A Theory of Finite Functions, Part I.
On finite trees associated to certain finite functions

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1. Introduction

1.1. Let A be a set of cardinality l, $l\in\omega$, $l\geqq 2$. For $n, m\in\omega$ we set $O_A^{(n,m)}=\{f|f: A^n\to A^m\}$ and $O_A^{(m)}=\bigcup_{n\in\omega} O_A^{(n,m)}$. Certain subsets of $O_A^{(m)}$, in particular, of $O_A^{(1)}$, are interesting for the very different mathematical theories of algebra, logic and computer science. For example, the celebrated result of I. Rosenberg picks up some subsets when enumerating maximal closed classes of $O_A^{(1)}$ [9]. Several special types of functions such as monotone (unate) and symmetric ones play a role in the theory of logic design [10], and in other applications of finite functions (cf. e.g. Dedekind's problem on freely generated lattices generated by finitely many generators). In the common part of logic and computer science, e.g. in the theory of theorem-proving and of semantics for programming languages, certain restrictions to logical formulae with prescribed forms seem to help in increasing efficiency [8].

1.2. One possible method for investigation the properties of these subsets is to associate special finite algebras (or more precisely finite graphs and trees) to the elements of $O_A^{(m)}$. There is a very common way of doing this: let the "parse tree" be associated to each function. By this correspondence several remarkable results have been established. The parse tree, however, mirrors mostly the syntactical features of the function at hand and very little can be learnt about the "semantics" of the mapping by the parse tree only. Here we suggest another tree-representation of finite functions — the valuation tree — and show the use by examples. Valuation trees are compressed forms of valuation tables (generalized truth tables) of functions (for $l=2$, see [5]). It should be mentioned that a more compact representation in graph forms can also be given, cf. [1] for $l=2$. Trees, however, seem to be more tractable in spite of or thanks to their redundancies. Clearly, valuation trees are completely semantically oriented and designed to contain all information about the action of a function.

1.3. The natural question arises what kinds of trees are associated to certain interesting subsets of $O_A^{(m)}$. Our main contribution in this first part of a series of papers is to present a uniform graphical property, the level-homogenity, to answer this question. As an illustration we apply the method for three well-known pre-primal subsets of $O_A^{(1)}$. In this paper we do not assume any algebraic structure on A except the ordering relation. From the second part, however, we shall endow some more operations to A, in fact, we suppose that A is a Post-algebra of order l and apply the results obtained in Part I to this case. Actually, we shall develop some optimization techniques for synthesizing Post formulae. Later parts are devoted partly to complexity questions where several estimates are established concerning the methods of Part I and II, and partly to different problems concerning finite functions.

1.4. The organization of this paper is as follows. In Section 2, we overview the notations used in this series of papers. In Section 3 we deal with trees and introduce several notions and notations concerning them. Some notions of this section will be used only in later parts, but is presented here for the sake of uniformity. Key notion of these considerations, the level-homogeneous tree, will be introduced in Section 4. This section deals with some auxiliary concepts, too. Finite functions enter in Section 5 where, after a general representation theorem, we investigate degenerate, order-preserving, value-preserving and permutation-preserving functions in terms of trees.

We note that this paper is selfcontained, i.e. no preliminary knowledge is assumed.

2. Preliminaires

2.1. Let ω be the set of finite ordinals, $\emptyset$ is the empty set. If $m\in\omega$, then we make use of the following notations: $\{m\}=\{0, 1, \dots, m-1\}$, $[m]=\{1, 2, \dots, m\}$, $[0]=\emptyset$, $[\omega]=\{1, 2, \dots\}$. We shall fix $2\leqq l<\omega$; $n, m\in\omega$ and the set A of cardinality l. Since A is finite, it can be identified with $\{l\}$. We shall usually use this identification. From now on, the letters l, m, n, A will always refer to these fixed sets. Let $<$ be the well-known total ordering on $\{l\}$ (and thus on A). We extend $<$ to the elements of $\{l\}^n$ (hence to A^n) componentwise. The elements of the set $O_A^{(n,m)}=\{f|f\colon A^n\to A^m\}$ will be called n-ary A-functions with m output. We make this concept independent of arity by setting $O_A^{(m)}=\bigcup_{n\in\omega} O_A^{(n,m)}$. If $f\in O_A^{(m)}$, then

$$f=(f_1, \dots, f_m) \quad \text{where} \quad f_i\in O_A^{(1)} \quad \text{for all} \quad i\in[m]; \quad \text{i.e.} \quad O_A^{(m)}=(O_A^{(1)})^m. \tag{1}$$

If $g\notin O_A^{(m)}$ and g is a function (a meta-function) of n arguments, then $e_1e_2 \dots e_n g$ will denote the application of g to the arguments $e_1, e_2, \dots, e_n$. This is to be distinguished from any application of a function $f\in O_A^{(m)}$ which will be displayed as $fe_1 \dots e_n$.

Let $f\in O_A^{(n,1)}$. By $f(x_i/\alpha)$ we mean a function in $O_A^{(n-1,1)}$ which is obtained from f by substituting α for each occurrence of x_i provided x_i occurs in f, otherwise let $f(x_i/\alpha)=f$, (and hence in $O_A^{(n,1)}$). $f^*(x_1/\alpha_1, \dots, x_n/\alpha_n)$ denotes the value of f under substituting its variables $x_1, \dots, x_n$ by $\alpha_1, \dots, \alpha_n$ in due course. In Part II we shall give a more detailed method for computing this value (by assuming that A is a Post algebra).

If $f \in O_A^{(n,m)}$, then we always assume that an ordering of variables occurring in f, say $x_1, x_2, \ldots, x_n$, is fixed. This convention will be essential from Section 5.

Let $f \in O_A^{(n,m)}$. Then, for every $n_1 > n$, f can be considered as a function of n_1 variables, i.e. $f \in O_A^{(n_1,m)}$ (cf. subsection 5.2).

