General products and equational classes of automata

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The aim of this paper is to characterize those equational classes of automata which are obtained by means of the general product. It will be seen that such classes can be given by "patterns" of identities to be called p-identities. Moreover, these equational classes are either very large or very small.

1. Preliminaries

Let $\mathfrak{A}=(A, F, \delta)$ be an automaton, where A is the state set, F is the input set and δ is the next-state function of $\mathfrak{A}$. As it is well known $\mathfrak{A}$ can be considered an F-unoid (F-algebra with unary operational symbols) $\mathfrak{A}=(A, F)$ such that $af=\delta(a,f)$ $(a\in A, f\in F)$. Further on it will be supposed that F is finite. If A is also finite then we speak about a *finite* F-unoid.

In the sequel F and F' with or without indices will denote finite sets of unary operational symbols.

As usual F^* will stand for the free monoid freely generated by F. If $p=f_1 \dots f_k \in F^*$ is a word and x is a variable then xp is the F-polynomial symbol $(\dots(xf_1)\dots)f_k$.

Let K be a class of F-unoids. Then the operators $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$ on K are defined as follows:

$\mathbf{H}(K)$: homomorphic images of unoids from K,

$\mathbf{S}(K)$: subunoids of unoids from K,

$\mathbf{P}(K)$: direct products of nonvoid families of unoids from K.

By Birkhoff's Theorem (cf. [3]): For a nonvoid class K of F-unoids $\mathbf{HSP}(K)$ is the smallest equational class containing K.

Next we recall the concept of the products of automata (cf. [1]).

Let $\mathfrak{A}_i=(A_i, F_i)$ $(i\in I)$ be a non-void family of unoids, F a finite set of operational symbols and

$$\varphi\colon \Pi(A_i|i\in I)\times F \to \Pi(F_i|i\in I)$$

a mapping. Take the F-unoid $\mathfrak{A}=(A, F)$ with $A=\Pi(A_i|i\in I)$ and $\mathrm{pr}_i(\mathbf{a}f)=\mathrm{pr}_i(\mathbf{a})\mathrm{pr}_i(\varphi(\mathbf{a},f))$ for arbitrary $\mathbf{a}\in A, f\in F$ and $i\in I$, where pr_i is the i^{th} projection. Then $\mathfrak{A}$ is the *(general) product* of $\mathfrak{A}_i$ $(i\in I)$ with respect to F and φ.

For arbitrary $\mathbf{a}\in A, f\in F$ and $i\in I$ let $\varphi_i(\mathbf{a},f)$ be the i^{th} component of $\varphi(\mathbf{a},f)$. If there exists a linear ordering $\leqq$ on I such that for every $i\in I$, φ_i is independent of its j^{th} component ($j\in I$) whenever $j\geqq i$ then $\mathfrak{A}$ is an *α_0-product*. Obviously, if $F_i=F$ and $\varphi_i(\mathbf{a},f)=f$ for arbitrary $i\in I$, $\mathbf{a}\in A$ and $f\in F$ then $\mathfrak{A}$ is the direct product of $\mathfrak{A}_i$ ($i\in I$). Let us note that the formations of the product, the α_0-product and the direct product are transitive. Moreover, further on for α_0-products in $\varphi_i(\mathbf{a},f)$ we shall indicate only those components on which φ_i may depend, i.e., f and $\mathrm{pr}_j(\mathbf{a})$ if $j<i$ ($j\in I$).

Let K be a class of unoids (not necessarily of the same type). Then

$\mathbf{P}_g(K)$ is the class of all general products of unoids from K,

$\mathbf{P}_{f\alpha_0}(K)$ is the class of all α_0-products of unoids from K with finitely many factors, and

$\mathbf{K}_F$ is the similarity class of F-unoids.

To determine unoid identities preserved by products we recall the concept of an l-free system.

Take a unoid $\mathfrak{A}=(A, F)$, an element $a\in A$ and an integer $l\geqq 0$. The system $(\mathfrak{A}, a)$ is *l-free* if $ap\neq aq$ whenever $p\neq q$ and $|p|, |q|\leqq l$ ($p, q\in F^*$), where $|p|$ denotes the length of p.

A state $a\in A$ is *ambiguous* if there are $f_1, f_2\in F$ such that $af_1\neq af_2$.

Obviously, every system $(\mathfrak{A}, a)$ is 0-free. Moreover, it easily follows from the proof of the Theorem in [2] that for a class K of unoids the following statements are equivalent:

(i) For an $l>0$ and all F there is an l-free system $(\mathfrak{A}, a)$ with $\mathfrak{A}=(A, F)\in\in\mathbf{P}_{f\alpha_0}(K)\cap K_F$,

(ii) K contains a $\mathfrak{B}=(B, F')$ such that for a $b\in B$ and a $p\in F'^*$ with $|p|=l-1$, bp is ambiguous. Therefore, if l is the greatest integer under which the above l-free system exists then for arbitrary $\mathfrak{B}=(B, F')\in K$, $b\in B, p\in F'^*$ with $|p|\geqq l$ and $f_1, f_2\in F'$, $bpf_1=bpf_2$.

