamma (b_0, \ldots, b_{k-1})$ implies $(a_0, \ldots, a_{k-1}) \tau(t) \bar{\pi}_{\Gamma'} (b_0, \ldots, b_{k-1}) \tau(t)$ for any $t \in T_n$ and $(a_0, \ldots, a_{k-1}), (b_0, \ldots, b_{k-1}) \in \prod_{t=0}^{k-1} A_t$, where $\Gamma' = \{j_s + |\tau(t)| (\mathrm{mod}\, k)\colon 1 \leqq s \leqq r\}$. Indeed, assume that $(a_0, \ldots, a_{k-1}) \bar{\pi}_\Gamma (b_0, \ldots, b_{k-1})$ and let $t \in T_n$ be arbitrary. Since $\mathbf{T}_n$ can be simulated isomorphically by $\mathbf{B}$ there exist $t_1, t_2, t_3 \in T_n$ such that

$$(a_0, \ldots, a_{k-1}) \tau(t) \tau(t_1) = (b_0, \ldots, b_{k-1}) \tau(t) \tau(t_1),$$

$$(a_0, \ldots, a_{k-1}) \tau(t) \tau(t_1) \tau(t_2) = (b_0, \ldots, b_{k-1}),$$

$$(b_0, \ldots, b_{k-1}) \tau(t) \tau(t_1) \tau(t_3) = (a_0, \ldots, a_{k-1}).$$

Let $\tau(t) = x_1 \ldots x_j$, $\tau(t_1) = x_{j+1} \ldots x_{j+u}$, $\tau(t_2) = y_1 \ldots y_v$ and $\tau(t_3) = z_1 \ldots z_w$. Introduce the following notations

$$q_{1t}^{(1)} = \varphi_t(a_{t-1(\mathrm{mod}\, k)}, x_1) \quad (t = 0, \ldots, k-1),$$

$$q_{lt}^{(1)} = \varphi_t(a_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(1)} \ldots q_{l-1t-1(\mathrm{mod}\, k)}^{(1)}, x_l) \quad (t = 0, \ldots, k-1), \quad (2 \leqq l \leqq j+u),$$

$$q_{1t}^{(2)} = \varphi_t(b_{t-1(\mathrm{mod}\, k)}, x_1) \quad (t = 0, \ldots, k-1),$$

$$q_{lt}^{(2)} = \varphi_t(b_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(2)} \ldots q_{l-1t-1(\mathrm{mod}\, k)}^{(2)}, x_l) \quad (t = 0, \ldots, k-1), \quad (2 \leqq l \leqq j+u),$$

$$p_{1t} = \varphi_t(a_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(1)} \ldots q_{j+ut-1(\mathrm{mod}\, k)}^{(1)}, y_1) \quad (t = 0, \ldots, k-1),$$

$$p_{lt} = \varphi_t(a_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(1)} \ldots q_{j+ut-1(\mathrm{mod}\, k)}^{(1)} p_{1t-1(\mathrm{mod}\, k)} \ldots p_{l-1t-1(\mathrm{mod}\, k)}, y_l)$$
$$(t = 0, \ldots, k-1), \quad (2 \leqq l \leqq v),$$

$$r_{1t} = \varphi_t(b_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(2)} \ldots q_{j+ut-1(\mathrm{mod}\, k)}^{(2)}, z_1) \quad (t = 0, \ldots, k-1),$$

$$r_{lt} = \varphi_t(b_{t-1(\mathrm{mod}\, k)} q_{1t-1(\mathrm{mod}\, k)}^{(2)} \ldots q_{j+ut-1(\mathrm{mod}\, k)}^{(2)} r_{1t-1(\mathrm{mod}\, k)} \ldots r_{l-1t-1(\mathrm{mod}\, k)}, z_l)$$
$$(t = 0, \ldots, k-1), \quad (2 \leqq l \leqq w).$$

Then, by the above equations, we have that for any $t \in \{0, \ldots, k-1\}$

(i) $$a_t q_{1t}^{(1)} \ldots q_{j+ut}^{(1)} = b_t q_{1t}^{(2)} \ldots q_{j+ut}^{(2)},$$

