Decomposition results concerning K-visit attributed tree transducers

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The concept of attributed tree transducer was introduced in [1], [4] and [6]. On the other hand, the 1-visit, pure K-visit and simple K-visit classes of attributed grammars were defined in [3] and [5]. In this paper, we formulate these properties for deterministic attributed tree transducers defined in [6] and prove some decomposition results. Namely, we show that each tree transformation induced by a pure K-visit attributed tree transducer can be induced by a bottom-up tree transducer followed by an 1-visit attributed tree transducer. Here, the bottom-up tree transducer can be substituted by a top-down one. Moreover, each tree transformation induced by a simple K-visit attributed tree transducer can be induced by a deterministic bottom-up tree transducer followed by an 1-visit attributed tree transducer.

1. Notions and notations

By a type we mean a finite set F of the form $F=\bigcup_{n<\omega} F_n$ where the sets F_n are pairwise disjoint and $F_0\neq\emptyset$.

For an arbitrary type F and set S the set of trees over S of type F is the smallest set $T_F(S)$ satisfying:

(i) $F_0\cup S\subseteq T_F(S)$,

(ii) $f(p_1, ..., p_n)\in T_F(S)$ whenever $f\in F_n, p_1, ..., p_n\in T_F(S)$ $(n>0)$. If $S=\emptyset$ then $T_F(S)$ is written T_F.

The set of all positive integers is denoted by N. Let N^* denote the free monoid generated by N, with identity λ.

For a tree $p(\in T_F(S))$ the depth $(\mathrm{dp}\,(p))$, root $(\mathrm{root}\,(p))$, the set of subtrees $(\mathrm{sub}\,(p))$ of p and paths $(\mathrm{path}\,(p))$ of p as a subset of N^* are defined as follows:

(i) $\mathrm{dp}\,(p)=0$, $\mathrm{sub}\,(p)=\{p\}$, $\mathrm{root}\,(p)=p$, $\mathrm{path}\,(p)=\{\lambda\}$ if $p\in F_0\cup S$,

(ii) $\mathrm{dp}\,(p)=1+\max\,\{\mathrm{dp}\,(p_i)|1\leq i\leq n\}$, $\mathrm{root}\,(p)=f$, $\mathrm{sub}\,(p)=\{p\}\cup(\bigcup(\mathrm{sub}\,(p_i)|\,1\leq i\leq n))$, $\mathrm{path}\,(p)=\{\lambda\}\cup\{iv|1\leq i\leq n,\ v\in\mathrm{path}\,(p_i)\}$ if $p=f(p_1, ..., p_n)$ $(n>0, f\in F_n)$. Subtrees of height 0 of a tree $p(\in T_F(S))$ are called leaves of p.

For each $p(\in T_F(S))$, $w(\in \text{path}(p))$ there is a corresponding label $\text{lb}_p(w)$ $(\in F\cup S)$ and a subtree $\text{str}_p(w)$ $(\in \text{sub}(p))$ in p which are defined by induction on the length of w:

(i) $\text{lb}_p(w)=\text{root}(p)$, $\text{str}_p(w)=p$ if $w=\lambda$,

(ii) $\text{lb}_p(w)=\text{lb}_{p_i}(v)$, $\text{str}_p(w)=\text{str}_{p_i}(v)$ if $w=iv$, $p=f(p_1, \ldots, p_n)$, $1\leqq i\leqq n$.

In the rest of this paper, F, G and H always mean types, moreover, the set of auxiliary variables $Z=\{z_0, z_1, \ldots\}$ and its subsets $Z_n=\{z_1, \ldots, z_n\}$ $(n=0, 1, \ldots)$ are kept fixed. Observe that $Z_0=\emptyset$. Let $n\geqq 0$ and $p\in T_F(Z_n)$. Substituting the elements $s_1, \ldots, s_n$ of a set S for $z_1, \ldots, z_n$ in p, respectively, we have another tree, which is in $T_F(S)$ and denoted by $p(s_1, \ldots, s_n)$. There is a distinguished subset $\hat{T}_F(Z_n)$ of $T_F(Z_n)$ defined as follows: $p\in\hat{T}_F(Z_n)$ if and only if each z_i $(1\leqq i\leqq n)$ appears in p exactly once.

We now turn to the definition of tree transducers. The terminology used here follows [2].

Subsets of $T_F\times T_G$ are called tree transformations. The domain of a tree transformation $\tau(\subseteqq T_F\times T_G)$ is denoted by $\text{dom}\,\tau$ and defined by $\text{dom}\,\tau=\{p\in T_F|(p, q)\in\tau$ for some $q\in T_G\}$. The composition $\tau_1\circ\tau_2$ of the tree transformations $\tau_1(\subseteqq T_F\times T_G)$ and $\tau_2(\subseteqq T_G\times T_H)$ is defined by $\tau_1\circ\tau_2=\{(p, q)|(p, r)\in\tau_1,\ (r, q)\in\tau_2$ for some $r\}$. If $\mathscr{C}_1$ and $\mathscr{C}_2$ are classes of tree transformations then their composition $\mathscr{C}_1\circ\mathscr{C}_2$ is the class $\mathscr{C}_1\circ\mathscr{C}_2=\{\tau_1\circ\tau_2|\tau_1\in\mathscr{C}_1, \tau_2\in\mathscr{C}_2\}$.

