Deterministic ascending tree automata II

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To the memory of my Mother

In [12] we started a systematic study of deterministic ascending (called also root-to-frontier or top-down) tree automata. The present second part is entirely devoted to the investigation of the product of such automata. We generalize the notion of the product of ordinary automata due to Gluskov [cf. 7] and that of the special products defined by Gécseg in [3]. Some other generalizations can be found in [9], [10] and [11] for the case of bottom-up (known also as frontier-to-root) tree automata.

1. Preliminaries

The reader is assumed to be familiar with the fundamental concepts concerning tree automata and tree transducers. To keep the size of the paper within reasonable limits we give only a brief account on notions defined elsewhere but used in our treatment, too. For terminology not defined here, see [1], [5] and/or [6].

The concepts of a *type* F, a *deterministic ascending F-algebra* $\mathfrak{A}=\langle A, F\rangle$, a *deterministic ascending F-automaton* $\mathbf{A}=\langle\mathfrak{A}, a', \mathbf{a}\rangle$ and *the forest* $T(\mathbf{A})\subseteqq T_{F,X_n}$ *recognized by* $\mathbf{A}$ are used in the same sense as in [12]. In the sequel F-algebra (F-automaton) means a deterministic ascending F-algebra (F-automaton). When F is not specified we speak simply about algebra and automaton. Furthermore, all algebras and automata are assumed to be finite and have no nullary operations.

Now we shall introduce some additional terminology. $|A|$ denotes the cardinality of the set A. A *rank type* R is a finite nonvoid subset of the set $N=\{1, 2, ...\}$ of natural numbers. The type F has rank type $R(F)=\{n|F_n\neq\varnothing\}$.

Let $\mathbf{A}=\langle\mathfrak{A}, a', \mathbf{a}\rangle$, $\mathbf{a}=(A^{(1)}, ..., A^{(n)})$ and $\mathbf{B}=\langle\mathfrak{B}, b', \mathbf{b}\rangle$, $\mathbf{b}=(B^{(1)}, ..., B^{(n)})$ be two F-automata with the associated algebras $\mathfrak{A}=\langle A, F\rangle$ and $\mathfrak{B}=\langle B, F\rangle$. Then $\mathfrak{B}$ is called a *subalgebra of* $\mathfrak{A}$ if $B\subseteqq A$ and for all $k\in R(F)$, $f\in F_k$ and $b\in B$, $f^{\mathfrak{A}}(b)=f^{\mathfrak{B}}(b)\in B^k$ holds. The automaton $\mathbf{A}$ is *connected* if all states $a\in A$ are reachable from the initial state a' by suitable operations. (For a formal description see [5].)

Next we recall some concepts and results from [5].

A *homomorphism of the algebra* $\mathfrak{A}$ into $\mathfrak{B}$ is a mapping $\varphi: A \to B$ such that

(i) for all $k \in R(F)$, $f \in F_k$ and $a \in A$, $f^{\mathfrak{B}}(\varphi(a)) = (\varphi(a_1); \dots; \varphi(a_k))$, where $(a_1, \dots, a_k) = f^{\mathfrak{A}}(a)$. If, in addition

(ii) $\varphi(a') = b'$ and

(iii) for all $i = 1, \dots, n$; $\varphi(A_i) = B_i$ and $\varphi^{-1}(B_i) = A_i$ hold, then φ is a *homomorphism of the automaton* **A** into **B**. In case of $\varphi(A) = B$ we call $\mathfrak{B}$ a *homomorphic image* of $\mathfrak{A}$. If φ is also bijective then it is called an *isomorphism*. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* and write $\mathfrak{A} \cong \mathfrak{B}$ if there exists an isomorphism $\varphi: \mathfrak{A} \to \mathfrak{B}$. The same terminology is used for automata.

A *congruence relation of the algebra* $\mathfrak{A}$ is defined as an equivalence relation ϱ on A such that

(i) for all $k \in R(F), f \in F_k$ and $a, a' \in A$, $a \varrho a'$ implies $a_i \varrho a'_i$ where $i = 1, \dots, k, f^{\mathfrak{A}}(a) = (a_1, \dots, a_k)$ and $f^{\mathfrak{A}}(a') = (a'_1, \dots, a'_k)$.

Moreover, ϱ is a *congruence relation of the automaton* **A** if the additional condition

(ii) for all $i = 1, \dots, n$ and $a \in A$, $a \in A^{(i)}$ implies $\varrho(a) \subseteq A^{(i)}$

holds.

NOTATION. The two trivial congruences of $\mathfrak{A}$ will be denoted by $\iota = A \times A$ and $\omega = \{(a, a) | a \in A\}$, respectively. $\mathfrak{A}$ is *simple* if it has only trivial congruences.