2. Identities preserved by general products

Let K be an arbitrary nonvoid class of unoids. Then for every F, $\mathbf{HSP}_g(K)\cap K_F$ is an equational class since $\mathbf{HSP}_g(K)=\mathbf{HSPP}_g(K)$ obviously holds. Moreover, it is easy to show that $\mathbf{HSP}_g(K)$ is closed under the general product.

Now we introduce special identities to characterize $\mathbf{HSP}_g(K)\cap K_F$. A *p-identity* is

(i) $m=n$, or

(ii) $(k, m)=(k, n)$

where m, n and k are non-negative integers. A unoid $\mathfrak{A}=(A, F)$ *satisfies* p-identity (i) if $\mathfrak{A}$ satisfies all identities $xg_1\dots g_m=yh_1\dots h_n$ for arbitrary $g_1, \dots, g_m, h_1, \dots, h_n\in F$. Moreover, $\mathfrak{A}$ satisfies (ii) if it satisfies all identities $xf_1\dots f_kg_1\dots g_m=xf_1\dots f_kh_1\dots h_n$ for arbitrary $f_1, \dots, f_k, g_1, \dots, g_m, h_1, \dots, h_n\in F$. In these cases we also say that (i) or (ii) *holds* in $\mathfrak{A}$.

For a class K of unoids denote by K^* the class of all p-identities holding in every unoid from K. Moreover, K^{**} stands for the class of all unoids which satisfy every p-identity in K^*. Then we have the following

Theorem. For arbitrary F and nonvoid class K of unoids $\mathbf{HSP}_g(K)\cap K_F = K^{**}\cap K_F = \mathbf{HSPP}_{f\alpha_0}(K)\cap K_F$.

Proof. Obviously p-identities are preserved under general products. Thus $K^{**}\supseteq \mathbf{HSP}_g(K)$. Therefore, to prove the Theorem it is enough to show that $\mathbf{HSPP}_{f\alpha_0}(K)\cap K_F\supseteq K^{**}\cap K_F$ which follows from statements (i) and (ii) below.

(i) Let $xg_1\dots g_m=yh_1\dots h_n$ be an F-identity satisfied by $\mathbf{HSPP}_{f\alpha_0}(K)\cap K_F$. Then the p-identity $m=n$ is in K^*.

(ii) Let $xf_1\dots f_kg_1\dots g_m=xf_1\dots f_kh_1\dots h_n$ be an F-identity holding in $\mathbf{HSPP}_{f\alpha_0}(K)\cap K_F$ such that $g_1\neq h_1$ if $m,n>0$. Then the p-identity $(k,m)=(k,n)$ is in K^*.

We shall prove (ii) only. Statement (i) can be shown in a similar way.

If for every l there are an $\mathfrak{A}=(A,F)\in\mathbf{P}_{f\alpha_0}(K)\cap K_F$ and an $a\in A$ such that $(\mathfrak{A},a)$ is l-free then in $\mathbf{HSPP}_{f\alpha_0}(K)\cap K_F$ only the trivial identities hold. Therefore, $\mathbf{HSPP}_{f\alpha_0}(K)\supseteq K_F$.

Next assume that l is the greatest integer for which there exist an $\mathfrak{A}=(A,F)\in\mathbf{P}_{f\alpha_0}(K)\cap K_F$ and an $a\in A$ such that $(\mathfrak{A},a)$ is l-free. Let the identity $xf_1\dots f_kg_1\dots g_m=xf_1\dots f_kh_1\dots h_n$ hold in $\mathbf{HSPP}_{f\alpha_0}(K)\cap K_F$ where $g_1\neq h_1$ if $m,n>0$. Suppose that the p-identity $(k,m)=(k,n)$ is not in K^*. Then we find a unoid $\mathfrak{A}'=(A',F')\in K$, an element $a'\in A'$ and operational symbols $f_1',\dots,f_k',g_1',\dots,g_m',h_1',\dots,h_n'\in F'$ under which

$$a'f_1'\dots f_k'g_1'\dots g_m'\neq a'f_1'\dots f_k'h_1'\dots h_n'.$$

Take the l-free system $(\mathfrak{A},a)$ above, and form the α_0-product $\mathfrak{B}=(B,F)$ of $\mathfrak{A}$ and $\mathfrak{A}'$ given by the function $\varphi\colon A\times A'\times F\to F\times F'$ such that φ_1 is the identity mapping of F. Moreover,

$$\varphi_2(af_1\dots f_i,f_{i+1})=f_{i+1}'\quad\text{if}\quad i\leqq l,$$

$$\varphi_2(af_1\dots f_kg_1\dots g_i,g_{i+1})=g_{i+1}'\quad\text{if}\quad k+i\leqq l$$

and

$$\varphi_2(af_1\dots f_kh_1\dots h_i,h_{i+1})=h_{i+1}'\quad\text{if}\quad k+i\leqq l.$$

In all other cases φ_2 is defined arbitrarily. Then in $\mathfrak{B}$ we have

$$(a,a')f_1\dots f_kg_1\dots g_m=(af_1\dots f_kg_1\dots g_m,a'f_1'\dots f_k'g_1'\dots g_m')\neq$$
$$\neq(af_1\dots f_kh_1\dots h_n,a'f_1'\dots f_k'h_1'\dots h_n')=(a,a')f_1\dots f_kh_1\dots h_n,$$

which is a contradiction. This ends the proof of the Theorem.