By a bottom-up tree transducer we mean a system $\mathbf{A}=(F, A, G, A', P)$ where A is a nonempty finite set, the set of states, $A'(\subseteqq A)$ is the set of final states, moreover, P is a finite set of rewriting rules of the form $f(a_1z_1, \ldots, a_kz_k)\to aq$ where $k\geqq 0$, $f\in F_k$, $a, a_1, \ldots, a_k\in A$, $q\in T_G(Z_k)$. $\mathbf{A}$ is said to be deterministic if different rules in P have different left sides. P can be used to define a binary relation $\underset{\mathbf{A}}{\Rightarrow}$ on the set $T_F(A\times T_G)$. The reflexive, transitive closure of $\underset{\mathbf{A}}{\Rightarrow}$ is denoted by $\overset{*}{\underset{\mathbf{A}}{\Rightarrow}}$ and called derivation. The exact definition can be found in [2]. The tree transformation induced by $\mathbf{A}$ is a relation $\tau_{\mathbf{A}}(\subseteqq T_F\times T_G)$ defined by

$$\tau_{\mathbf{A}} = \{(p, q)|p\overset{*}{\underset{\mathbf{A}}{\Rightarrow}} aq \quad \text{for some} \quad a(\in A')\}.$$

A top-down tree transducer is again a system $\mathbf{A}=(F, A, G, A', P)$ which differs from the bottom-up one only in the form of the rewriting rules. Here, P is a finite set of rules of the form $af(z_1, \ldots, z_k)\to q(a_1z_{i_1}, \ldots, a_lz_{i_l})$ where $k, l\geqq 0$, $f\in F_k$, $a, a_1, \ldots, a_l\in A$, $1\leqq i_1, \ldots, i_l\leqq k$, $q\in\hat{T}_G(Z_l)$. Moreover, A' is called the set of initial states. The relation $\underset{\mathbf{A}}{\Rightarrow}$ can now be defined on the set $T_G(A\times T_F)$ and its reflexive, transitive closure is again denoted by $\overset{*}{\underset{\mathbf{A}}{\Rightarrow}}$ (c.f. [2]). The tree transformation induced by $\mathbf{A}$ is a relation $\tau_{\mathbf{A}}(\subseteqq T_F\times T_G)$ defined by

$$\tau_{\mathbf{A}} = \{(p, q)|ap\overset{*}{\underset{\mathbf{A}}{\Rightarrow}} q \quad \text{for some} \quad a(\in A')\}.$$

The following concept of attributed tree transducer was defined in [6]. We repeat this definition, with a slightly different formalism, because this new one seems to be simpler. Moreover, we allow not only the completely defined but the partially defined case as well.

By a deterministic attributed tree transducer, or shortly DATT, we mean a system $\mathbf{A}=(F, A, G, a_0, P, \mathrm{rt})$ defined as follows:

(a) A is a finite set, the set of attributes, which is the union of the disjoint sets A_s and A_i where A_s is called the set of synthesized attributes, A_i is called the set of inherited attributes;

(b) $a_0 \in A_s$;

(c) rt is a partial mapping from A_i to T_G;

(d) P is a finite set of rewriting rules of the form

$$af(z_1, \dots, z_k) \leftarrow \bar{q}(a_1 z_{j_1}, \dots, a_l z_{j_l}) \tag{1}$$

where $k, l \geqq 0$, $f \in F_k$, $\bar{q} \in \hat{T}_G(Z_l)$, $a \in A_s$, $0 \leqq j_1, \dots, j_l \leqq k$, $a_r \in A_i$ if $j_r = 0$ and $a_r \in A_s$ if $1 \leqq j_r \leqq k$ $(r=1, \dots, l)$ as well as rules of the form

$$a(z_j, f) \leftarrow \bar{q}(a_1 z_{j_1}, \dots, a_l z_{j_l}) \tag{2}$$

where $f \in F_k$ for some $k (\geqq 1)$, $l \geqq 0$, $a \in A_i$, $1 \leqq j \leqq k$, $\bar{q} \in \hat{T}_G(Z_l)$, $0 \leqq j_1, \dots, j_l \leqq k$ and a_r is the same as above $(r=1, \dots, l)$. Any two different rules of P are required to have different left sides.