(ii) $$a_t q_{1t}^{(1)} \ldots q_{j+ut}^{(1)} p_{1t} \ldots p_{vt} = b_t,$$

(iii) $$b_t q_{1t}^{(2)} \ldots q_{j+ut}^{(2)} r_{1t} \ldots r_{wt} = a_t.$$

Now let us denote by $(a_0^{(0)}, \ldots, a_{k-1}^{(0)})$, $(b_0^{(0)}, \ldots, b_{k-1}^{(0)})$ the states $(a_0, \ldots, a_{k-1})$, $(b_0, \ldots, b_{k-1})$ and $(a_0^{(i)}, \ldots, a_{k-1}^{(i)})$, $(b_0^{(i)}, \ldots, b_{k-1}^{(i)})$ the states $(a_0, \ldots, a_{k-1}) x_1 \ldots x_i$, $(b_0, \ldots, b_{k-1}) x_1 \ldots x_i$ $(i = 1, \ldots, j)$, respectively. To prove our statement we show that $(a_0, \ldots, a_{k-1}) \bar{\pi}_\Gamma (b_0, \ldots, b_{k-1})$ implies $(a_0^{(i)}, \ldots, a_{k-1}^{(i)}) \pi_{\Gamma_i} (b_0^{(i)}, \ldots, b_{k-1}^{(i)})$ for any $0 \leqq i \leqq j$, where $\Gamma_i = \{j_s + i (\mathrm{mod}\, k)\colon 1 \leqq s \leqq r\}$. We proceed by induction on i. $(a_0^{(0)}, \ldots, a_{k-1}^{(0)}) \pi_{\Gamma_0} (b_0^{(0)}, \ldots, b_{k-1}^{(0)})$ obviously holds. Now assume that our statement has been proved for $i-1$ $(1 \leqq i \leqq j)$. Then from $(a_0^{(i-1)}, \ldots, a_{k-1}^{(i-1)}) \pi_{\Gamma_{i-1}} (b_0^{(i-1)}, \ldots, b_{k-1}^{(i-1)})$ it follows that

$$a^{(i-1)}_{j_s-\binom{m}{2}+l+i-1 (\mathrm{mod}\, k)} = b^{(i-1)}_{j_s-\binom{m}{2}+l+i-1 (\mathrm{mod}\, k)} \quad \left(l = 1, \ldots, \binom{m}{2}\right), \quad (s = 1, \ldots, r).$$

Therefore, by the definition of $q_{lt}^{(1)}, q_{lt}^{(2)}$ we have that

$$q^{(1)}_{ij_s-\binom{m}{2}+l+i-1 (\mathrm{mod}\, k)} = q^{(2)}_{ij_s-\binom{m}{2}+l+i-1 (\mathrm{mod}\, k)} \quad \left(l = 2, \ldots, \binom{m}{2}+1\right), \quad (s=1, \ldots, r)$$

and thus $a^{(i)}_{j_s-\binom{m}{2}+l+i (\mathrm{mod}\, k)} = b^{(i)}_{j_s-\binom{m}{2}+l+i (\mathrm{mod}\, k)} \left(l=1, \ldots, \binom{m}{2}-1\right), (s=1, \ldots, r)$.

