On a representation of deterministic uniform root-to-frontier tree transformations

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The concepts of products and complete systems of finite automata can be generalized for ascending algebras in a natural way (see [4]). Results in finite automata theory imply that for most types of products there are no finite complete systems of ascending algebras. Therefore, it is reasonable to investigate a weaker form of completeness to be called m-completeness when tree transformations are represented up to a finite but not bounded height. In this paper we give necessary and sufficient conditions under which a system of ascending algebras is m-complete for the class of all deterministic uniform root-to-frontier tree transformations with respect to different kinds of products. Moreover, we show the existence of such finite m-complete systems.

1. Notions and notations

The terms "node of a tree" and "subtree at a given node of a tree" will be used in an informal and obvious way.

The symbol R will stand for a nonvoid finite rank type with $0 \notin R$.

By a *path* of rank type R we mean a word over $U(R) = \bigcup(\{(m, 1), \ldots, (m, m)\} | m \in R)$. The set of all paths with rank type R will be denoted by $\mathrm{pt}(R)$.

Take a ranked alphabet Σ of rank type R, a tree $p \in F_\Sigma(X_n)$ and a path $u \in \mathrm{pt}(R)$. Then the *realization* $u(p)$ of u in p (if it exists) is defined in the following way:

1. if $u=e$ then $u(p)=e$ and u *ends* in p at the root of p,
2. if $u=u_1(m, i)$, $u_1(p)$ exists, u_1 ends in p at the node d of p labelled by σ and $\sigma \in \Sigma_m$ then $u(p)=u_1(p)(\sigma, i)$ and u ends in p at the i^{th} descendent of d.

For $U \subseteq \mathrm{pt}(R)$ and $T \subseteq F_\Sigma(X_n)$ $(n \geqq 1)$ let $U(T) = \{u(p) | u \in U, p \in T\}$. One can easily see, that for arbitrary $n \geqq 1$, $\mathrm{pt}(R)(F_\Sigma(X_n)) = U(\Sigma)^*$, where $U(\Sigma) = \bigcup(\{(\sigma, 1), \ldots, (\sigma, m)\} | \sigma \in \Sigma_m, m > 0)$.

Let Σ be an operator domain with $\Sigma_0 = \emptyset$. A (deterministic) *ascending Σ-algebra* $\mathscr{A}$ is a pair consisting of a nonempty set A and a mapping that assigns to every operator $\sigma \in \Sigma$ an m-ary ascending operation $\sigma^{\mathcal{A}}: A \to A^m$, where m is the arity of σ. The mapping $\sigma \to \sigma^{\mathcal{A}}$ will not be mentioned explicitely, but we write $\mathcal{A} = (A, \Sigma)$. If Σ is not specified then we speak about an ascending algebra. The ascending Σ-algebra $\mathcal{A}$ is finite if both A and Σ are finite. Moreover, $\mathcal{A}$ has *rank type* R if Σ is of rank type R. The class of all finite ascending Σ-algebras of rank type R will be denoted by $K(R)$. If there is no danger of confusion then we omit $\mathcal{A}$ in $\sigma^{\mathcal{A}}$.

In this paper by an algebra we mean a finite deterministic ascending algebra.

A *(deterministic) root-to-frontier* ΣX_n*-recognizer* or a $(D)R\Sigma X_n$-recognizer, for short, is a system $\mathbf{A} = (\mathcal{A}, a_0, X_n, \mathbf{a})$, where

(1) $\mathcal{A} = (A, \Sigma)$ is a finite Σ-algebra,
(2) $a_0 \in A$ is the *initial state*,
(3) $\mathbf{a} = (A^{(1)}, \ldots, A^{(n)}) \in P(A)^n$ is the *final-state vector*.

Next we recall the concept of a tree transducer.

A *root-to-frontier tree transducer (R-transducer)* is a system $\mathfrak{A} = (\Sigma, X_n, A, \Omega, Y_m, A', P)$, where

(1) Σ and Ω are ranked alphabets,
(2) X_n and Y_m are the *frontier alphabets*,
(3) A is a ranked alphabet consisting of unary operators, the *state set* of $\mathfrak{A}$. (It is assumed that A is disjoint with all other sets in the definition of $\mathfrak{A}$, except A'.)
(4) $A' \subseteq A$ is the set of *initial states*,
(5) P is a finite set of *productions* of the following two types:
(i) $ax_i \to q$ $(a \in A, x_i \in X_n, q \in F_\Omega(Y_m))$,
(ii) $a\sigma \to q$ $(a \in A, \sigma \in \Sigma_l, l \geqq 0, q \in F_\Omega(Y_m \cup A\Xi_l))$. ($\Xi = \{\xi_1, \xi_2, \ldots\}$ is the set of auxiliary variables.)

The *transformation* induced by $\mathfrak{A}$ will be denoted by $\tau_{\mathfrak{A}}$.

The R-transducer $\mathfrak{A}$ is *deterministic* if $A' = \{a_0\}$ is a singleton and there are no distinct productions in P with the same left side. Moreover, the R-transducer $\mathfrak{A}$ is *uniform* if each production $a\sigma \to q$ $(a \in A, \sigma \in \Sigma_l, l \geqq 0, q \in F_\Omega(Y_m \cup A\Xi_l))$ can be written in the form $a\sigma \to \bar{q}(a_1\xi_1, \ldots, a_l\xi_l)$ for some $\bar{q} \in F_\Omega(Y_m \cup \Xi_l)$. In this paper by a transducer we shall mean a deterministic uniform R-transducer. One can easily see that for every transducer $\mathfrak{A} = (\Sigma, X_n, A, \Omega, Y_m, a_0, P)$ there exists a transducer $\mathfrak{B} = (\Sigma, X_n, B, \Omega', Y_m, b_0, P')$ such that (i) for arbitrary $b \in B$ and $\sigma \in \Sigma_m$ with $m > 0$ there is exactly one production in P' with left side $b\sigma$, and (ii) $\tau_{\mathfrak{B}} = \tau_{\mathfrak{A}}$. In the sequel we shall confine ourselves to transducers having property (i) and $\Sigma_0 = \emptyset$.