From now on, for the sake of convenience we shall use the following notation for each element x of the set $N \cup \{0\}$

$$\bar{x} = \begin{cases} x & \text{if } x \in N \\ \lambda & \text{if } x = 0. \end{cases}$$

Let $p \in T_F$. We can define the relation $\underset{p,\mathbf{A}}{\Longleftarrow}$ on the set $T_G(A \times \mathrm{path}(p))$ in the following way. For $q, r (\in T_G(A \times \mathrm{path}(p)))$ $q \underset{p,\mathbf{A}}{\Longleftarrow} r$ if r is obtained from q by substituting the tree $\bar{q}((a_1, v_1), \dots, (a_l, v_l))$ for some leaf $(a, w) (\in A \times \mathrm{path}(p))$ of q if either the condition (a) or (b) holds:

(a) (i) $a \in A_s$,
(ii) $\mathrm{lb}_p(w) = f (\in F_k$ for some $k \geqq 0)$,
(iii) the rule (1) is in P,
(iv) $v_r = w\bar{j}_r$ $(r=1, \dots, l)$;

(b) (i) $a \in A_i$,
(ii) $w = vj$ for some $j (\in N)$,
(iii) $\mathrm{lb}_p(v) = f (\in F_k$ for some $k \geqq 1)$,
(iv) the rule (2) is in P,
(v) $v_r = v\bar{j}_r$ $(r=1, \dots, l)$.

Observe that a leaf of q which is in $A_i \times \{\lambda\}$ can never be substituted because, for such a leaf, neither (a) nor (b) can hold. Therefore we define the relation "$\underset{p,\mathbf{A}}{\Longleftarrow}$ concerning rt" which contains $\underset{p,\mathbf{A}}{\Longleftarrow}$ in the following manner: $q \underset{p,\mathbf{A}}{\Longleftarrow} r$ concerning rt if either $q \underset{p,\mathbf{A}}{\Longleftarrow} r$ or r is obtained from q by substituting $\mathrm{rt}(a)$ (if it exists) for a leaf $(a, \lambda) (\in A_i \times \{\lambda\})$ of q. Let the n-th power, transitive closure, reflexive, transitive closure of $\underset{p,\mathbf{A}}{\Longleftarrow}$ be denoted by $\overset{n}{\underset{p,\mathbf{A}}{\Longleftarrow}}$, $\overset{+}{\underset{p,\mathbf{A}}{\Longleftarrow}}$, $\overset{*}{\underset{p,\mathbf{A}}{\Longleftarrow}}$, respectively, and similarly for the relation $\underset{p,\mathbf{A}}{\Longleftarrow}$ concerning rt. We can now define the tree transformation $\tau_{\mathbf{A}}(\subseteq T_F \times T_G)$ induced by **A** in the following way

$$\tau_{\mathbf{A}} = \{(p, q) | (a_0, \lambda) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} q \text{ concerning rt}\}.$$

An example for a DATT can be found in [6]. The relation $\overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}}$ is called derivation. The length $\operatorname{lt}(\alpha)$ of a derivation $\alpha = q \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} r$ is defined as the integer n for which $q \overset{n}{\underset{p,\mathbf{A}}{\Leftarrow}} r$.

In the rest of this paper, by a DATT we always mean a noncircular DATT (see [6]).

Before going on, we make an observation which will often be used without reference. Let $p \in T_F$, $w \in \operatorname{path}(p)$, $l \geqq 0$, $q \in \hat{T}_G(Z_l)$, $a \in A_s$, $a_1, \ldots, a_l \in A_i$ and let $\operatorname{str}_p(w)$ be denoted by p_w.

Suppose that

$$(a, w) \overset{n}{\underset{p,\mathbf{A}}{\Leftarrow}} q((a_1, w), \ldots, (a_l, w)) \tag{3}$$

and there is no step in (3), in which, a leaf in $A_i \times \{w\}$ is substituted. Then

$$(a, \lambda) \overset{n}{\underset{p_w,\mathbf{A}}{\Leftarrow}} q((a_1, \lambda), \ldots, (a_l, \lambda))$$

and the converse also holds.

The classes of all tree transformations induced by top-down tree transducers, (deterministic) bottom-up tree transducers, deterministic attributed tree transducers are denoted by $\mathcal{T}$, $(\mathcal{D})\mathcal{B}$, $\mathcal{D}\mathcal{A}$, respectively.

2. K-visit attributed tree transducers

Let $\mathbf{A}(=(F, A, G, a_0, P, \text{rt}))$ be a DATT and let $K(\geqq 1)$ be an integer.

By a partition of A we mean a sequence $((I_1, S_1), \ldots, (I_l, S_l))$ where I_j (S_j) are pairwise disjoint subsets of A_i (A_s) whose union is A_i (A_s). Let $\Phi_K(A)$ denote the set of all partitions of A with $l \leqq K$.