Now, if $a^{(i)}_{j_s+i (\mathrm{mod}\, k)} = b^{(i)}_{j_s+i (\mathrm{mod}\, k)}$ for all $1 \leqq s \leqq r$ then we get that $(a_0^{(i)}, \ldots, a_{k-1}^{(i)}) \pi_{\Gamma_i} (b_0^{(i)}, \ldots, b_{k-1}^{(i)})$ and so we are ready. In the opposite case there exists an index $s \in \{1, \ldots, r\}$ such that $a^{(i)}_{j_s+i (\mathrm{mod}\, k)} \neq b^{(i)}_{j_s+i (\mathrm{mod}\, k)}$. Let us denote by f the index $j_s+i \pmod k$. Then $a_f^{(i)} \neq b_f^{(i)}$. From this, by $q_{if}^{(1)} = q_{if}^{(2)}$, it follows that $a_f^{(i-1)} \neq b_f^{(i-1)}$ and $a_f^{(i-1)} q_{if}^{(1)} \neq b_f^{(i-1)} q_{if}^{(1)}$. Now let $h = \min \left(j+u-i, \binom{m}{2}-1\right)$. Then, by $a^{(i)}_{f-\binom{m}{2}+l (\mathrm{mod}\, k)} = b^{(i)}_{f-\binom{m}{2}+l (\mathrm{mod}\, k)} \left(l=1, \ldots, \binom{m}{2}-1\right)$, we have that $q_{i+lf}^{(1)} = q_{i+lf}^{(2)} \left(l=1, \ldots, \binom{m}{2}-1\right)$. Therefore, $q_{i+1f}^{(1)} \ldots q_{i+hf}^{(1)} = q_{i+1f}^{(2)} \ldots q_{i+hf}^{(2)}$. Now we show that $a_f^{(i)} q_{i+1f}^{(1)} \ldots q_{i+hf}^{(1)} = b_f^{(i)} q_{i+1f}^{(1)} \ldots q_{i+hf}^{(1)}$. Indeed, if $h = i+u-i$ then we get the required equality from (i). If $h = \binom{m}{2}-1$ then let us consider the sets M_l $(l=0, \ldots, h)$ defined by $M_0 = \{a_f^{(i)}, b_f^{(i)}\}$ and $M_l = M_{l-1} q_{i+lf}^{(1)}$ $(l=1, \ldots, h)$. If $|M_l| = 1$ for some $l \in \{1, \ldots, h\}$ then $a_f^{(i)} q_{i+1f}^{(1)} \ldots q_{i+lf}^{(1)} = b_f^{(i)} q_{i+1f}^{(1)} \ldots q_{i+lf}^{(1)}$ and thus $a_f^{(i)} q_{i+1f}^{(1)} \ldots q_{i+hf}^{(1)} = b_f^{(i)} q_{i+1f}^{(1)} \ldots q_{i+hf}^{(1)}$. Therefore, it is enough to consider the case for which $|M_l| = 2$ for all $l \in \{0, \ldots, h\}$. If $M_g = M_l$ for some $0 \leqq g < l \leqq h$ then $M_g p = M_l$ where $p = q_{i+g+1f}^{(1)} \ldots q_{i+lf}^{(1)}$. But in this case it can be seen easily that the automaton $\mathbf{A}_f$ has the property required in (2) which is a contradiction. Now consider the case for which $|M_l| = 2$ for all $l \in \{0, \ldots, h\}$ and the sets M_l $(l=0, \ldots, h)$ are pairwise different. It is not difficult to see that from (ii) and (iii) it follows that for any $a, b \in \bigcup_{l=0}^{h} M_l$ there exists an input word p of $\mathbf{A}_f$ with $ap = b$. From this, by the definition m, we obtain that $\left| \bigcup_{l=0}^{h} M_l \right| = m' < m$. Thus we got that a set with cardinality $m' (< m)$ has $\binom{m}{2}$ pairwise different subsets of two elements which is a contradiction. Therefore, we have proved that $a_f^{(i)} q_{i+1f}^{(1)} \ldots q_{i+hf}^{(1)} = b_f^{(i)} q_{i+1f}^{(1)} \ldots q_{i+hf}^{(1)}$. In this case, by (i), (ii), (iii), it can be seen easily that the automaton $\mathbf{A}_f$ with the states $a_f^{(i-1)}, b_f^{(i-1)}$ has the property required in (3) which is a contradiction. So we get a contradiction from the assumption $a^{(i)}_{j_s+i (\mathrm{mod}\, k)} \neq b^{(i)}_{j_s+i (\mathrm{mod}\, k)}$ for some $s \in \{1, \ldots, r\}$. Therefore, $a^{(i)}_{j_s+i (\mathrm{mod}\, k)} = b^{(i)}_{j_s+i (\mathrm{mod}\, k)}$ for all $s \in \{1, \ldots, r\}$ and thus $(a_0^{(i)}, \ldots, a_{k-1}^{(i)}) \pi_{\Gamma_i} (b_0^{(i)}, \ldots, b_{k-1}^{(i)})$. From this, by $i=j$ we obtain that $(a_0, \ldots, a_{k-1}) x_1 \ldots x_j \pi_{\Gamma_j} (b_0, \ldots, b_{k-1}) x_1 \ldots x_j$ i.e. $(a_0, \ldots, a_{k-1}) \tau(t) \pi_{\Gamma'} (b_0, \ldots, b_{k-1}) \tau(t)$. On the other hand $(a_0, \ldots, a_{k-1}) \tau(t)$, $(b_0, \ldots, b_{k-1}) \tau(t) \in \bar{B}$ and thus $(a_0, \ldots, a_{k-1}) \tau(t) \bar{\pi}_{\Gamma'} (b_0, \ldots, b_{k-1}) \tau(t)$ which ends the proof of the statement.