The cardinality of a set H is denoted by card H. $\mathcal{P}H$ is the powerset of H. If f is a function defined on H and $H' \subset H$, then $f \restriction H'$ is the restriction of f onto H'. Range f and Dom f denote the range and domain of the function f, respectively.

We shall omit all indices without any remark unless confusion can occur. In this paper $O_A^{(n,m)}$ will be denoted by $O_l^{(n,m)}$ to emphasize that no algebraic operations are present on A. All considerations apply for arbitrary $m > 0$, however, for the sake of simplicity, we often give definitions and assertions in the case $m = 1$, only. If generalization for larger m is not straightforward then we shall explicitly discuss it.

3. Trees

3.1. Let V be an arbitrary set and $\varrho: V \to \{l+1\}$. The pair (V, ϱ) is called an l-ary *pretree* (ranked set). We set $E_{V,\varrho} = \{(v, i) \mid v \in V \wedge i \in [v\varrho]\}$. The function ϱ is the *rank function* of the pregraph; $v\varrho$ is the *rank* of v in (V, ϱ) provided $v \in V$. $E_{V,\varrho}$ is the set of *edges*.

The triplet $T = ((V, \varrho), \sigma, (\varepsilon_1, \ldots, \varepsilon_m))$ is an *m-rooted l-ary tree* if and only if (V, ϱ) is an l-ary pretree; $\sigma: E \to V$; $\varepsilon_1, \ldots, \varepsilon_m \in V$ and the following (Peano-like) conditions are satisfied:

(i) σ is a bijection.

(ii) Range $\sigma \cap \{\varepsilon_i \mid i \in [m]\} = \emptyset$.

(iii) If $V' \subset V$ is such that $\{\varepsilon_i \mid i \in [m]\} \subset V'$ and $[V']_\sigma \subset V$, where $[V']_\sigma$ denotes the closure of V' under σ, then $V' = V$.

The elements of V are called *points* of T; the point ε_i $(i \in [m])$ is the i-th *root* and σ is the *successor function* of T.

Note, that $m = 0$ implies $V = \emptyset$. We shall use the name *leaf* for an element of $0\varrho^{-1}$ (of a given tree), where $a\varrho^{-1}$ denotes the total inverse of ϱ on a. Clearly, card $V \in [\omega]$ entails $0\varrho^{-1} \neq \emptyset$. From now on, we always assume that card $V \in [\omega]$ and $m \neq 0$.

We remark, that m-rooted trees are usually defined in a different way (cf. [2]). The definition presented here is originated from C. C. Elgot et al. and is proved equivalent to the more common one used in the literature in [6].

3.2. We define the *immediate successors* vD_T^1 and the successors vD_T of v in T as follows:

$$vD_T^1 = \{v' \mid v' \in V \wedge \exists i (i \in [v\varrho] \wedge v' = (v, i)\sigma)\} \tag{2}$$

and

$$vD_T = \{v' \mid v' \in V \wedge (\exists n \in \omega, \exists f\colon [n+1] \to V)(1f = v \wedge (n+1)f = v' \wedge \wedge (\forall j \in [n])((j+1)f \in (jf)D_T^1))\}. \tag{3}$$

In particular, $v \in vD_T$, i.e. $\varepsilon_i D_T \neq \emptyset$ provided $i \in [m]$, and for all $v \in V$, there exists a unique $i \in [m]$ such that $v \in \varepsilon_i D_T$. The following assertion is immediate by definitions.

Lemma 1. *Let T be a tree and $\varepsilon_i \in V$, $i \in [m]$, furthermore assume that $v \in \hat{\varepsilon}_i D_T$. Then, there exist exactly one n $(n \in \omega)$ and exactly one f such that*

$$1f = \varepsilon_i \wedge (n+1)f = v \wedge (\forall j \in [n])((j+1)f \in (jf)D_T^1) \tag{4}$$

holds.

If the conditions of Lemma 1 are fulfilled for v, then n and f, determined uniquely by (4) and the remark preceeding the assertion, are called the *level of* v and the *derivation function of* v, respectively. We shall use the notations, $v\lambda_T$ for the level of v and vd_T for the sequence $(1f, 2f, \ldots, (n+1)f)$, the *derivation* of v in T. By a *path* we mean a derivation of a leaf $v \in 0\varrho^{-1}$. We shall denote the set of all paths of T by P_T. Clearly, card $P_T =$ card $0\varrho^{-1}$. For each $p = (1f, 2f, \ldots$ $\ldots, (n+1)f) \in P_T$, there exist a unique i and a sequence $(k_1, \ldots, k_n)$ such that $1f = \varepsilon_i$ and for all $j \in [n]$, $(jf, k_j)\sigma = (j+1)f$, hence we can use the pair $(i, (k_1, \ldots, k_n))$ to identify paths. Note, that the set of paths in T completely determines T, thus P_T and T can be identified and is actually done at several points of this paper.

3.3. We define $T\lambda$, the *level of* T, as follows:

$$T\lambda = n \Leftrightarrow (\forall v \in 0\varrho^{-1})((v\lambda_T \leqq n) \wedge (\exists v \in 0\varrho^{-1})(v\lambda_T = n))$$

i.e., $T\lambda$ is the least element of ω such that every leaf of T has level less than or equal to n. The tree T is exactly of level n if and only if

$$(\forall v \in 0\varrho^{-1})(v\lambda_T = n).$$

3.4. The m-rooted l-ary tree T exactly of level n is *complete* if and only if $(\forall v \in V)(v\varrho = l)$. It follows that in an m-rooted l-ary complete tree T,

$$\text{card } P_T = m \cdot l^n.$$

The following observation is trivial but very useful.