Now let $f \in F_k$ $(k \geqq 0)$, $\mathbf{e}^i \in \Phi_K(A)$ with $\mathbf{e}^i = ((I_1^i, S_1^i), \ldots, (I_{l_i}^i, S_{l_i}^i))$ $(i = 0, 1, \ldots, k)$. The oriented graph $D_f(\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^k)$ is defined as follows. Its nodes are the symbols I_j^λ, S_j^λ $(j = 1, \ldots, l_0)$ and the symbols I_j^i, S_j^i $(i = 1, \ldots, k,\ j = 1, \ldots, l_i)$. Edges are oriented for each

(i) $j(=1, \ldots, l_0)$ from I_j^λ to S_j^λ;

(ii) $j(=1, \ldots, l_0 - 1)$ from S_j^λ to I_{j+1}^λ;

(iii) $i(=1, \ldots, k)$, $j(=1, \ldots, l_i)$ from I_j^i to S_j^i;

(iv) $i(=1, \ldots, k)$, $j(=1, \ldots, l_i - 1)$ from S_j^i to I_{j+1}^i;

(v) $j(=1, \ldots, l_0)$, $a(\in S_j^0)$ from $X_r^{i_s}$ to S_j^λ if there is a rule $af(z_1, \ldots, z_k) \leftarrow q(a_1 z_{i_1}, \ldots, a_l z_{i_l})$ in P for which $a_s \in X_r^{i_s}$ under some $s(=1, \ldots, l)$, $r(=1, \ldots, l_{i_s})$, $X \in \{I, S\}$;

(vi) $i(=1, \ldots, k)$, $j(=1, \ldots, l_i)$, $a(\in I_j^i)$ from $X_r^{i_s}$ to I_j^i if there is a rule $a(z_i, f) \leftarrow q(a_1 z_{i_1}, \ldots, a_l z_{i_l})$ in P with $a_s \in X_r^{i_s}$ under some s, r, X defined as in (v).

The graph $D_f(e^0, e^1, \ldots, e^k)$ corresponds to the concept of partition graph for a production of an attribute grammar, which concept was introduced in [5].

Let $p(=f(p_1, \ldots, p_k)) \in T_F$ $(k>0, f \in F_k)$ and consider a mapping π: path $(p) \to \Phi_K(A)$. The mappings π^i: path $(p_i) \to \Phi_K(A)$ are defined by $\pi^i(w) = \pi(iw)$ $(i = 1, \ldots, k,\ w \in \text{path}(p_i))$.

Now, let again $p \in T_F$ and π: path $(p) \to \Phi_K(A)$. The oriented graph $D_p(\pi)$ is defined by induction on dp (p):

(i) if $p = f(\in F_0)$ with $\pi(\lambda) = e$ then $D_p(\pi) = D_f(e)$;

(ii) if $p = f(p_1, \ldots, p_k)$ $(k>0,\ f \in F_k)$ with $\pi(\lambda) = e$, $\pi(i) = e^i$ $(i = 1, \ldots, k)$ then $D_p(\pi) = D_f(e, e^1, \ldots, e^k) \cup (\cup (D'_{p_i}(\pi^i) | 1 \leq i \leq k))$ where $D'_{p_i}(\pi^i)$ is obtained from $D_{p_i}(\pi^i)$ by "multiplying its nodes by i", that is, the nodes of $D'_{p_i}(\pi^i)$, are the symbols X_r^{iw} where X_r^w are nodes of $D_{p_i}(\pi^i)$, moreover, there is an edge from X_r^{iw} to Y_s^{iv} in $D'_{pi}(\pi^i)$ iff there is an edge from X_r^w to Y_s^v in $D_{p_i}(\pi^i)$. Nodes and edges of graphs are combined as sets.

Definition 1. We say that $\mathbf{A}$ is pure K-visit, if for each $p(\in \text{dom}\, \tau_{\mathbf{A}})$ there exists a π: path $(p) \to \Phi_K(A)$ with acyclic $D_p(\pi)$.

To support this definition, the following observation can be made. If $D_p(\pi)$ is acyclic then a computation sequence (see in [5] for attribute grammars) can be constructed, which induces a K-visit tree-walking attribute evaluation strategy on p.

Definition 2. Suppose that to each $f(\in F)$ there corresponds an element e^f of $\Phi_K(A)$ and let $\Pi_K = \{e^f | f \in F\}$. $\mathbf{A}$ is said to be simple K-visit concerning Π_K if for each $p(\in \text{dom}\, \tau_{\mathbf{A}})$ there exists a π: path $(p) \to \Pi_K$ for which the following two conditions hold:

(i) if $\text{lb}_p(w) = f$ then $\pi(w) = e^f$ $(w \in \text{path}(p))$,

(ii) $D_p(\pi)$ is acyclic.

$\mathbf{A}$ is simple K-visit, if it is simple K-visit concerning some Π_K.

The classes of all tree transformations induced by pure, simple K visit DATTs are denoted by $\mathcal{DA}_{PK}$, $\mathcal{DA}_{SK}$, respectively. Observe, that $\Phi_1(A) = \{(A_i, A_s)\}$ so, in the particular case $K=1$, the two properties defined above are identical. Therefore $\mathcal{DA}_{P1} = \mathcal{DA}_{S1}$ and they can be denoted by $\mathcal{DA}_1$.