To a transducer $\mathfrak{A} = (\Sigma, X_n, A, \Omega, Y_m, a_0, P)$ we can correspond an $R\Sigma X_n$-recognizer $\mathbf{A} = (\mathcal{A}, a_0, X_n, \mathbf{a})$ with $\mathcal{A} = (A, \Sigma)$ and $\mathbf{a} = (A^{(1)}, \ldots, A^{(n)})$, where

(1) for arbitrary $l > 0$, $\sigma \in \Sigma_l$, $a \in A$ and $(a_1, \ldots, a_l) \in A^l$ if $(a_1, \ldots, a_l) = \sigma^{\mathcal{A}}(a)$ then $a\sigma \to q(a_1\xi_1, \ldots, a_l\xi_l) \in P$ for some $q \in F_\Omega(Y_m \cup \Xi_l)$,

(2) $a \in A^{(i)}$ $(1 \leqq i \leqq n)$ if and only if $ax_i \to q \in P$ for some $q \in F_\Omega(Y_m)$.

The class of all recognizers obtained from $\mathfrak{A}$ in the above way will be denoted by $\mathrm{rec}(\mathfrak{A})$.

Now take an $R\Sigma X_n$-recognizer $\mathbf{A}=(\mathcal{A}, a_0, X_n, \mathbf{a})$ with $\mathcal{A}=(A, \Sigma)$ and $\mathbf{a}=(A^{(1)}, \ldots, A^{(n)})$. Define a transducer $\mathfrak{A}=(\Sigma, X_n, A, \Omega, Y_m, a_0, P)$ by

$$P = \{ax_i \to q^{(a,i)} | a\in A^{(i)},\ q^{(a,i)}\in F_\Omega(Y_m),\ i = 1, \ldots, n\} \cup$$
$$\cup \{a\sigma \to q^{(a,\sigma)}(a_1\xi_1, \ldots, a_l\xi_l) | a\in A,\ \sigma\in\Sigma_l,\ l > 0,$$
$$(a_1, \ldots, a_l) = \sigma^{\mathcal{A}}(a),\ q^{(a,\sigma)}\in F_\Omega(Y_m\cup\Xi_l)\},$$

where the ranked alphabet Ω, the integer m and the trees on the right sides of the productions in P are fixed arbitrarily. Denote by $\mathrm{tr}(\mathbf{A})$ the class of all transducers obtained from $\mathbf{A}$ in the above way. Obviously, for arbitrary transducer $\mathfrak{A}$ and $\mathbf{A}\in\mathrm{rec}(\mathfrak{A})$ the inclusion $\mathfrak{A}\in\mathrm{tr}(\mathbf{A})$ holds. Therefore, we have

Statement 1. For every transducer $\mathfrak{A}$ there exists a recognizer $\mathbf{A}$ such that $\mathfrak{A}\in\mathrm{tr}(\mathbf{A})$.

Next we recall the concept of a product of ascending algebras (see [4]).

Let $\Sigma, \Sigma^1, \ldots, \Sigma^k$ be ranked alphabets of rank type R, and consider the Σ^i-algebras $\mathcal{A}_i=(A_i, \Sigma^i)$ $(i=1, \ldots, k)$. Furthermore, let

$$\psi = \{\psi_m\colon A_1\times\ldots\times A_k\times\Sigma_m \to \Sigma^1_m\times\ldots\times\Sigma^k_m | m\in R\}$$

be a family of mappings. Then by the *product* of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ with respect to ψ we mean the Σ-algebra $\psi(\mathcal{A}_1, \ldots, \mathcal{A}_k, \Sigma)=\mathcal{A}=(A, \Sigma)$ with $A=A_1\times\ldots\times A_k$ and for arbitrary $m\in R$, $\sigma\in\Sigma_m$ and $\mathbf{a}\in A$

$$\sigma^{\mathcal{A}}(\mathbf{a}) = \big((\mathrm{pr}_1(\sigma_1^{\mathcal{A}_1}(\mathrm{pr}_1(\mathbf{a}))), \ldots, \mathrm{pr}_1(\sigma_k^{\mathcal{A}_k}(\mathrm{pr}_k(\mathbf{a})))), \ldots$$
$$\ldots, (\mathrm{pr}_m(\sigma_1^{\mathcal{A}_1}(\mathrm{pr}_1(\mathbf{a}))), \ldots, \mathrm{pr}_m(\sigma_k^{\mathcal{A}_k}(\mathrm{pr}_k(\mathbf{a}))))\big),$$

where $(\sigma_1, \ldots, \sigma_k)=\psi_m(\mathbf{a}, \sigma)$ and $\mathrm{pr}_i(\mathbf{a})$ $(1\leqq i\leqq k)$ denotes the i^{th} component of $\mathbf{a}$.