Lemma 2. *Let T_1 and T_2 be arbitrary m-rooted l-ary complete trees of level n. Then T_1 and T_2 are isomorphic.*

Let T be an m-rooted l-ary tree of level n and let $h \in [n]$. We say that T is *complete on level h* if and only if $(\forall v \in V_T)(v\lambda = h \Rightarrow v\varrho = l)$.

3.5. Let T_1 and T_2 be two m-rooted l-ary trees exactly of level n. We say T_1 is a *subtree* of T_2 if and only if $P_{T_1} \subset P_{T_2}$ and for all $p = (1f, 2f, \ldots, (n+1)f) \in P_{T_1}$, if for some $i \in [m]$, $(n+1)f \in \varepsilon_i D_{T_2}$, then $(n+1)f \in \varepsilon_i D_{T_1}$. Note, that if T_1 is a subtree of T_2, then it may well happen that T_1 is not a subalgebra of T_2, and vica versa. If there is a subtree T' in T_2 such that T' is isomorphic to T_1, then we say T_1 is *embeddable* in T_2. Obviously, every m-rooted l-ary tree exactly of level n is embeddable in an (m-rooted l-ary) complete tree. The embedding is, up to isomorphism, unique by definition and Lemma 2.

3.6. Let T be an m-rooted l-ary tree exactly of level n and let $P \subset P_T$. P defines, in the natural way, an m-rooted l-ary tree exactly of level n which is a subtree of T, *the subtree of T determined by P.* This subtree is unique and we denote it by T_P.

Let T be an m-rooted l-ary tree exactly of level n. Let $p=(i,(k_1,\ldots,k_n))\in P_T$, $q=(j,(h_1,\ldots,h_n))\in P_T$. We let $p\sim q\Leftrightarrow(\forall s\in[n])(k_s=h_s)$. Clearly, $\sim$ is an equivalence relation. Set $\tilde{p}=\{q\,|\,q\in P_T\wedge p\sim q\}$ and $\tilde{P}_T=\{\tilde{p}\,|\,p\in P_T\}$. The 1-rooted l-ary tree exactly of level n determined by $\tilde{P}_T$ is named the *compressed form of* T and is denoted by T^c. Let $P\subset P_T$, then the subtree T^c_P of T^c determined by P is called the *compressed-subtree of* T *determined by* P. Note, however, that this name is a somewhat misleading: T^c_P is not a subtree of T in the very sense of 3.5.

3.7. Let $T=((V,\varrho),\sigma,(\varepsilon_1,\ldots,\varepsilon_m))$ be an m-rooted l-ary tree of level n, $v\in 0\varrho^{-1}$. Let us suppose that $v\lambda_T=h$, $h<n$. Let V_1 be a set of new points with cardinality $\sum_{i\in[n-h]} l^i$. The tree $T^E_v=((V\cup V_1,\varrho'),\sigma',(\varepsilon_1,\ldots,\varepsilon_m))$ is defined as follows: $\varrho'\restriction V=\varrho$, and for all $w\in V_1$, $w\varrho'=l$; $\sigma'\restriction V=\sigma$ and σ' is extended to V_1 in such a way that T^E_v is a tree ($f\restriction H$ denotes the restriction of the function of f to the set H). Roughly speaking, the tree T^E_v is obtained from T by identifying the root of a 1-rooted l-ary complete tree of level $n-h$ to v. Let $\{v_1,\ldots,v_s\}\subset 0\varrho^{-1}$ be that set of leaves, the level of which is strictly less than n. Let $T_0=T$ and for every $r\in[s]$, $T_r=(T_{r-1})^E_{v_r}$. Then, T_s is unique up to isomorphism and is called *the extended form of* T, in notation T^E. Clearly, T^E is an m-rooted l-ary tree exactly of level n.

3.8. Let T be a complete m-rooted l-ary tree of level n and define the *index function* $\delta\colon P_T\to\{ml^n\}$ by the formula

$$p\delta=(i-1)l^n+\sum_{j\in[n]}k_j\cdot l^{n-j} \tag{5}$$

where p is determined by the pair $(i,(k_1,\ldots,k_n))$. Clearly, δ is a bijection, hence for each $k\in\{m\cdot l^n\}$, there exists $p\in P_T$ such that $p\delta=k$. If p is determined by the pair $(i,(k_1,\ldots,k_n))$ then we shall make use of the following notations $p=k\delta^{-1}$, $(i,(k_1,\ldots,k_n))\Delta=k$, $k\Delta^{-1}=(i,(k_1,k_2,\ldots,k_n))$. We use also the *compressed index function* $\delta^c\colon P_T\to\{l^n\}$ defined by

$$p\delta^c=\sum_{j\in[n]}k_jl^{n-j}. \tag{6}$$

If T is not complete but is exactly of level n, then $\delta=\delta'\restriction P_T$, where δ' is the index function defined on the complete tree in which T is embeddable. It is obvious, that P_T determines a unique subset of $\{m\cdot l^n\}$; the notations introduced above apply in the natural way. If T is of level n but is not exactly of level n, then we extend δ as follows: $\delta^E\colon P_T\to\mathcal{P}\{ml^n\}$; for $p=(1f,2f,\ldots,(h+1)f)\in P_T$, let $p\delta^E=\{p'\delta\,|\,p'\in P_{T^E}\wedge p'=(s_1,s_2,\ldots,s_{n+1})$ such that for all $j\in[h+1],s_j=jf\}$. It follows, that if $h=n$, then $p\delta^E=p\delta$.

δ^E, the *extended index function,* is well defined since δ is a bijection. It follows that δ^E is injective as well and thus we can employ the natural generalizations of $(\delta^E)^{-1}$, Δ^{-1}, Δ to those k which are in the range of δ^E.

3.9. Let T be an m-rooted l-ary tree. The pair (T,τ) is called a *terminated* (m-rooted, l-ary) tree if and only if $\tau\colon P_T\to\{l\}$.