Theorem 3. For each $K(\geq 1)$, $\mathcal{DA}_{PK} \subset \mathcal{B} \circ \mathcal{DA}_1$.

Proof. Let $\mathbf{A}(=(F, A, G, a_0, P, \text{rt}))$ be a pure K-visit DATT. Consider the bottom-up tree transducer $\mathbf{B}(=(F, B, \bar{F}, B', P'))$ where

(a) $B = B' = \Phi_K(A)$;

(b) for each $m(\geq 0)$, $\bar{F}_m$ is defined as follows $\langle f; e, e^1, \ldots, e^k \rangle \in \bar{F}_m$ if and only if

(i) $f \in F_k$ for some $k(\geq 0)$,

(ii) $e, e^1, \ldots, e^k \in \Phi_K(A)$,

(iii) $m = l_1 + \ldots + l_k$ where l_i is the number of components of e^i $(i = 1, \ldots, k)$,

(iv) $D_f(e, e^1, \ldots, e^k)$ is acyclic;

(c) for each $m(\geq 0)$, $\langle f;\ e, e^1, \ldots, e^k \rangle$ $(\in \bar{F}_m)$ the rule

$$f(e^1 z_1, \ldots, e^k z_k) \to e \langle f;\ e, e^1, \ldots, e^k \rangle (\overbrace{z_1, \ldots, z_1}^{l_1 \text{ times}}, \ldots, \overbrace{z_k, \ldots, z_k}^{l_k \text{ times}})$$

is in P'.

Moreover, let the DATT $\mathbf{C}=(\bar{F}, C, G, c_0, P'', \mathrm{rt}'')$ be defined as follows

(a) $C_s=A_s$, $C_i=A_i$, $c_0=a_0$, $\mathrm{rt}''=\mathrm{rt}$;

(b) P'' is constructed in the following way. Let $m\geqq 0$, $\langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle \in \bar{F}_m$ with $\mathbf{e}=((I_1, S_1), \ldots, (I_l, S_l))$ and $\mathbf{e}^j=((I_1^j, S_1^j), \ldots, (I_{l_j}^j, S_{l_j}^j))$ $(1\leqq j\leqq k)$. For each $a(\in C_s)$ let the rule $a\langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle\,(z_1, \ldots, z_m)\leftarrow q(a_1 z_{i_1}, \ldots, a_s z_{i_s})$ be in P'' if the following conditions hold:

(i) $$af(z_1, \ldots, z_k) \leftarrow q(a_1 z_{j_1}, \ldots, a_s z_{j_s})\in P,$$

(ii) $$i_r = \begin{cases} j_r(=0) & \text{if } a_r\in A_i \qquad (r=1, \ldots, s) \\ l_1+\ldots+l_{j_r-1}+n & \text{if } a_r\in S_n^{j_r} \text{ for some } n(=1, \ldots, l_{j_r}). \end{cases}$$

Moreover, for each $j(=1, \ldots, k)$, $n(=1, \ldots, l_j)$, $a(\in I_1^j\cup\ldots\cup I_n^j)$ let the rule $a(z_i, \langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle)\leftarrow q(a_1 z_{i_1}, \ldots, a_s z_{i_s})$ be in P'' if

(i) $$a(z_j, f) \leftarrow q(a_1 z_{j_1}, \ldots, a_s z_{j_s})\in P,$$

(ii) $$i = l_1+\ldots+l_{j-1}+n,$$

$$i_r = \begin{cases} j_r(=0) & \text{if } a_r\in A_i \qquad (r=1, \ldots, s) \\ l_1+\ldots+l_{j_r-1}+u & \text{if } a_r\in S_u^{j_r} \text{ for some } u=(1, \ldots, l_{j_r}). \end{cases}$$

The 1-visit property of $\mathbf{C}$ can be shown in the following manner. In [3], it was proved that an attributed grammar is 1-visit iff each of its brother graphs is acyclic. We can formulate the concept of the brother graph for DATTs and can easily show that each brother graph of $\mathbf{C}$ is acyclic.

The proof of the next lemma can be performed by a simple induction on $\mathrm{dp}\,(p)$.

Lemma 4. Let $p\in T_F$, $\mathbf{e}\in B$. Then $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$ for some $\bar{q}(\in T_{\bar{F}})$ if and only if there exists a $\pi\colon \mathrm{path}\,(p)\to\Phi_K(A)$ with $\pi(\lambda)=\mathbf{e}$ and acyclic $D_p(\pi)$.