To define special types of products let us write ψ_m in the form $\psi_m=(\psi_m^{(1)}, \ldots, \psi_m^{(k)})$ where for arbitrary $\mathbf{a}\in A$ and $\sigma\in\Sigma_m$, $\psi_m(\mathbf{a}, \sigma)=(\psi_m^{(1)}(\mathbf{a}, \sigma), \ldots, \psi_m^{(k)}(\mathbf{a}, \sigma))$. We say that $\mathcal{A}$ is an α_i*-product* $(i=0, 1, \ldots)$ if for arbitrary j $(1\leqq j\leqq k)$ and $m\in R$, $\psi_m^{(j)}$ is independent of its u^{th} component if $i+j\leqq u\leqq k$. If $\Sigma^1=\ldots=\Sigma^k=\Sigma$ and $\psi_m(\mathbf{a}, \sigma)=(\sigma, \ldots, \sigma)$ for arbitrary $m\in R$, $\sigma\in\Sigma_m$ and $\mathbf{a}\in A$ then $\mathcal{A}$ is the *direct product* of $\mathcal{A}_1, \ldots, \mathcal{A}_k$. In the case of an α_i-product in $\psi_m^{(j)}$ we shall indicate only those variables on which $\psi_m^{(j)}$ may depend.

One can see easily that the formation of the product, α_0-product and direct product is associative. (This is not true for the α_i-product with $i>0$.)

Let $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, a_0, P)$ and $\mathfrak{B}=(\Sigma, X_u, B, \Omega, Y_v, b_0, P')$ be two transducers and $m\geqq 0$ an integer. We write $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$ if $\tau_{\mathfrak{A}}(p)=\tau_{\mathfrak{B}}(p)$ for every $p\in F_\Sigma^m(X_u)$, where $F_\Sigma^m(X_u)$ denotes the set of all trees from $F_\Sigma(X_u)$ with height less than or equal to m.

Take a class K of algebras of rank type R. We say that K is *metrically complete* (*m-complete,* for short) with respect to the product (α_i-product) if for arbitrary transducer $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, a_0, P)$ and integer $m\geqq 0$ there exist a product (α_i-product) $\mathcal{B}=(B, \Sigma)$ of algebras from K, an element $b_0\in B$ and a vector $\mathbf{b}\in P(B)^u$ such that $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$ for some $\mathfrak{B}\in\mathrm{tr}(\mathbf{B})$, where $\mathbf{B}=(\mathcal{B}, b_0, X_u, \mathbf{b})$.

Let $\mathcal{A}=(A, \Sigma)$ be an arbitrary algebra from $K(R)$. We correspond to $\mathcal{A}$ a semiautomaton $s(\mathcal{A})=(I_{\mathcal{A}}, A, \delta_{\mathcal{A}})$, where $I_{\mathcal{A}}=U(\Sigma)$ and for arbitrary $a\in A$ and $(\sigma, i)\in I_{\mathcal{A}}$, $\delta_{\mathcal{A}}(a, (\sigma, i))=\mathrm{pr}_i(\sigma^{\mathcal{A}}(\mathbf{a}))$.

Take a Σ-algebra $\mathscr{A}=(A, \Sigma)\in K(R)$, an element $a\in A$ and an integer $m\geqq 0$. We say that the system $(\mathscr{A}, a)$ is *m-free* if the initial semiautomaton $s(\mathscr{A}, a)=(I_{\mathscr{A}}, A, a, \delta_{\mathscr{A}})$ is m-free. (For the definition of m-free semiautomata, see [1]. In [1] initial semiautomata are called initial automata. Moreover, here it is not supposed that $s(\mathscr{A}, a)$ is connected.)

For the system $(\mathscr{A}, a)$ and integer $m\geqq 0$ set $A_a^{(m)}=\{\delta_{\mathscr{A}}(a, p)|p\in I_{\mathscr{A}}^*, |p|\leqq m\}$, where $|p|$ denotes the length of p. Moreover, $\delta_{\mathscr{A}}(a, e)=a$ and $\delta_{\mathscr{A}}(a, p(\sigma, i))=\delta_{\mathscr{A}}(\delta_{\mathscr{A}}(a, p), (\sigma, i))$ $(p\in I_{\mathscr{A}}^*, (\sigma, i)\in I_{\mathscr{A}})$.

Let $(\mathscr{A}, a)$ and $(\mathscr{B}, b)$ be two systems with $\mathscr{A}=(A, \Sigma)$, $\mathscr{B}=(B, \Sigma)\in K(R)$. A mapping φ of $A_a^{(m)}$ onto $B_b^{(m)}$ is an *m-homomorphism* of $(\mathscr{A}, a)$ onto $(\mathscr{B}, b)$ if it satisfies the following conditions:

(1) $\varphi(a)=b$,
(2) $\varphi(\sigma^{\mathscr{A}}(a'))=\sigma^{\mathscr{B}}(\varphi(a'))$ $(a'\in A_a^{(m-1)}, \sigma\in\Sigma_l, l>0)$.

If the above φ is also one-to-one then we speak about an *m-isomorphism* and say that $(\mathscr{A}, a)$ and $(\mathscr{B}, b)$ are *m-isomorphic*. In notation, $(\mathscr{A}, a)\overset{m}{\simeq}(\mathscr{B}, b)$.

One can easily prove the following statements.

Statement 2. Let $\mathscr{A}=(A, \Sigma)$, $\mathscr{B}=(B, \Sigma)\in K(R)$ and $a\in A$, $b\in B$ be arbitrary. For an integer $m\geqq 0$, $(\mathscr{B}, b)$ is an m-homomorphic image of $(\mathscr{A}, a)$ if and only if $s(\mathscr{B}, b)$ is an m-homomorphic image of $s(\mathscr{A}, a)$.

Statement 3. Let $(\mathscr{A}, a)$ and $(\mathscr{B}, b)$ be the systems of Statement 2. For arbitrary $m\geqq 0$,

(1) if $(\mathscr{A}, a)$ is m-free then $(\mathscr{B}, b)$ is an m-homomorphic image of $(\mathscr{A}, a)$,
(2) if $(\mathscr{A}, a)$ is m-free and m-isomorphic to $(\mathscr{B}, b)$ then $(\mathscr{B}, b)$ is also m-free, and
(3) if both $(\mathscr{A}, a)$ and $(\mathscr{B}, b)$ are m-free then they are m-isomorphic.