Next we show that $\mathbf{HSP}_g(K)\cap K_F$ has a finite basis. As it has been noted if for arbitrary l and F' there are an $\mathfrak{A}=(A,F')\in\mathbf{HSP}_g(K)$ and an $a\in A$ such that $(\mathfrak{A},a)$ is l-free then only the trivial identities hold in $\mathbf{HSP}_g(K)\cap K_F$. Thus we may assume that there exists such a maximal l which is also denoted by l.

(To check that the F-identities determined by the systems of p-identities below form a basis observe the existence of an l-free system $(\mathfrak{A},a)$ with $\mathfrak{A}\in\mathbf{HSP}_g(K)\cap K_F$ such that for arbitrary $\mathfrak{B}=(B,F)\in\mathbf{HSP}_g(K)\cap K_F$ and $b\in B$ the mapping $\varphi\colon\{a\}\to\{b\}$ can be extended to a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.)

I. K^* contains no p-identities of form $m=n$.

I. 1. There is a p-identity $(k,m)=(k,n)$ in K^* with $m<n$.

a) k_1 is minimal among all k occurring in p-identities $(k,m)=(k,n)$ from K^* with $m<n$,

b) m_1 is minimal among all m occurring in p-identities $(k_1, m)=(k_1, n)$ from K^* with $m<n$,

c) n_1 is minimal among all n occurring in p-identities $(k_1, m_1)=(k_1, n)$ from K^* with $m_1<n$,

d) k_2 is minimal among all k occurring in nontrivial* p-identities $(k, m)= =(k, m)$ from K^*,

e) m_2 is minimal among all m occurring in nontrivial p-identities $(k_1, m)= =(k_1, m)$ from K^*.

Then a suitable basis can be given in form

$$(k_1, m_1) = (k_1, n_1),\ (k_2^{(1)}, m_2^{(1)}) = (k_2^{(1)}, m_2^{(1)}),\ \dots,\ (k_2^{(r)}, m_2^{(r)}) = (k_2^{(r)}, m_2^{(r)}),$$

where $k_2^{(1)}=k_2$, $m_2^{(1)}=m_2$, $k_2^{(1)}<\dots<k_2^{(r)}<k_2+m_2$ and $k_2+m_2\geqq m_2^{(1)}>\dots>m_2^{(r)}$. (Note that $k_1, k_2\leqq l$ and $k_1+n_1, k_2+m_2>l$.)

I. 2. K^* contains no p-identity $(k, m)=(k, n)$ with $m<n$. Then there is a basis of form $(k_2^{(1)}, m_2^{(1)})=(k_2^{(1)}, m_2^{(1)}), \dots, (k_2^{(r)}, m_2^{(r)})=(k_2^{(r)}, m_2^{(r)})$ $(k_2^{(1)}=k_2$, $m_2^{(1)}=m_2$, $k_2^{(1)}<\dots<k_2^{(r)}<k_2+m_2$, $m_2^{(r)}<\dots<m_2^{(1)}\leqq k_2+m_2)$ where k_2 and m_2 are obtained by d) and e) in I. 1.

II. K^* has a p-identity $m=n$.

Let m_1 be minimal among all m occurring in p-identities $m=n$ from K^*. Moreover let k_2 and m_2 be given by d) and e) in I. Then one of the bases has the form

$$m_1 = m_1,\ (k_2^{(1)}, m_2^{(1)}) = (k_2^{(1)}, m_2^{(1)}),\ \dots,\ (k_2^{(r)}, m_2^{(r)}) = (k_2^{(r)}, m_2^{(r)}),$$

where again $k_2^{(1)}=k_2$, $m_2^{(1)}=m_2$, $k_2^{(1)}<\dots<k_2^{(r)}<k_2+m_2$ and $k_2+m_2\geqq m_2^{(1)}>\dots>m_2^{(r)}$.

If K consists of finitely many finite unoids then a finite basis can be given effectively. Therefore, for such a K and a finite $\mathfrak{A}=(A, F)$ it is decidable whether $\mathfrak{A}$ is contained by $\mathbf{HSP}_g(K)\cap K_F$.

Finally, it can be shown by a slight modification of the proof that the Theorem remains valid for infinite F, too.

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* A p-identity of form $(k, m)=(k, m)$ is *trivial* if $m=0$.