Since $n > m^{\binom{m}{2}}$ there exists a subset $\Gamma \subseteq \{0, \ldots, k-1\}$ such that $\bar{\pi}_\Gamma \neq \Delta_{\bar{B}}$, where $\Delta_{\bar{B}}$ denotes the identity relation on $\bar{B}$. Therefore, the set $C = \{\Gamma : \Gamma \subseteq \{0, \ldots, k-1\}, \Gamma \neq \emptyset, \bar{\pi}_\Gamma \neq \Delta_{\bar{B}}\}$ is nonempty. Then let us denote by $\Gamma = \{j_1, \ldots, j_r\}$ such an element of C for which $|\Gamma|$ is maximal. Since $\bar{\pi}_\Gamma \neq \Delta_{\bar{B}}$ there exist $u \neq v \in \{0, \ldots, n-1\}$ with $\mu(u) \bar{\pi}_\Gamma \mu(v)$. Consider the element $t_1 \in T_n$ defined by $t_1(u) = v$, $t_1(v) = u$ and $t_1(l) = l$ if $l \in \{0, \ldots, n-1\} \setminus \{u, v\}$. By the isomorphic simulation, we have that $\mu(u)\tau(t_1)=\mu(v)$, $\mu(v)\tau(t_1)=\mu(u)$ and $\mu(l)\tau(t_1)=\mu(l)$ if $l\in\{0,\dots,n-1\}\setminus\{u,v\}$. On the other hand $\mu(u)\bar{\pi}_\Gamma\mu(v)$ and thus $\mu(u)\tau(t_1)\bar{\pi}_{\Gamma'}\mu(v)\tau(t_1)$, where $\Gamma'=\{j_s+|\tau(t_1)|(\mathrm{mod}\ k)\colon 1\leqq s\leqq r\}$. Therefore, $\mu(u)\bar{\pi}_{\Gamma'}\mu(v)$. It is clear that the mapping $\beta_1\colon t\to t+|\tau(t_1)|(\mathrm{mod}\ k)$ $(t=0,\dots,k-1)$ is a permutation of the set $\{0,\dots,k-1\}$ and thus $|\Gamma|=|\Gamma'|$. By the maximality of $|\Gamma|$ we have that $\Gamma'\subseteq\Gamma$ and thus $\Gamma=\Gamma'$. This means that the mapping β_1 fixes the set Γ, i.e. $\beta_1(\Gamma)=\Gamma$, where $\beta_1(\Gamma)$ denotes the set $\{\beta_1(t)\colon t\in\Gamma\}$. On the other hand it is not difficult to see that β_1 fixes a subset M of the set $\{0,\dots,k-1\}$ if and only if

$$M=\{i,\ i+|\tau(t_1)|(\mathrm{mod}\ k),\ \dots,\ i+(f-1)|\tau(t_1)|(\mathrm{mod}\ k)\}$$