The next statement is also obvious.

Statement 4. Take two systems $(\mathscr{A}, a)$ and $(\mathscr{B}, b)$ $(\mathscr{A}=(A, \Sigma)$, $\mathscr{B}=(B, \Sigma)\in K(R)$, $a\in A, b\in B)$. Moreover, let $m\geqq 0$ be an integer. If $(\mathscr{B}, b)$ is an m-homomorphic image of $(\mathscr{A}, a)$ then for arbitrary $u\geqq 0$, $\mathbf{b}\in P(B)^u$, $\mathbf{B}=(\mathscr{B}, b, X_u, \mathbf{b})$ and $\mathfrak{B}=(\Sigma, X_u, B, \Omega, Y_v, b, P')\in\operatorname{tr}(\mathbf{B})$ there exist an $\mathbf{a}\in P(A)^u$, an $\mathbf{A}=(\mathscr{A}, a, X_u, \mathbf{a})$ and an $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, a, P)\in\operatorname{tr}(\mathbf{A})$ such that $\tau_{\mathfrak{B}}\overset{m}{=}\tau_{\mathfrak{A}}$.

Let $(\mathscr{A}, a)$ be a system with $\mathscr{A}=(A, \Sigma)\in K(R)$ and $a\in A$ an element. We say that for an integer $m\geqq 0$ the algebra $\mathscr{B}=(B, \Sigma)$ *m-isomorphically represents* $(\mathscr{A}, a)$ if there exists a $b\in B$ such that $(\mathscr{A}, a)\overset{m}{\simeq}(\mathscr{B}, b)$.

The α_i-product and the α_j-product $(i, j\geqq 0)$ will be called *metrically equivalent (m-equivalent)* provided that a system of algebras is m-complete with respect to the α_i-product if and only if it is m-complete with respect to the α_j-product. The m-equivalence between an α_i-product and the product is defined similarly.

Finally, we shall suppose that every finite index set $I=\{i_1, \ldots, i_k\}$ is given together with a (fixed) ordering of its elements. Furthermore, for arbitrary system $\{a_{i_j}|i_j\in I\}$, $(a_{i_j}|i_j\in I)$ is the vector $(a_{i_1}, a_{i_2}, \ldots, a_{i_k})$ if $i_1<i_2<\ldots<i_k$ is the ordering of I.

For terminology not defined here, see [2] and [3].

2. Metrically complete systems

In this section we give necessary and sufficient conditions for a system of ascending algebras to be m-complete with respect to the α_i-products ($i=0, 1, \ldots$) and the product. We shall see that the α_i-products are m-equivalent to each other and they are m-equivalent to the product.

We start with

Theorem 1. A system $K \subseteq K(R)$ is m-complete with respect to the product (α_i-product) if and only if for every $m \geqq 0$ each m-free system $(\mathscr{A}, a)$ with $\mathscr{A} \in K(R)$ can be represented m-isomorphically by a product (α_i-product) of algebras from K.

Proof. The sufficiency is obvious by Statements 3 and 4.

To prove the necessity take an arbitrary m-free system $(\mathscr{A}, a_0)$ with $\mathscr{A} = (A, \Sigma) \in K(R)$. Consider the transducer $\mathfrak{A} = (\Sigma, X_n, A, \Omega, A \times X_n, a_0, P)$, where $n > 1$ is an arbitrary natural number, $\Omega_l = A \times \Sigma_l$ ($l > 0$) and P consists of the following productions:

(1) $a x_i \to (a, x_i)$ $(a \in A,\ x_i \in X_n)$,
(2) $a\sigma \to (a, \sigma)(a_1 \xi_1, \ldots, a_l \xi_l)$ $(a \in A,\ \sigma \in \Sigma,\ l > 0,\ \sigma^{\mathscr{A}}(a) = (a_1, \ldots, a_l))$.

Let $\mathscr{B} = (B, \Sigma)$ be a product (α_i-product) of algebras from K such that for a $\mathfrak{B} = (\Sigma, X_n, B, \Omega, A \times X_n, b_0, P') \in \mathrm{tr}(\mathbf{B})$ we have $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$, where $\mathbf{B} = (\mathscr{B}, b_0, X_n, \mathbf{b})$ $(b_0 \in B,\ \mathbf{b} \in P(B)^n)$. We show that $(\mathscr{B}, b_0)$ is m-free. This, by Statement 3, will imply that $(\mathscr{A}, a_0) \overset{m}{\cong} (\mathscr{B}, b_0)$.

First of all observe that $\mathfrak{A}$ is a totally defined, linear, nondeleting transducer inducing a one-to-one transformation. Moreover, in a tree $\tau_{\mathfrak{A}}(p)$ with $h(p) \leqq m$ no subtree occurs more than once. Therefore, by $\tau_{\mathfrak{A}} \overset{m}{=} \tau_{\mathfrak{B}}$, all productions occurring in a derivation $b_0 p \Rightarrow^* q$ $(p \in F_\Sigma(X_n),\ q \in F_\Omega(X_n \times A))$ with $h(p) \leqq m$ are linear and nondeleting. Thus, we have the following relation between derivations in $\mathfrak{A}$ and $\mathfrak{B}$. Let $u \in \mathrm{pt}(R)$ be a path with $|u| \leqq m$. Take a tree $p \in F_\Sigma(X_n)$ with $h(p) \leqq m$, and assume that $u(p)$ is defined, it ends in p at the node d, p' is the subtree of p at d, $\bar{p}(\xi_1)$ is obtained from p by replacing the occurrence of p' at d by ξ_1, $\delta_{\mathscr{A}}(a_0, u(p)) = a$ and $\delta_{\mathscr{B}}(b_0, u(p)) = b$. Then the following derivations are valid:

$$a_0 p = a_0 \bar{p}(p') \Rightarrow^*_{\mathfrak{A}} q_1(ap') \Rightarrow^*_{\mathfrak{A}} q_1(q') = q$$

and

$$b_0 p = b_0 \bar{p}(p') \Rightarrow^*_{\mathfrak{B}} q_2(bp') \Rightarrow^*_{\mathfrak{B}} q_2(q'') = q,$$

where $a_0 \bar{p}(\xi_1) \Rightarrow^*_{\mathfrak{A}} q_1(a\xi_1)$, $b_0 \bar{p}(\xi_1) \Rightarrow^*_{\mathfrak{B}} q_2(b\xi_1)$ $(q_1, q_2 \in F_\Omega(A \times X_n \cup \xi_1))$ and $ap' \Rightarrow^*_{\mathfrak{A}} q'$, $bp' \Rightarrow^*_{\mathfrak{B}} q''$ $(q', q'' \in F_\Omega(A \times X_n))$. (Observe that ξ_1 occurs exactly once in q_1 and q_2.) Furthermore, if $v_1 \in \mathrm{pt}(R)$ is the path such that $v_1(q_1)$ ends in q_1 at the node labelled by ξ_1 and $v_2 \in \mathrm{pt}(R)$ is the path for which $v_2(q_2)$ ends in q_2 at the node labelled by ξ_1 then $v_2(q_2)$ is a subword of $v_1(q_1)$.

Now assume that $(\mathscr{B}, b_0)$ is not m-free, that is there are two distinct words $u, v \in I^*_{\mathscr{B}}$ $(= I^*_{\mathscr{A}})$ such that $|u|, |v| \leqq m$ and $\delta_{\mathscr{B}}(b_0, u) = \delta_{\mathscr{B}}(b_0, v) = b$. Let $\bar{u}, \bar{v} \in \mathrm{pt}(R)$ be paths and $p_1, p_2 \in F_\Sigma(X_n)$ trees such that $\bar{u}(p_1) = u$, $\bar{v}(p_2) = v$, $h(p_1)$, $h(p_2) \leqq m$, u ends in p_1 at the node d_1 and v ends in p_2 at the node d_2. Replace in p_1 and p_2 the subtrees at d_1 resp. d_2 by x_1, and denote by $\bar{p}_1$ resp. $\bar{p}_2$ the resulting trees. Moreover, let $\delta_{\mathscr{A}}(a_0, u)=a_1$ and $\delta_{\mathscr{A}}(a_0, v)=a_2$. (Note that $a_1 \neq a_2$ since $u \neq v$ and $(\mathscr{A}, a_0)$ is m-free.) Then, by the choice of $\mathfrak{A}$, if $q_1, q_2 \in F_\Omega(A \times X_n)$ are obtained by the derivations $a_0\bar{p}_1 \Rightarrow^*_{\mathfrak{A}} q_1$ and $a_0\bar{p}_2 \Rightarrow^*_{\mathfrak{A}} q_2$ then $\bar{u}(q_1)$ ends in q_1 at a node labelled by (a_1, x_1) and $\bar{v}(q_2)$ ends in q_2 at a node labelled by (a_2, x_1). Moreover, by $\tau_{\mathfrak{B}} \overset{m}{=} \tau_{\mathfrak{A}}$, $b_0\bar{p}_1 \Rightarrow^*_{\mathfrak{B}} q_1$ and $b_0\bar{p}_2 \Rightarrow^*_{\mathfrak{B}} q_2$ hold also. From this, taking into consideration our observation concerning the relation between derivations in $\mathfrak{A}$ and $\mathfrak{B}$, we get that at the ends of $\bar{u}(q_1)$ and $\bar{v}(q_2)$ the same label should occur which is a contradiction.

The next theorem gives necessary conditions for a system of ascending algebras to be m-complete with respect to the product.

Theorem 2. Let $K \subseteq K(R)$ be a system which is m-complete with respect to the product. Then the following conditions are satisfied:

(i) for arbitrary integer $m \geqq 0$, path $\bar{u} \in \mathrm{pt}(R)$ with $|\bar{u}|=m$, rank $l \in R$ and natural number $1 \leqq i \leqq l$ there exist an $\mathscr{A}=(A, \Sigma') \in K$, an $a_0 \in A$, $\sigma_1, \sigma_2 \in \Sigma'_l$ and a $u \in \bar{u}(F_{\Sigma'}(X_1))$ such that $\delta_{\mathscr{A}}(a_0, u(\sigma_1, i)) \neq \delta_{\mathscr{A}}(a_0, u(\sigma_2, i))$,

(ii) for arbitrary integer $m \geqq 0$, path $\bar{u} \in \mathrm{pt}(R)$ with $|\bar{u}|=m$, rank $l \in R$ $(l>1)$ and integers $1 \leqq i < j \leqq l$ there exist an $\mathscr{A}=(A, \Sigma) \in K$, an $a_0 \in A$, a $\sigma \in \Sigma_l$ and a $u \in \bar{u}(F_\Sigma(X_1))$ such that $\delta_{\mathscr{A}}(a_0, u(\sigma, i)) \neq \delta_{\mathscr{A}}(a_0, u(\sigma, j))$.