Lemma 5. Let $p\in T_F, \bar{q}\in T_{\bar{F}}, q\in \hat{T}_G(Z_s), a_1, \ldots, a_s\in A_i, \mathbf{e}\in B$ with $\mathbf{e}=((I_1, S_1), \ldots$ $\ldots, (I_l, S_l))$ and let $a\in S_j$ for some $j(=1, \ldots, l)$. Suppose that $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$ and $(a, \lambda) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} q((a_1, \lambda), \ldots, (a_s, \lambda))$. Then $a_1, \ldots, a_s\in I_1\cup\ldots\cup I_j$.

Proof. It follows from the previous lemma that there exists a $\pi\colon \mathrm{path}\,(p)\to$ $\to\Phi_K(A)$ with $\pi(\lambda)=\mathbf{e}$ and acyclic $D_p(\pi)$. Suppose that, say, $a_1\in I_k$ where $k>j$. Then, by the definition of $D_p(\pi)$, there is a path from I_k^λ to S_j^λ in $D_p(\pi)$ due to the dependency edges of $D_p(\pi)$. On the other hand, there is a path from S_j^λ to I_k^λ in $D_p(\pi)$ because $k>j$, which contradicts the fact that $D_p(\pi)$ is acyclic.

Lemma 6. Let $a\in A_s$, $p\in T_F$, $\bar{q}\in T_{\bar{F}}$, $q\in T_G(A_i\times\{\lambda\})$, $\mathbf{e}\in B$. Suppose that $(a, \lambda) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} q$ and $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$. Then $(a, \lambda) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} q$.

Proof. The proof can be performed by induction on $\mathrm{dp}\,(p)$.

(a) Let $\mathrm{dp}\,(p)=0$ i.e. $p=f(\in F_0)$. Then by supposition, $af\leftarrow q'(a_1 z_0, \ldots, a_s z_0)\in P$ $(s\geqq 0, q'\in \hat{T}_G(Z_s), a_1, \ldots, a_s\in A_i)$, $q=q'((a_1, \lambda), \ldots, (a_s, \lambda))$, moreover, $f\to\mathbf{e}\langle f; \mathbf{e}\rangle\in P'$ and $\bar{q}=\langle f; \mathbf{e}\rangle$. Therefore, by the definition of $\mathbf{C}$, $a\langle f; \mathbf{e}\rangle\leftarrow q'(a_1 z_0, \ldots, a_s z_0)\in P''$.

(b) Now let $\mathrm{dp}(p)>0$ that is $p=f(p_1, \dots, p_k)$ $(k>0, f\in F_k)$. Here, $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$ can be written in the form

$$p = f(p_1, \dots, p_k) \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} f(\mathbf{e}^1\bar{q}_1, \dots, \mathbf{e}^k\bar{q}_k) \underset{\mathbf{B}}{\Rightarrow}$$

$$\mathbf{e}\langle f;\ \mathbf{e}, \mathbf{e}^1, \dots, \mathbf{e}^k\rangle(\overbrace{\bar{q}_1, \dots, \bar{q}_1}^{l_1 \text{ times}}, \dots, \overbrace{\bar{q}_k, \dots, \bar{q}_k}^{l_k \text{ times}}) = \mathbf{e}\bar{q}$$

with

$$\mathbf{e}^j = ((I_1^j, S_1^j), \dots, (I_{l_j}^j, S_{l_j}^j)) \quad (j = 1, \dots, k).$$

First we can prove the following

STATEMENT. Let $1\leqq j\leqq k$, $1\leqq n\leqq l_j$, $b\in I_1^j\cup\dots\cup I_n^j$, $t\in T_G(A_i\times\{\lambda\})$ and suppose that the relation $\beta=(b, j) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t$ holds. Then $(b, i) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} t$ where $i=l_1+\dots+l_{j-1}+n$.

The proof of this statement can be done by an induction on $\mathrm{lt}(\beta)$. When $\mathrm{lt}(\beta)=1$ then $b(z_j, f)\leftarrow t'(b_1z_0, \dots, b_sz_0)\in P$ $(s\geqq 0, t'\in T_G(Z_s), b_1, \dots, b_s\in A_i)$ and $t=t'((b_1, \lambda), \dots, (b_s, \lambda))$ so, by the definition of $\mathbf{C}$, $b(z_i, f)\leftarrow t'(b_1z_0, \dots, b_sz_0)\in P''$.

When $\mathrm{lt}(\beta)>1$ then β can be written in the following form

$$(b, j) \underset{p,\mathbf{A}}{\Leftarrow} t'((b_1, j_1), \dots, (b_s, j_s)) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t'(t_1, \dots, t_s)=t$$

where

$s\geqq 0$, $t'\in \hat{T}_G(Z_s)$, $b_1, \dots, b_s\in A$, $t_1, \dots, t_s\in T_G(A_i\times\{\lambda\})$, $b(z_j, f)\leftarrow t'(b_1z_{j_1}, \dots, b_sz_{j_s})\in P$.