for some $i\in\{0,1,\dots,\text{g.c.d.}\,(k,|\tau(t_1)|)-1\}$ or M is equal to an union of such sets, where g.c.d. $(k,|\tau(t_1)|)$ denotes the greatest common divisor of the numbers $k,|\tau(t_1)|$ and $f=k/\text{g.c.d}\,(k,|\tau(t_1)|)$. Furthermore, it is clear that the considered sets $m_i=\{i,\ i+|\tau(t_1)|(\mathrm{mod}\ k),\ \dots,\ i+(f-1)|\tau(t_1)|(\mathrm{mod}\ k)\}$ form a partition of $\{0,\dots,k-1\}$. Thus assume that $\Gamma=\bigcup_{t=1}^{g} m_{i_t}$. Now consider the set $\bar{B}\setminus\{\mu(u),\mu(v)\}$. Since $n\geqq 3$ there exists an element $w\in\{0,\dots,n-1\}$ such that $\mu(w)\in\bar{B}\setminus\{\mu(u),\mu(v)\}$. Let us denote by t_2 a cyclic permutation from T_n with $t_2(u)=v$ and $t_2(v)=w$. By the isomorphic simulation we have that $\mu(u)\tau(t_2)=\mu(v)$ and $\mu(v)\tau(t_2)=\mu(w)$. On the other hand $\mu(u)\bar{\pi}_\Gamma\,\mu(v)$. Therefore, $\mu(u)\tau(t_2)\bar{\pi}_{\Gamma'}\,\mu(v)\tau(t_2)$ where $\Gamma'=\{j_s+|\tau(t_2)|(\mathrm{mod}\ k)\colon 1\leqq s\leqq r\}$. Since the mapping $\beta_2\colon t\to t+|\tau(t_2)|(\mathrm{mod}\ k)$ $(t=0,\dots,k-1)$ is a permutation of $\{0,\dots,k-1\}$ we obtain that $|\Gamma|=|\Gamma'|$. Now we distinguish two cases.

First assume that $\Gamma=\Gamma'$. Then it is not difficult to see that $\mu(u)\bar{\pi}_\Gamma\,\mu(l)$ holds for any $l\in\{0,\dots,n-1\}$ which contradicts the maximality of $|\Gamma|$.

Now assume that $\Gamma\neq\Gamma'$. Observe that $\Gamma'=\bigcup_{t=1}^{g}\beta_2(m_{i_t})$ and $\beta_2(m_{i_t})=m_{i_t+|\tau(t_2)|(\mathrm{mod}\ \text{g.c.d.}\,(k,|\tau(t_1)|))}$. Therefore, from $|\Gamma|=|\Gamma'|$ and $\Gamma\neq\Gamma'$ it follows that there exists an index $j\in\{0,\dots,\text{g.c.d.}\,(k,|\tau(t_1)|)-1\}$ with $m_j\cap\Gamma=\emptyset$ and $m_j\subseteq\Gamma'$. On the other hand $\mu(v)\bar{\pi}_{\Gamma'}\mu(w)$ and thus $\mu(v)\tau(t_1)\bar{\pi}_{\Gamma''}\mu(w)\tau(t_1)$ where $\Gamma''=\beta_1(\Gamma')$. By $\mu(v)\tau(t_1)=\mu(u)$ and $\mu(w)\tau(t_1)=\mu(w)$ we obtain that $\mu(u)\bar{\pi}_{\Gamma''}\,\mu(w)$. Since β_1 fixes the sets m_i $(i=0,\dots,\text{g.c.d.}\,(k,|\tau(t_1)|)-1)$ we have that $m_j\subseteq\Gamma''$. Then $j\in\Gamma'$ and $j\in\Gamma''$ and thus

$$a_{vj-\binom{m}{2}+l(\mathrm{mod}\,k)}=a_{wj-\binom{m}{2}+l(\mathrm{mod}\,k)}\quad \left(l=1,\dots,\binom{m}{2}\right),$$

$$a_{wj-\binom{m}{2}+l(\mathrm{mod}\,k)}=a_{uj-\binom{m}{2}+l(\mathrm{mod}\,k)}\quad \left(l=1,\dots,\binom{m}{2}\right).$$

From this it follows that $j\in\Gamma$ which is a contradiction. This ends the proof of the necessity.