Proof. We start with the necessity of (i). Assume that there are $m \geqq 0$, $\bar{u} \in \mathrm{pt}(R)$ with $|\bar{u}|=m$, $l \in R$ and $1 \leqq i \leqq l$ such that for arbitrary $\mathscr{A}=(A, \Sigma') \in K$, $a_0 \in A$, $\sigma_1, \sigma_2 \in \Sigma'_l$ and $u \in \bar{u}(F_{\Sigma'}(X_1))$ the equation $\delta_{\mathscr{A}}(a_0, u(\sigma_1, i))=\delta_{\mathscr{A}}(a_0, u(\sigma_2, i))$ holds. Take a ranked alphabet Σ of rank type R such that Σ_l contains two distinct elements σ and σ'. Moreover, consider a product $\mathscr{B}=(B, \Sigma)=\psi(\mathscr{A}_1, \ldots, \mathscr{A}_k, \Sigma)$ $(\mathscr{A}_i=(A_i, \Sigma^i) \in K, i=1, \ldots, k)$ and an element $\mathbf{b}_0 \in B$. We show that the system $(\mathscr{B}, \mathbf{b}_0)$ is not $(m+1)$-free.

First of all let us introduce a notation. Consider the above product $\mathscr{B}$ and define the mappings $\psi^i\colon B \times F_\Sigma(X_n) \to F_{\Sigma^i}(X_n)$ $(i=1, \ldots, k;\ n \geqq 0)$ in the following way: for arbitrary $\mathbf{b} \in B$ and $p \in F_\Sigma(X_n)$

(1) if $p=x_j$ $(1 \leqq j \leqq n)$ then $\psi^i(\mathbf{b}, p)=x_j$,

(2) if $p=\sigma(p_1, \ldots, p_l)$ then $\psi^i(\mathbf{b}, p)=\sigma_i(\psi^i(\mathbf{b}_1, p_1), \ldots, \psi^i(\mathbf{b}_l, p_l))$, where $(\sigma_1, \ldots, \sigma_k)=\psi_l(\mathbf{b}, \sigma)$ and $(\mathbf{b}_1, \ldots, \mathbf{b}_l)=\sigma^{\mathscr{B}}(\mathbf{b})$.

One can see easily that for arbitrary $\mathbf{b} \in B$, $p \in F_\Sigma(X_n)$ and $\bar{u} \in \mathrm{pt}(R)$ the equation $\delta_{\mathscr{B}}(\mathbf{b}, \bar{u}(p))=(\delta_{\mathscr{A}_1}(\mathrm{pr}_1(\mathbf{b}), \bar{u}(\psi^1(\mathbf{b}, p))), \ldots, \delta_{\mathscr{A}_k}(\mathrm{pr}_k(\mathbf{b}), \bar{u}(\psi^k(\mathbf{b}, p))))$ holds.

Now take two trees $p, q \in F_\Sigma(X_1)$ such that $(\bar{u}(l, i))(p)=u(\sigma, i)$ and $(\bar{u}(l, i))(q)=u(\sigma', i)$. For every $j(=1, \ldots, k)$ let $(\bar{u}(l, i))(\psi^j(\mathbf{b}_0, p))=u_j(\sigma^{(j)}, i)$ and $(\bar{u}(l, i))(\psi^j(\mathbf{b}_0, q))=v_j(\bar{\sigma}^{(j)}, i)$. By the definition of the product, the equations $u_j=v_j$ $(j=1, \ldots, k)$ obviously hold. Moreover,

$$\delta_{\mathscr{B}}(\mathbf{b}_0, u(\sigma, i)) = (\delta_{\mathscr{A}_1}(\mathrm{pr}_1(\mathbf{b}_0), u_1(\sigma^{(1)}, i)), \ldots, \delta_{\mathscr{A}_k}(\mathrm{pr}_k(\mathbf{b}_0), u_k(\sigma^{(k)}, i)))$$

and

$$\delta_{\mathscr{B}}(\mathbf{b}_0, u(\sigma', i)) = (\delta_{\mathscr{A}_1}(\mathrm{pr}_1(\mathbf{b}_0), u_1(\bar{\sigma}^{(1)}, i)), \ldots, \delta_{\mathscr{A}_k}(\mathrm{pr}_k(\mathbf{b}_0), u_k(\bar{\sigma}^{(k)}, i))).$$

But, by our assumptions, $\delta_{\mathscr{A}_j}(\mathrm{pr}_j(\mathbf{b}_0), u_j(\sigma^{(j)}, i))=\delta_{\mathscr{A}_j}(\mathrm{pr}_j(\mathbf{b}_0), u_j(\bar{\sigma}^{(j)}, i))$ for every j $(1 \leqq j \leqq k)$, i.e., $\delta_{\mathscr{B}}(\mathbf{b}_0, u(\sigma, i))=\delta_{\mathscr{B}}(\mathbf{b}_0, u(\sigma', i))$. Therefore, $(\mathscr{B}, \mathbf{b}_0)$ is not $(m+1)$-free which, by Theorem 1, implies that K is not m-complete with respect to the product.

The necessity of (ii) can be shown in a similar way.

Theorem 3. If a system $K \subseteq K(R)$ satisfies the conclusions of Theorem 2 then K is m-complete with respect to the α_0-product.

Proof. Let Σ be a fixed ranked alphabet of rank type R. We shall show by induction on m that for every integer $m \geqq 0$ there are an α_0-product $\mathscr{B}=(B, \Sigma)$ of algebras from K and an element $\mathbf{b} \in B$ such that $(\mathscr{B}, \mathbf{b})$ is m-free. This, by Theorem 1, will end the proof of Theorem 3.