3.10. Let us define the following function $\xi^{(l)}\colon\omega\to\omega^{l-1}$; for $k\in\omega$ let $k\xi^{(l)}=(\xi_1^{(l)},\xi_2^{(l)},\ldots,\xi_{l-1}^{(l)})$ where $\xi_i^{(l)}$, $i\in[l-1]$ is the number of occurrences of i in the l-ary expansion of k. Let T be an m-rooted l-ary tree exactly of level n. The pair (T, ξ) is called a *ξ-augmented* (m-rooted, l-ary) *tree* if and only if $\xi\colon P_T \to \omega^{l-1}$ is defined by $p\xi = (p\delta^c)\xi^{(l)}$. The following assertion can be proved by an easy induction.

Lemma 3. *Let (T, ξ) be a ξ-augmented m-rooted l-ary tree, $p = (1f, \ldots, (n+1)f) \in P_T$ and $p\xi = (\xi_1^{(l)}, \ldots, \xi_{l-1}^{(l)})$. Then, for all $s \in [l-1]$,* card $\{(jf, s) \mid j \in [n] \wedge (jf, s)\sigma = (j+1)f\} = \xi_s^{(l)}$. In other words, if $p = (i, (k_1, \ldots, k_n))$, then $\xi_s^{(l)}$ gives the number of k_j such that $k_j = s$.

4. Homogeneous trees

4.1. Let T be a 1-rooted l-ary tree exactly of level n; $j \in [n]$. T is called *λ-homogeneous* (to shorten the term level-homogeneous) *on level* j if and only if

$$(\forall v_1, v_2 \in V,\ \forall h, k \in [l])\big((v_1\lambda = v_2\lambda = j \wedge v_1 \neq v_2 \wedge (v_1, h) \in E) \Rightarrow \Rightarrow ((v_2, k) \in E \Leftrightarrow k = h)\big). \tag{7}$$

An equivalent formalization of (7) is the following

$$(\forall v_1, v_2 \in V,\ \forall k \in [l])\big((v_1\lambda = v_2\lambda = j \wedge v_1 \neq v_2) \Rightarrow \Rightarrow ((v_1, k) \in E \Leftrightarrow (v_2, k) \in E)\big). \tag{8}$$

T is *λ-homogeneous* if and only if, for all $j \in [n]$, T is λ-homogeneous on level j. Clearly, any path $p \in P_T$ considered as a tree, any complete tree and any tree exactly of level 1 is λ-homogeneous.

Let r be a binary relation on $\{l\}$ and T a 1-rooted l-ary tree exactly of level n. We can extend r to paths of T by defining $(p, q) \in \bar{r} \Leftrightarrow$ for all $j \in [n]$, $(p_j, q_j) \in r$, where $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n) \in P_T$.

The following assertion, although it is trivial, gives some insight into the very nature of λ-homogeneous trees.

Lemma 4. *Let r be an arbitrary binary relation on $\{l\}$, let T be a 1-rooted l-ary tree exactly of level n, and let $\bar{r}$ denote the extension of r to P_T defined as above. Then for every $p \in P_T$, the set $\{p' \mid p' \in P_T \wedge (p, p') \in \bar{r}\}$ uniquely determines a λ-homogeneous subtree of T.*

Proof. It follows that $\{p' \mid p' \in P_T \wedge (p, p') \in \bar{r}\}$ defines a unique subtree of T; let $T_{r;p}$ denote this subtree, and let $p = (k_1, \ldots, k_n)$. Let us suppose, that $v_1, v_2 \in V_{T_{r,p}}$ such that $v_1 \neq v_2$ and $v_1\lambda = v_2\lambda = h$ for some $h \in [n]$. Then, $v_1\varrho = v_2\varrho$ and for all $j \in [v_1\varrho]$, $(v_1, j) \in E \Leftrightarrow (j, k_h) \in r \Leftrightarrow (v_2, j) \in E$. But then $((v_1, j) \in E \Rightarrow (v_2, j) \in E) \Leftrightarrow (j, k_h) \in r$, hence $T_{r,p}$ is λ-homogeneous on level h. Being h arbitrary, we have that $T_{r,p}$ is λ-homogeneous.

If r is nonempty and total (i.e. $\forall x \exists y((x, y) \in r)$), then $T_{r,p}$ is not empty. We also note, that the converse of the lemma is not true; more precisely, if T_1 is a λ-homogeneous subtree of T, then it may well happen that there is no binary relation r on $\{l\}$ such that $T_1 = T_{r,p}$ for an appropriate $p \in P_T$.

In particular, if r is a partial ordering or is a non-trivial equivalence or $r = \{(x, \pi x) \mid x \in \{l\}\}$ where π is a permutation of $\{l\}$ with l/q cycles of the same prime length q, then $T_{r,p}$ is λ-homogeneous by Lemma 4. All of these relations are total so if $r \neq \emptyset$, then $T_{r,p} \neq \emptyset$ for any $p \in P_T$. This observation establishes some links between λ-homogeneous trees and (three) types of maximal closed classes exhibited by Rosenberg's completeness theorem. The main interest of this paper is, however, to use λ-homogeneous subtrees of a tree to portrait some elementary properties of the function to which the tree at hand is associated by Theorem 13 below, hence we do not provide similar results for the other (three) types of maximal closed classes. Instead, we study further λ-homogeneous trees. The following lemmata are immediate.

Lemma 5. *Let T be a 1-rooted l-ary tree exactly of level n. If T_1 is a λ-homogeneous subtree of T, then there exists a maximal λ-homogeneous subtree T_2 of T containing T_1; i.e. $P_{T_1} \subset P_{T_2} \subset P_T$ and T_2 is not a subtree of any λ-homogeneous subtree of T containing T_1 other than T_2.*

Note, that T_2 is not unique in general.