Then, by the definition of $\mathbf{C}$, $b(z_i, \langle f; \mathbf{e}, \mathbf{e}^1, \dots, \mathbf{e}^k\rangle)\leftarrow t'(b_1z_{i_1}, \dots, b_sz_{i_s})\in P''$ where

$$i_r = \begin{cases} j_r(=0) & \text{if } \ b_r\in A_i \qquad\qquad (r = 1, \dots, s) \\ l_1+\dots+l_{j_r-1}+v & \text{if } \ b_r\in S_v^{j_r} \ \text{ for some } \ v(=1, \dots, l_{j_r}). \end{cases}$$

Now let $r(=1, \dots, s)$ be such an index for which $b_r\in S_v^{j_r}$ and so $1\leqq j_r\leqq k$. Then the relation $(b_r, j_r) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t_r$ can be written in the form $(b_r, j_r) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t_r'((c_1, j_r), \dots$ $\dots, (c_u, j_r)) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t_r'(\bar{t}_1, \dots, \bar{t}_u)=t_r$ for some $u(\geqq 0)$, $t_r'(\in \hat{T}_G(Z_u))$, $c_1, \dots, c_u(\in A_i)$, $\bar{t}_1, \dots$ $\dots, \bar{t}_u(\in T_G(A_i\times\{\lambda\})$ and we can suppose that the derivation $(b_r, j_r) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} t_r'((c_1, j_r), \dots$ $\dots, (c_u, j_r))$ has no such a step, in which, a leaf in $A_i\times\{j_r\}$ substituted. Then $(b_r, \lambda) \overset{*}{\underset{p_{j_r},\mathbf{A}}{\Leftarrow}} t_r'((c_1, \lambda), \dots, (c_u, \lambda))$ so, by the induction hypothesis concerning $\mathrm{dp}(p)$, we have $(b_r, \lambda) \overset{*}{\underset{\bar{q}_{j_r},\mathbf{C}}{\Leftarrow}} t_r'((c_1, \lambda), \dots, (c_u, \lambda))$ which means that $(b_r, i_r) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} t_r'((c_1, i_r), \dots$ $\dots, (c_u, i_r))$ because $\mathrm{lb}_{\bar{q}}(i_r)=\bar{q}_{j_r}$. On the other hand, by Lemma 5, $c_1, \dots, c_u\in I_1^{j_r}\cup\dots$ $\dots\cup I_v^{j_r}$, moreover, the length of each of the derivations $(c_1, j_r) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} \bar{t}_1, \dots$ $\dots, (c_u, j_r) \overset{*}{\underset{p,\mathbf{A}}{\Leftarrow}} \bar{t}_u$ is less than $\mathrm{lt}(\beta)$ so we have $(c_1, i_r) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} \bar{t}_1, \dots, (c_u, i_r) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} \bar{t}_u$, that is $(b_r, i_r) \overset{*}{\underset{\bar{q},\mathbf{C}}{\Leftarrow}} t_r'(\bar{t}_1, \dots, \bar{t}_u)=t_r$.

If r is such an index for which $b_r\in A_i$ and so $j_r=0$ then $t_r=(b_r, \lambda)$, therefore

$(b_r, \bar{i}_r) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} t_r$ again. All that means that

$$(b, i) \underset{\bar{q}, \mathbf{C}}{\Leftarrow} t'((b_1, \bar{i}_1), \ldots, (b_s, \bar{i}_s)) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} t'(t_1, \ldots, t_s) = t$$

. proving our statement.

Now we return to the induction step of the lemma. The relation $(a, \lambda) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q$ can be written in the form

$$(a, \lambda) \underset{p, \mathbf{A}}{\Leftarrow} q'((a_1, \bar{j}_1), \ldots, (a_s, \bar{j}_s)) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q'(q_1, \ldots, q_s) = q$$

where $s \geqq 0$, $q' \in \hat{T}_G(Z_s)$, $a_1, \ldots, a_s \in A$, $q_1, \ldots, q_s \in T_G(A_i \times \{\lambda\})$ and $af(z_1, \ldots, z_k) \leftarrow$ $\leftarrow q'(a_1 z_{j_1}, \ldots, a_s z_{j_s})$ is in P. Then, by the definition of $\mathbf{C}$, the rule $a\langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle$ $(z_1, \ldots, z_m) \leftarrow q(a_1 z_{i_1}, \ldots, a_s z_{i_s})$ is in P'' where $m = l_1 + \ldots + l_k$ and

$$i_r = \begin{cases} j_r (= 0) & \text{if} \quad a_r \in A_i \qquad\qquad (r = 1, \ldots, s) \\ l_1 + \ldots + l_{j_r - 1} + n & \text{if} \quad a_r \in S_n^{j_r} \quad \text{for some} \quad n (= 1, \ldots, l_{j_r}). \end{cases}$$