The next theorem holds for the generalized ν_i-product if $i>1$.

Theorem 3. *A system Σ of automata is isomorphically S-complete with respect to the generalized ν_i-product $(i>1)$ if and only if Σ contains an automaton which has two different states a,b and input words p,q such that $ap=b$ and $bq=a$.*

Proof. The necessity is obvious. Conversely, assume that Σ satisfies the condition of Theorem 3 by **A**. Let $n\geqq 3$ be arbitrary and take the generalized ν_2-product

$\mathbf{A}^n(X, \varphi, \gamma)$ where $X=\{x_1, \ldots, x_6\}$ and the mappings γ, φ are defined in the following way: for any $t \in \{0, \ldots, n-1\}$

$$\gamma(t) = \{t, t-1 \pmod{n}\},$$

$$\varphi_t(a, a, x_1) = pq,\ \varphi_t(a, b, x_1) = q,\ \varphi_t(b, a, x_1) = p,$$

$$\varphi_0(a, a, x_2) = \varphi_0(b, a, x_2) = p,\ \varphi_0(a, b, x_2) = q,\ \varphi_1(a, a, x_2) = pq,\ \varphi_1(a, b, x_2) = q,$$

$$\varphi_1(b, a, x_2) = p, \quad \varphi_t(u, v, x_2) = \begin{cases} pq & \text{if } v = a, \\ qp & \text{if } v = b, \quad (t = 2, \ldots, n-1), \end{cases}$$

$$\varphi_t(u, v, x_3) = \begin{cases} pq & \text{if } v = a, \\ qp & \text{if } v = b, \quad (t = 0, 1), \end{cases}$$

$$\varphi_t(u, v, x_3) = \begin{cases} p & \text{if } v = a, \quad u = b, \\ pq & \text{if } v = a, \quad u = a, \\ qp & \text{if } v \neq a \quad (t = 2, \ldots, n-1), \end{cases}$$

$$\varphi_0(a, a, x_4) = \varphi_0(b, a, x_4) = pq, \quad \varphi_0(a, b, x_4) = qp, \quad \varphi_0(b, b, x_4) = q,$$

$$\varphi_t(u, v, x_4) = \begin{cases} pq & \text{if } v = a, \\ qp & \text{if } v = b, \quad (t = 1, \ldots, n-1), \end{cases}$$

$$\varphi_t(u, v, x_5) = \begin{cases} pq & \text{if } v = a, \\ qp & \text{if } v = b, \quad (t = 0, 1), \end{cases}$$

$$\varphi_t(u, v, x_5) = \begin{cases} q & \text{if } u = v = b, \\ qp & \text{if } u = a, \quad v = b, \\ pq & \text{if } v = a, \quad (t = 2, \ldots, n-1), \end{cases}$$

$$\varphi_0(a, a, x_6) = \varphi_0(b, a, x_6) = p, \ \varphi_0(a, b, x_6) = qp,$$

$$\varphi_1(a, a, x_6) = \varphi_1(b, a, x_6) = pq, \quad \varphi_1(a, b, x_6) = q,$$

$$\varphi_t(u, v, x_6) = \begin{cases} pq & \text{if } v = a, \\ qp & \text{if } v = b, \quad (t = 2, \ldots, n-1). \end{cases}$$

In the remaining cases $\varphi_t(u, v, x_j)$ is an arbitrary input word from $\{p, q\}$. Now consider the mappings:

$$\mu\colon \begin{array}{rl} 0 & \to (b, a, \ldots, a), \\ 1 & \to (a, b, \ldots, a), \\ \vdots & \\ n-1 & \to (a, a, \ldots, b), \end{array} \qquad \tau\colon \begin{array}{l} t_1 \to x_1, \\ t_2 \to x_2 x_3^{n-3} x_4 x_5, \\ t_3 \to x_6 x_3^{n-3} x_4 x_5. \end{array}$$

It is not difficult to see that the automaton $\mathbf{T}'_n$ can be simulated isomorphically by $\mathbf{A}^n(X, \varphi, \gamma)$ under μ and τ.

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