Lemma 6. *Let T be a 1-rooted l-ary tree exactly of level n, let r be a nonempty reflexive binary relation on $\{l\}$. Then, for every $p \in P_T$, the tree $T_{r,p}$ is the unique maximal λ-homogeneous subtree of T which contains p.*

Proof. Since r is reflexive, $p \in P_{T_{r,p}}$. λ-homogenity and uniqueness follow from Lemma 4. It remains to prove that $T_{r,p}$ is maximal. It is, however, trivial by definition since if for some $p' \in P_T$, $(p, p') \in \bar{r}$ then $p' \in P_{T_{r,p}}$, hence no λ-homogeneous subtree of T exists which contains p and $T_{r,p}$ properly.

4.2. Let T be a terminated m-rooted l-ary tree and let $t \subset \{l\}$. T is said to be *τ-homogeneous with respect to* (in short w.r.t.) t if and only if $(\forall p \in P_T)(p\tau \in t)$. T is called *quasi τ-homogeneous w.r.t.* t if and only if

$$(\exists p \in P_T)\big(p\tau \notin t \wedge (\forall p' \in P_T)(p'\tau \notin t \Rightarrow p = p')\big).$$

In particular, if $t \in \{l\}$, then T is τ-homogeneous w.r.t. t if and only if $(\forall p \in P_T)$ $(p\tau = t)$ and T is quasi τ-homogeneous w.r.t. t if and only if for all but one p in P_T, $p\tau = t$.

Let T be a terminated m-rooted l-ary tree and let r be a partial ordering on $\{l\}$; $\bar{r}$ is the expansion of r to P_T. T is *τ-increasing w.r.t.* r if and only if

$$(\forall p, p')\big((p, p') \in \bar{r} \Rightarrow (p\tau, p'\tau) \in r\big).$$

Lemma 7. *Let T be a 1-rooted l-ary terminated tree exactly of level n. Let T_1 be a λ-homogeneous subtree of T which is τ-homogeneous w.r.t. some $t \subset \{l\}$. Then there exists a maximal λ-homogeneous subtree of T which contains T_1 and is τ-homogeneous w.r.t. t.*

Lemma 8. *Let T be a 1-rooted l-ary terminated tree exactly of level n; let r be a partial ordering on $\{l\}$. Then, for every $p \in P_T$, there exists a maximal λ-homogeneous subtree T_1 of T such that*

(i) $p \in P_{T_1} \subset P_{T_{r,p}} \subset P_T$,

(ii) *T_1 is τ-increasing w.r.t. r.*

Lemma 9. *Let T be a 1-rooted terminated l-ary tree exactly of level n; let r be a nontrivial equivalence relation on $\{l\}$. Then, for every $p \in P_T$, there exists a maximal λ-homogeneous subtree T_1 of T such that*

(i) $p\in P_{T_1}\subset P_{T_{r,p}}\subset P_T$,
(ii) T_1 *is* τ*-homogeneous w.r.t.* r; i.e. $(\forall p, p'\in P_{T_1})((p, p')\in\bar{r}\Rightarrow(p\tau, p'\tau)\in r)$.

Note, that T_1 is not unique in general in either of the above three lemmata. Proofs are immediate by finiteness of trees.

Lemma 10. *Let* T *be a* 1*-rooted* l*-ary terminated tree exactly of level* n. *Let* T_1 *be a* λ*-homogeneous subtree of* T *which is* τ*-homogeneous w.r.t. some* $t\in\{l\}$, *and assume that for some* $v\in V_{T_1}$, $v\lambda=j$ *and for* $k_1, k_2\in\{l\}$, $k_1\neq k_2$, *we have both* $(v, k_1)\in E_{T_1}$ *and* $(v, k_2)\in E_{T_1}$. *Let* $p=(b_1, \ldots, b_{j-1}, k_1, b_{j+1}, \ldots, b_n)$ *and* $q=(b_1, \ldots, b_{j-1}, k_2, b_{j+1}, \ldots, b_n)$. *Then,* $p\in P_{T_1}\Leftrightarrow q\in P_{T_1}$.

Proof. Let $v_1, v_2\in V_{T_1}, v_1\lambda=v_2\lambda=h$. Let $p=(1f, 2f, \ldots, (n+1)f)$, $q=(1g, 2g, \ldots, (n+1)g)$. Let us suppose that, $hf=v_1, hg=v_2$. Let the root of the tree be ε. Then $1f=\varepsilon=1g$, moreover for $h<j$, $hf=hg$ by simple induction. For $h=j$, we have $v_1=v_2$ and $(v_1, k_1)\in E_{T_1}$, $(v_2, k_2)\in E_{T_1}$ by assumption. For $h>j$, we have $(v_1, b_h)\in E_{T_1}\Leftrightarrow(v_2, b_h)\in E_{T_1}$ by λ-homogenity.

Lemma 11. *Let* T *be a* 1*-rooted* l*-ary terminated tree exactly of level* n. *Let* T_1 *be a* λ*-homogeneous subtree of* T *which is* τ*-homogeneous w.r.t. some* $t\in\{l\}$ *and is complete on level* j *for some* $j\in[n]$. *Then every path of the form* $(b_1, \ldots, b_{j-1}, k, b_{j+1}, \ldots, b_n)$ *with fixed* $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n\in\{l\}$ *and arbitrary* $k\in\{l\}$ *is in* P_{T_1}.

Proof. It follows from Lemma 10 by an easy induction.

Let $t\subset\{l\}$ and define r_t by

$$(\forall l_1, l_2\in\{l\})((l_1, l_2)\in r_t\Leftrightarrow l_1\in t\wedge l_2\in t).$$

Clearly, r_t is an equivalence relation. The following assertion is immediate.