Let $r (= 1, \ldots, s)$ be an index for which $a_r \in S_n^{j_r}$ for some $n (= 1, \ldots, l_{j_r})$ and so $1 \leqq j_r \leqq k$. Then the relation $(a_r, \bar{j}_r) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q_r$ can be written in the form $(a_r, j_r) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q_r'((b_1, j_r), \ldots, (b_u, j_r)) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q_r'(\bar{q}_1, \ldots, \bar{q}_u) = q_r$ for some $u \geqq 0$, $q_r' \in \hat{T}_G(Z_u)$, $b_1, \ldots, b_u \in A_i$, $\bar{q}_1, \ldots, \bar{q}_u \in T_G(A_i \times \{\lambda\})$. We can again suppose, that there is no step in the derivation $(a_r, j_r) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q_r'((b_1, j_r), \ldots, (b_u, j_r))$, in which, a leaf in $A_i \times \{j_r\}$ is substituted. Therefore $(a_r, \lambda) \overset{*}{\underset{p_{j_r}, \mathbf{A}}{\Leftarrow}} q_r'((b_1, \lambda), \ldots, (b_u, \lambda))$ from which, by Lemma 5, $b_1, \ldots, b_u \in I_1^{j_r} \cup \ldots \cup I_n^{j_r}$ and, by the induction hypothesis on $\mathrm{dp}(p)$, we get $(a_r, \lambda) \overset{*}{\underset{\bar{q}_{j_r}, \mathbf{C}}{\Leftarrow}} q_r'((b_1, \lambda), \ldots, (b_u, \lambda))$ that is $(a_r, i_r) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} q_r'((b_1, i_r), \ldots, (b_u, i_r))$. On the other hand, by the statement, we have $(b_1, i_r) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} \bar{q}_1, \ldots, (b_u, i_r) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} \bar{q}_u$ which means that $(a_r, i_r) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} q_r'(\bar{q}_1, \ldots, \bar{q}_u) = q_r$.

If $r (= 1, \ldots, s)$ is such an index for which $a_r \in A_i$ and so $j_r = 0$ then it is clear that $q_r = (a_r, \lambda)$, therefore $(a_r, \bar{i}_r) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} q_r$ again. The two cases of r and $a\langle f; \mathbf{e}, \mathbf{e}^1, \ldots, \mathbf{e}^k\rangle (z_1, \ldots, z_m) \leftarrow q'(a_1 z_{i_1}, \ldots, a_s z_{i_s}) \in P''$ together prove that

$$(a, \lambda) \underset{\bar{q}, \mathbf{C}}{\Leftarrow} q'((a_1, \bar{i}_1), \ldots, (a_s, \bar{i}_s)) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} q'(q_1, \ldots, q_s) = q.$$

This ends the proof of Lemma 6.

The proof of the next lemma is essentially the converse of the previous one.

Lemma 7. Let $a \in A_s$, $p \in T_F$, $\bar{q} \in T_F$, $q \in T_G(A_i \times \{\lambda\})$, $\mathbf{e} \in B$. Suppose that $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$ and $(a, \lambda) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} q$. Then $(a, \lambda) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q$.

Now we are ready to prove our theorem. Suppose that $(p, q) \in \tau_{\mathbf{A}}$ that is $(a_0, \lambda) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q$ concerning rt. Because $\mathbf{A}$ is K-visit, by Lemma 4, there exist $\bar{q} \in T_F$

and $\mathbf{e} \in B$ with $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$, therefore, by Lemma 6, $(a_0, \lambda) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} q$ concerning rt″, hence $(p, q) \in \tau_{\mathbf{B}} \circ \tau_{\mathbf{C}}$. Conversely, by $(p, q) \in \tau_{\mathbf{B}} \circ \tau_{\mathbf{C}}$ we have a $\bar{q} \in T_F$ for which $p \overset{*}{\underset{\mathbf{B}}{\Rightarrow}} \mathbf{e}\bar{q}$ under some $\mathbf{e}(\in B)$ and $(a_0, \lambda) \overset{*}{\underset{\bar{q}, \mathbf{C}}{\Leftarrow}} q$ concerning rt″. Then, by Lemma 7, we have $(a_0, \lambda) \overset{*}{\underset{p, \mathbf{A}}{\Leftarrow}} q$ concerning rt. The fact, that the inclusion is strict follows from the proof of Theorem 4.1 of [6]. This ends the proof of Theorem 3.

After studying the proof of the previous theorem two observation can be made. On the one hand, instead of the bottom-up tree transducer **B** we can have a top-down one which can be constructed by reversing the rewriting rules of **B**. Although this top-down one does not induce the same tree transformation as **B**, the following will be valid.

Corollary 8. $\mathcal{DA}_{PK} \subset \mathcal{T} \circ \mathcal{DA}_1$.

On the other hand it also seems that if **A** is simple K-visit then a deterministic bottom-up tree transducer can be constructed, so we have

Corollary 9. $\mathcal{DA}_{SK} \subset \mathcal{DB} \circ \mathcal{DA}_1$.

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(Received Oct. 26, 1982)