If $m=0$ then our claim is obviously valid. Let us suppose that our statement has been proved for an $m \geqq 0$, and take a product $\mathscr{A}=(A, \Sigma)$ of algebras from K and an element $a \in A$ such that $(\mathscr{A}, a)$ is m-free. By our assumption, for every $\bar{u}=\bar{u}_1(l, i)$ $(\bar{u}_1 \in \mathrm{pt}(R), l \in R, 1 \leqq i \leqq l)$ there are an $\mathscr{A}^{(\bar{u})}=(A^{(\bar{u})}, \Sigma^{(\bar{u})}) \in K$, an $a^{(\bar{u})} \in A^{(\bar{u})}$, two operators $\sigma_1, \sigma_2 \in \Sigma_l^{(\bar{u})}$ and a $u_1 \in \bar{u}_1(F_\Sigma(X_1))$ such that $\delta_{\mathscr{A}^{(\bar{u})}}(a^{(\bar{u})}, u_1(\sigma_1, i)) \neq \delta_{\mathscr{A}^{(\bar{u})}}(a^{(\bar{u})}, u_1(\sigma_2, i))$. Moreover, for arbitrary $\bar{u}=\bar{u}_1(l, i)$, $\bar{v}=\bar{u}_1(l, j)$ $(\bar{u}_1 \in \mathrm{pt}(R)$, $l \in R$, $l>1$, $1 \leqq i<j \leqq l)$ there are an $\mathscr{A}^{(\bar{u}, \bar{v})}=(A^{(\bar{u}, \bar{v})}, \Sigma^{(\bar{u}, \bar{v})})$, an $a^{(\bar{u}, \bar{v})} \in A^{(\bar{u}, \bar{v})}$, a $u_1 \in \bar{u}_1(F_\Sigma(X_1))$ and a $\bar{\sigma} \in \Sigma_l^{(\bar{u}, \bar{v})}$ such that $\delta_{\mathscr{A}^{(\bar{u}, \bar{v})}}(a^{(\bar{u}, \bar{v})}, u_1(\bar{\sigma}, i)) \neq \delta_{\mathscr{A}^{(\bar{u}, \bar{v})}}(a^{(\bar{u}, \bar{v})}, u_1(\bar{\sigma}, j))$. Consider an index set I consisting of all pairs (u, v) where $u, v \in U(\Sigma)^*$, $u \neq v$, $|u|=m+1$ and $|v| \leqq m+1$. For the pair (u, v) with $u=u'(\sigma, i) \in \bar{u}(F_\Sigma(X_1))$ and $v=v'(\sigma^*, j)$ if $u' \neq v'$ or $\sigma \neq \sigma^*$ take the α_0-product $\mathscr{A}^{(u, v)}=\psi^{(u, v)}(\mathscr{A}, \mathscr{A}^{(\bar{u})}, \Sigma)=(A^{(u, v)}, \Sigma)$, where $\psi^{(u, v)}$ is defined in the following way. For every $s \in R$, $\psi_s^{(u, v)(1)}$ is the identity mapping on Σ_s. If $w=w_1(\sigma', j)$ $(\sigma' \in \Sigma_k)$ is a proper subword of u' and $w'=w_1'(\sigma'', j)$ is the subword of u_1 with $|w'|=|w|$ then let

$$\psi_k^{(u, v)(2)}(\delta_{\mathscr{A}}(a, w_1), \sigma')=\sigma''.$$

In all other cases, except $\psi_l^{(u, v)(2)}(\delta_{\mathscr{A}}(a, u'), \sigma)$, $\psi_s^{(u, v)(2)}$ $(s \in R)$ is given arbitrarily in accordance with the definition of the α_0-product. Since $u' \neq v'$ or $\sigma \neq \sigma^*$ and $(\mathscr{A}, a)$ is m-free $\delta_{\mathscr{A}^{(u, v)}}((a, a^{(\bar{u})}), v)$ is defined. Now let

$$\psi_l^{(u, v)(2)}(\delta_{\mathscr{A}}(a, u'), \sigma)=\begin{cases}\sigma_1 & \text{if } \delta_{\mathscr{A}^{(u, v)}}((a, a^{(\bar{u})}), v)=(a_1, a_2) \\ & \text{and } \delta_{\mathscr{A}^{(\bar{u})}}(a^{(\bar{u})}, u_1(\sigma_1, i)) \neq a_2 \\ \sigma_2 & \text{otherwise.}\end{cases}$$

Obviously, $(\mathscr{A}^{(u, v)}, a^{(u, v)})$ with $a^{(u, v)}=(a, a^{(\bar{u})})$ is m-free and $\delta_{\mathscr{A}^{(u, v)}}(a^{(u, v)}, u) \neq \delta_{\mathscr{A}^{(u, v)}}(a^{(u, v)}, v)$.

Now assume that $u'=v'$ and $\sigma=\sigma^*$; that is $u=u'(\sigma, i) \in \bar{u}(F_\Sigma(X_1))$ and $v=u'(\sigma, j) \in \bar{v}(F_\Sigma(X_1))$. Take the α_0-product $\mathscr{A}^{(u, v)}=\psi^{(u, v)}(\mathscr{A}, \mathscr{A}^{(\bar{u}, \bar{v})}, \Sigma)=(A^{(u, v)}, \Sigma)$, where $\psi^{(u, v)}$ is given as follows. Again for every $s \in R$, $\psi_s^{(u, v)(1)}$ is the identity mapping on Σ_s. If $w=w_1(\sigma', t)$ $(\sigma' \in \Sigma_k)$ is a proper subword of u' and $w'=w_1'(\sigma'', t)$ is the subword of u_1 with $|w'|=|w|$ then let $\psi_k^{(u, v)(2)}(\delta_{\mathscr{A}}(a, w_1), \sigma')=\sigma''$. Moreover, $\psi_l^{(u, v)(2)}(\delta_{\mathscr{A}}(a, u'), \sigma)=\bar{\sigma}$. In any other cases $\psi_s^{(u, v)(2)}$ $(s \in R)$ is given arbitrarily in accordance with the definition of the α_0-product. Since $(\mathscr{A}, a)$ is m-free $\mathscr{A}^{(u, v)}$ is well defined. Again, $(\mathscr{A}^{(u, v)}, a^{(u, v)})$ with $a^{(u, v)}=(a, a^{(\bar{u}, \bar{v})})$ is m-free and $\delta_{\mathscr{A}^{(u, v)}}(a^{(u, v)}, u) \neq \delta_{\mathscr{A}^{(u, v)}}(a^{(u, v)}, v)$.