Lemma 12. *Let* T *be a* 1*-rooted* l*-ary tree exactly of level* n. *Let us fix* $t\subset\{l\}$ *and let* $p\in P_T, q\in P_T$. *Then* $T_{r_t,p}=T_{r_t,q}\Leftrightarrow(p, q)\in\bar{r}_t$.

It follows from Lemmata 4, 5, 12, that r_t determines a unique maximal λ-homogeneous subtree of T. We shall denote it by T_{r_t}.

4.3. Let T be a terminated ξ-augmented m-rooted l-ary tree. T is *ξ-homogeneous* if and only if

$$(\forall a\in\omega^{l-1})(\exists t\in\{l\})(a\xi^{-1}\subset t\tau^{-1}).$$

4.4. Some further considerations concerning different types of homogenity will appear in later parts. In particular, the notions of anti-λ-homogeneous trees and of combs will be introduced and investigated.

5. Representation of finite functions by terminated trees

5.1. Let $f\in O_l^{(n,m)}$ and let T be an m-rooted l-ary complete tree of level n. We define a terminated tree for f, $T_f=(T, \tau)$, as follows. Let $k\in\{ml^n\}$ be arbitrary and $k\Delta^{-1}=(i, (k_1, \ldots, k_n))$. Then, let

$$(k\delta^{-1})\tau=f^*(x_1/k_1, \ldots, x_n/k_n). \tag{9}$$

By Lemmata 1, 2, the definition (9) is correct.

Theorem 13. *Let $f\in O_l^{(n,m)}$. Then every m-rooted l-ary complete terminated tree $T_f=(T,\tau)_f$ for f is isomorphic to a terminated tree $T'_f=(T',\tau')_f$ with $V\subset O_f^{(n,1)}$.*

Proof. It follows from Lemma 2 that any two m-rooted l-ary complete terminated trees of level n for f are isomorphic. It is sufficient therefore to prove that there exists a terminated tree $(T',\tau')_f$ for f with $V\subset O_l^{(n,1)}$, which is m-rooted, l-ary, complete and of level n. We define (T',τ') by recurrence. Let $i\in[m]$ and $\varepsilon_i=f_i(x_1,\dots,x_n)$ where f_i is the i-th component of f. If $g(x_1/k_1,\dots,x_{h-1}/k_{h-1})x_h x_{h+1}\dots x_n$ is defined as a point of $V\cap\varepsilon_i D_T$ on level h, then let

$$g(x_1/k_1,\dots,x_{h-1}/k_{h-1})x_hx_{h+1}\dots x_n\varrho=l \quad\text{and}\quad g(x_1/k_1,\dots,x_{h-1}/k_{h-1})x_hx_{h+1}\dots x_nD^1_{T'}=$$
$$=\{g(x_1/k_1,\dots,x_{h-1}/k_{h-1},x_h/k)x_{h+1}\dots x_n\,|\,k\in\{l\}\}$$

and for all $k\in\{l\}$,

$$(g(x_1/k_1,\dots,x_{h-1}/k_{h-1})x_hx_{h+1}\dots x_n,k)\sigma=g(x_1/k_1,\dots,x_h/k)x_{h+1}\dots x_n.$$

We stop this recursion on level n, where no point depends on any variables; i.e. eveiy points on level n is of the form $g(x_1/k_1,\dots,x_n/k_n)$. The leaves of the tree obtained are the points on level n. If p is a path in this tree, then $p\tau'$ is defined by (9). It is not hard to see that V hence $T'=((V,\varrho),\sigma,(\varepsilon_1,\dots,\varepsilon_m))$ are well defined. Clearly, T' is m-rooted, l-ary complete tree of level n, and (T',τ') is for f.

The terminated tree T'_f, defined uniquely up to isomorphism by Theorem 13 is called the *tree associated to* f (recall that T'_f is defined after fixing an ordering of the variables of f; it is clear that T'_f depends heavily on this ordering). In the sequel we simply write T_f to denote the tree associated to f.

From now on in this section we shall assume that $m=1$. The general case can be treated in a similar way at the expense of some complication of technical details.

5.2. Let $f\in O_l^{(n,1)}$. $fx_1\dots x_j\dots x_n$ is *partially degenerate in* x_j if and only if for arbitrary $b_1,\dots,b_{j-1},b_{j+1},\dots,b_n\in\{l\}$, there exist $k_1,k_2\in\{l\}$, $k_1\neq k_2$ such that $f^*b_1\dots b_{j-1}k_1b_{j+1}\dots b_n=f^*b_1\dots b_{j-1}k_2b_{j+1}\dots b_n$. If for all $k_1,k_2\in\{l\}$ this equation holds, then f is called *degenerate in* x_j. Let $PD_j^{(n,1)}$ and $D_j^{(n,1)}$ denote the sets of functions (in $O_l^{(n,1)}$) partially degenerate and degenerate in x_j, respectively. The set of nondegenerate functions is defined by $ND^{(n,1)}=O_l^{(n,1)}-\bigcup_{j\in[n]}D_j^{(n,1)}$.

Theorem 14. *Let $f\in O_l^{(n,1)}$ and let $(T,\tau)=T_f$. Then, the following two assertions are equivalent. For $j\in[n]$,*

(i) $f\in PD_j^{(n,1)}$.

(ii) *For every maximal λ-homogeneous subtree T_1 of T_f, which is τ-homogeneous w.r.t. some $t\in\{l\}$, there exist $k_1,k_2\in\{l\}$, $k_1\neq k_2$ such that T_1 contains the edges (v,k_1) and (v,k_2) for all $v\in V_{T_1}$, $v\lambda=j$.*

Proof. Let $f\in PD_j^{(n,1)}$ and assume that T_1 is a maximal λ-homogeneous subtree of T_f which is τ-homogeneous w.r.t. some $t\in\{l\}$. By definition, for all

$p, p' \in P_{T_1}$, we have $f^*(p)=f^*(p')$. Let $p=(b_1, ..., b_j, ..., b_n) \in P_{T_1}$. Since f is partially degenerate in x_j and T_1 is maximal, there exists an $a \in \{l\}$ such that $a=b_j$ and $p'=(b_1, ..., a, ..., b_n) \in P_{T_1}$. Let $k_1=b_j, k_2=a$. Then we obtain, that for some v on level j, (v, k_1) and (v, k_2) are in E_{T_1}. T_1 is λ-homogeneous, hence for all $v' \in V_{T_1}, v'\lambda=j$ we have $(v', k_1) \in E_{T_1}$ and $(v', k_2) \in E_{T_2}$.

Conversely, assume that for every maximal λ-homogeneous subtree T_1 of T which is τ-homogeneous w.r.t. some $t \in \{l\}$, there exist $k_1, k_2 \in \{l\}$ such that $k_1 \neq k_2$ and E_{T_1} contains (v, k_1) and (v, k_2) for all $v \in V_{T_1}$ on level j. Let $p=(b_1, ..., b_j, ..., b_n) \in P_{T_j}$ be arbitrary. By Lemma 7, there exists a maximal λ-homogeneous subtree T_1 of T which is τ-homogeneous w.r.t. $p\tau$. By assumption, there exist $k_1, k_2 \in \{l\}$ such that $k_1 \neq k_2$ and $(v, k_1) \in E_{T_1}$, $(v, k_2) \in E_{T_2}$ for all $v \in V_{T_1}$ on level j. By Lemma 10, $p'=(b_1, ..., k_1, ..., b_n)$ and $p''=(b_1, ..., k_2, ..., b_n)$ are in P_{T_1}. Then, by τ-homogenity, $f^*b_1...k_1...b_n=f^*b_1...k_2...b_n$, hence $f \in PD_j^{(n,1)}$.

Theorem 15. *Let $f \in O_l^{(n,1)}$ and let $(T, \tau)=T_f$. Then, the following two assertions are equivalent. For $j \in [n]$,*

(i) $f \in D_j^{(n,1)}$.

(ii) *Every maximal λ-homogeneous subtree T_1 of T which is τ-homogeneous w.r.t. some $t \in \{l\}$ is complete on level j.*

Proof. Let $f \in D_j^{(n,1)}$. Then, by definition, for arbitrary fixed $b_1, ..., b_{j-1}, b_{j+1}, ..., b_n \in \{l\}$, and for all $k_1, k_2 \in \{l\}, k_1 \neq k_2$ we have $f^*b_1...k_1...b_n=f^*b_1...k_2...b_n$. Consider all paths of the form $(b_1, ..., k, ..., b_n)$ where k varies over $\{l\}$. It is easily seen, that these paths gives rise to a λ-homogeneous subtree T_1 of T which is complete on level j. Clearly, any maximal λ-homogeneous subtree T_2 of T containing T_1 is again complete on level j, hence those maximal λ-homogeneous subtrees of T which are τ-homogeneous w.r.t. some $t \in \{l\}$, namely w.r.t. $f^*b_1...k_1...b_n$ and contain T_1 are complete on level j. Since $b_1, b_2, ..., b_{j-1}, b_{j+1}, ..., b_n$ are chosen arbitrarily, it follows that every maximal λ-homogeneous subtree T_1 of T which is τ-homogeneous w.r.t. some $t \in \{l\}$ is complete on level j.

Conversely, assume that every maximal λ-homogeneous subtree T_1 of T which is τ-homogeneous w.r.t. some $t \in \{l\}$ is complete on level j. Consider all paths of the form $(b_1, ..., k, ..., b_n)$ for fixed $b_1, ..., b_{j-1}, b_{j+1}, ..., b_n \in \{l\}$ and for all $k \in \{l\}$. These paths form a λ-homogeneous subtree T_1 of T which is complete on level j. But T_1 is contained in a maximal λ-homogeneous subtree of T which is τ-homogeneous w.r.t. some t and complete on level j by Lemma 11. It follows, that $f^*b_1...b_{j-1}kb_{j+1}...b_n=t$ for all $k \in \{l\}$.

Corollary 16. *Let $f \in O_l^{(n,1)}$ and let $(T, \tau)=T_f$. Then, the following two assertions are equivalent:*

(i) $f \in ND^{(n,1)}$.

(ii) *No naximal λ-homogeneous subtree T_1 of T exists such that T_1 is τ-homogeneous w.r.t. some $t \in \{l\}$ and complete on some level $j, j \in [n]$.*

Degenerate and partially degenerate functions will be investigated further in the next part [3].

5.3. Let $f \in O_l^{(n,1)}$. Let r be a partial ordering on $\{l\}$. $fx_1 ... x_j ... x_n$ is *r-preserving in* x_j if and only if for arbitrary $b_1, ..., b_{j-1}, b_{j+1}, ..., b_n \in \{l\}$ and for all $k_1, k_2 \in \{l\}$, $(k_1, k_2) \in r$ entails

$$(f^* b_1 \dots b_{j-1} k_1 b_{j+1} \dots b_n, f^* b_1 \dots b_{j-1} k_2 b_{j+1} \dots b_n) \in r.$$

If $X \subset \{x_1, \dots, x_n\}$, then we say that f is *r-preserving in* X if and only if f is r-preserving in x_j for all $x_j \in X$. f is called *r-preserving* if and only if f is r-preserving in $\{x_1, \dots, x_n\}$. We shall denote by $M_{j,r}^{(n,1)}$, $M_{X,r}^{(n,1)}$, $M_r^{(n,1)}$, the sets of functions which are r-preserving in x_j, in X and in $x_1, \dots, x_n$, respectively.

Theorem 17. *Let $f \in O_l^{(n,1)}$ and let r be a partial ordering on $\{l\}$. Then, the following two assertions are equivalent. For $j \in [n]$,*

(i) $f \in M_{j,r}^{(n,1)}$.