Finally, take the direct product $\mathscr{B}=(B, \Sigma)=\Pi(\mathscr{A}^{(u, v)} \mid (u, v) \in I)$ and the vector $\mathbf{b}=(a^{(u, v)} \mid (u, v) \in I)$. Then $(\mathscr{B}, \mathbf{b})$ is $(m+1)$-free. Indeed, for two different words $u, v \in U(\Sigma)^*$ if $|u|, |v|<m+1$ then $\delta_{\mathscr{B}}(\mathbf{b}, u) \neq \delta_{\mathscr{B}}(\mathbf{b}, v)$ since they differ in all of their components, and if $|u|=m+1$ and $|v| \leqq m+1$ then $\delta_{\mathscr{B}}(\mathbf{b}, u)$ and $\delta_{\mathscr{B}}(\mathbf{b}, v)$

are different at least in their $(u, v)^{\text{th}}$ components. Since the direct product is a special α_0-product and the formation of the α_0-product is associative $\mathcal{B}$ is an α_0-product of algebras from K.

From Theorems 2 and 3 we get

Corollary 4. For arbitrary $i, j \geqq 0$ the α_i-product and the α_j-product are m-equivalent to each other and they are m-equivalent to the product.

We now give an algorithm to decide for a finite $K \subseteq K(R)$ whether K is m-complete with respect to the product.

Take an algebra $\mathcal{A}=(A, \Sigma) \in K$. For arbitrary $l \in R$ and $1 \leqq i \leqq l$ set $A^{(l,i)} = \{a \in A \mid \mathrm{pr}_i(\sigma_1^{\mathcal{A}}(a)) \neq \mathrm{pr}_i(\sigma_2^{\mathcal{A}}(a))$ for some $\sigma_1, \sigma_2 \in \Sigma_l\}$. Moreover, for every $a \in A$ let $L_a^{(l,i)}$ be the language recognized by the automaton $\mathcal{A}_a^{(l,i)} = (I_{\mathcal{A}}, A, a, \delta_{\mathcal{A}}, A^{(l,i)})$. Furthermore, let $L_{\mathcal{A}}^{(l,i)} = \bigcup (L_a^{(l,i)} \mid a \in A)$ and $L^{(l,i)} = \bigcup (L_{\mathcal{A}}^{(l,i)} \mid \mathcal{A} \in K)$. For arbitrary $l \in R$ $(l>1)$ and $1 \leqq i < j \leqq l$ define $L^{(l,i,j)}$ in a similar way with $A^{(l,i,j)} = \{a \in A \mid \mathrm{pr}_i(\sigma^{\mathcal{A}}(a)) \neq \mathrm{pr}_j(\sigma^{\mathcal{A}}(a))$ for some $\sigma \in \Sigma_l\}$ instead of $A^{(l,i)}$. Finally, denote by $\bar{\Sigma}$ the union of all ranked alphabets belonging to algebras from K, and take the language homomorphism $\varphi\colon U(\bar{\Sigma})^* \to U(R)^*$ given by $\varphi(\sigma, i) = (k, i)$ $(\sigma \in \bar{\Sigma}, r(\sigma) = k)$, where $r(\sigma)$ denotes the rank of σ. Then, by Theorems 2 and 3, K is m-complete with respect to the product if and only if

(1) for arbitrary $l \in R$ and $1 \leqq i \leqq l$, $\varphi(L^{(l,i)}) = U(R)^*$,
(2) for arbitrary $l \in R$ $(l>1)$ and $1 \leqq i < j \leqq l$, $\varphi(L^{(l,i,j)}) = U(R)^*$.

The validity of these equations is decidable effectively.

Finally, for a given rank type R we give a one-element system which is m-complete with respect to the product. Let Σ be a ranked alphabet of rank type R such that for every $l \in R$, $\Sigma_l = \{\sigma_1^{(l)}, \sigma_2^{(l)}\}$. Assume that the greatest natural number in R is n. Take the Σ-algebra $\mathcal{A} = (A, \Sigma)$, where $A = \{a_0, \ldots, a_n\}$, $\sigma_1^{(l)}(a_i) = (a_{i+1 (\mathrm{mod}\, n+1)}, \ldots, a_{i+1 (\mathrm{mod}\, n+1)})$ $(l \in R, i = 0, 1, \ldots, n)$, $\sigma_2^{(l)}(a_n) = (a_n, a_{n-1}, \ldots, a_{n-l+1})$ $(l \in R)$ and for arbitrary $l \in R$ and a_i with $i \neq n$, $\sigma_2^{(l)}(a_i)$ is defined arbitrarily. $(i+1 \,(\mathrm{mod}\, n+1)$ denotes the least residue of $i+1$ modulo $n+1.)$ One can see easily that the system $K = \{\mathcal{A}\}$ satisfies the conclusions of Theorem 2.

References

[1] GÉCSEG, F. and I. PEÁK, *Algebraic theory of automata*, Akadémiai Kiadó, Budapest, 1972.
[2] GÉCSEG, F. and M. STEINBY, Minimal ascending tree automata, *Acta Cybernet.*, v. 4, 1978, pp. 37—44.
[3] GÉCSEG, F. and M. STEINBY, *Tree automata*, Akadémiai Kiadó, Budapest, to appear.
[4] VIRÁGH, J., Deterministic ascending tree automata II, *Acta Cybernet.*, to appear.

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