(ii) *For every $p = (p_1, \dots, p_j, \dots, p_n) \in P_{T_f}$, the subtree generated by $\{q | q = (p_1, \dots, q_j, \dots, p_n) \wedge (p_j, q_j) \in r \wedge q_j \in \{l\}\}$ is τ-increasing w.r.t. r.*

Proof. Trivial.

Theorem 18. *Let $f \in O_l^{(n,1)}$, $X \subset \{x_1, \dots, x_n\}$, $X = \{x_{j_1}, \dots, x_{j_k}\}$ and let r be a partial ordering on $\{l\}$. The following two assertions are equivalent:*

(i) $f \in M_{X,r}^{(n,1)}$.

(ii) *For every $p = (p_1, \dots, p_{j_1-1}, p_{j_1}, p_{j_1+1}, \dots, p_{j_k-1}, p_{j_k}, p_{j_k+1}, \dots, p_n) \in P_{T_f}$, the subtree generated by $\{q | q = (p_1, \dots, p_{j_1-1}, q_{j_1}, p_{j_1+1}, \dots, p_{j_k-1}, q_{j_k}, q_{j_k+1}, \dots \dots, p_n) \wedge q_{j_1}, \dots, q_{j_k} \in \{l\} \wedge (\forall s \in [k])\ ((p_{j_s}, q_{j_s}) \in r)\}$ is τ-increasing w.r.t. r.*

Proof. It follows from Theorem 17, by easy induction.

Theorem 19. *Let $f \in O_l^{(n,1)}$ and let r be a partial ordering on $\{l\}$. Then, the following two assertions are equivalent:*

(i) $f \in M_r^{(n,1)}$.

(ii) *For every $p \in P_{T_f}$, the (unique) maximal λ-homogeneous subtree of T_f generated by p and r is τ-increasing w.r.t. r.*

Proof. By Lemma 4, $T_{r,p}$ is λ-homogeneous and is obviously maximal. Taking $X = \{x_1, \dots, x_n\}$, Theorem 19 follows from Theorem 18 since if $p = (p_1, \dots, p_n)$, then $T_{r,p} = \{q | q = (q_1, \dots, q_n) \wedge (\forall i \in [n])\ (q_i \in \{l\} \wedge (p_i, q_i) \in r)$.

This characterization of r-preserving functions will be used later to estimate the cardinality of $M_r^{(n,1)}$ [4].

5.4. Let $t \subset \{l\}$ and define

$$T_t^{(n,1)} = \{f | f \in O_l^{(n,1)} \wedge (\forall a)(f^* a \in t) \tag{10}$$

the set of *t-valued* functions. We have immediately:

Theorem 20. *Let $f \in O_l^{(n,1)}$, $t \subset \{l\}$. Then, the following two assertions are equivalent:*

(i) $f \in T_t^{(n,1)}$.

(ii) *T_f is τ-homogeneous w.r.t. t.*

Theorem 20 will be used in later parts to establish strong decidability of some finite-valued sentential calculi in which the elements of t are designated and, using some additional arguments, to prove the strong completeness of some finite-valued predicate logics.

Let $t \subset \{l\}$, and define

$$Q_t^{(n,1)} = \{f | f \in O_l^{(n,1)} \wedge (\exists a \in \{l\}^n)(f^* a \notin t) \wedge (\forall b \in \{l\}^n)(f^* b \notin t \Leftrightarrow b = a)\}$$

the set of *quasi t-valued* functions. Elements of $Q_t^{(n,1)}$ are natural generalizations of functions associated to conditional sentences (Horn sentences, or quasi-equations) of the two-valued propositional logic. They have almost all of the nice properties of the two-valued functions associated to Horn sentences and hence it is of some interest to characterize them by trees. We have immediately

Theorem 21. *Let* $f \in O_l^{(n,1)}$ *and* $t \subset \{l\}$. *Then, the following two assertions are equivalent:*

(i) $f \in Q_t^{(n,1)}$.

(ii) T_f *is quasi* τ*-homogeneous w.r.t.* r.

Let $t \subset \{l\}$ and r_t be the equivalence relation generated by t. We set

$$P_t^{(n,1)} = \{f | f \in O_l^{(n,1)} \wedge (\forall a, b \in \{l\}^n)((a, b) \in \bar{r}_t \Rightarrow (f^* a, f^* b) \in r_t)\},$$

the set of *t-preserving* functions. The following claim is trivial.

Theorem 22. *Let* $f \in O_l^{(n,1)}$ *and* $t \subset \{l\}$. *Then, the following two assertions are equivalent:*

(i) $f \in P_t^{(n,1)}$.

(ii) T_{r_t}, *the subtree of* T_f *determined by* r_t *is* τ*-homogeneous w.r.t.* r_t.

5.5. Let $f \in O_l^{(n,1)}$ and π be a permutation of the set $[n]$. f *preserves* π if and only if for all $a_1, ..., a_n \in \{l\}$, we have $f^* a_1 \dots a_n = f^* a_{\pi(1)} \dots a_{\pi(n)}$. Let $S^{(n,1)} = \{f | f \in O_l^{(n,1)}$ and f preserves all permutations π of the set $[n]\}$.

Theorem 23. *Let* $f \in O_l^{(n,1)}$ *and let* (T, τ) *be a terminated* ξ*-augmented* 1*-rooted* l*-ary tree associated to* f. *Then the following two assertions are equivalent:*

(i) $f \in S^{(n,1)}$.

(ii) (T, τ) *is* ξ*-homogeneous.*

Proof. The theorem follows immediately from the well-known fact [7], that

$$f \in S^{(n,1)} \Leftrightarrow (\forall a \in \omega^{l-1}, \forall p_1, p_2 \in a\xi^{-1})(p_1 \tau = p_2 \tau).$$

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