For any state $a \in A$ let $\mathbf{A}(a)$ denote the automaton $\langle \mathfrak{A}, a, \mathbf{a} \rangle$. The state a is called a *0-state* if $T(\mathbf{A}(a)) = \emptyset$. We say that **A** is *normalized* if, for all $k \in R(F), f \in F_k$ and $a \in A$, either all of the components of $f^{\mathfrak{A}}(a)$ are 0-states or none of them is a 0-state. The automaton **A** is *minimal* if $|A| \leqq |B|$ whenever $T(\mathbf{A}) = T(\mathbf{B})$.

The following results are from [5].

Proposition 1.1. If **B** is a homomorphic image of **A**, then $T(\mathbf{A}) = T(\mathbf{B})$.

Proposition 1.2. If the minimal automaton **A** is equivalent to the normalized and connected automaton **B** then **A** is a homomorphic image of **B**.

In the rest of this paper we consider only algebras belonging to the class $K(R)$ of all finite algebras of the fixed rank type R. Let $F, F^1, \dots, F^k$ be ranked alphabets of rank type R and consider the F^i-algebras $\mathfrak{A}_i = (A_i, F^i)$ $(i = 1, \dots, k)$. Furthermore, let

$$\psi: A_1 \times \dots \times A_k \times F \to F^1 \times \dots \times F^k$$

be an arity-preserving mapping, i.e., for every $m \in R, f \in F_m$ and $\mathbf{a} \in \prod_{i=1}^{k} A_i$, $\psi(\mathbf{a}, f) = (f^1, \dots, f^k)$ implies $f^i \in F^i_m$ $(i = 1, \dots, k)$. Then by the *general product* or, shortly *G-product* of $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ with respect to the *feedback function* ψ we mean the F-algebra $\mathfrak{A} = \langle A, F \rangle = \prod_{i=1}^{k} \mathfrak{A}_i[F, \psi]$ with $A = \prod_{i=1}^{k} A_i$ and for arbitrary $m \in R$, $f \in F_m$ and $\mathbf{a} \in A$

$$f^{\mathfrak{A}}(\mathbf{a}) = \big((\pi_1(f^1(\pi_1(\mathbf{a}))), \dots, \pi_1(f^k(\pi_k(\mathbf{a})))), \dots$$

$$\dots(\pi_m(f^1(\pi_1(\mathbf{a}))), \dots, \pi_m(f^k(\pi_k(\mathbf{a}))))\big),$$

where $(f^1, \dots, f^k) = \psi(\mathbf{a}, f)$ and π_l denotes the l^{th} projection.

To define special types of products let us write ψ in the form $\psi=(\psi^{(1)}, \ldots, \psi^{(k)})$, where for arbitrary $\mathbf{a}\in A$ and $f\in F_m$, $\psi(\mathbf{a},f)=(\psi^{(1)}(\mathbf{a},f), \ldots, \psi^{(k)}(\mathbf{a},f))$. We say that $\mathfrak{A}$ is an *α_i-product* ($i=0, 1, \ldots$) if for arbitrary j $(1\leqq j\leqq k)$ $\psi^{(j)}$ is independent of its u^{th} component if $i+j\leqq u\leqq k$. If ψ is independent of $\prod_{i=1}^{k} A_i$, i.e., ψ is a mapping of F into $\prod_{i=1}^{k} F^i$, then $\mathfrak{A}$ is a *quasidirect-product* (shortly Q-product).

Let θ-product mean any of the α_i-products, the Q-product or the G-product. Now take a class K of algebras. Then $H_\theta(K)$ denotes the class of all algebras which can be given as homomorphic images of subalgebras of θ-products of algebras from K. Similarly, $I_\theta(K)$ stands for the class consisting of all algebras which are isomorphic to subalgebras of θ-products of algebras from K. The class K is *homomorphically (isomorphically)* complete with respect to the θ-product if $H_\theta(K)=$ $=K(R)$ $(I_\theta(K)=K(R))$ holds. Finally, K is *forest complete* with respect to the θ-product if for every forest $T\subseteq T_{F,X_n}$ recognizable by deterministic ascending automata there exists a θ-product $\mathfrak{A}=\langle A, F\rangle$ of algebras from K and an automaton $\mathbf{A}=\langle\mathfrak{A}, a', \mathbf{a}\rangle$ satisfying $T(\mathbf{A})=T$.

2. Some general properties of the products

It is obvious that every isomorphically θ-complete system is homomorphically θ-complete as well. For the converse we note

Remark 2.1. For every θ there exists a homomorphically θ-complete system $M\subseteq K(R)$ which is not isomorphically θ-complete.

To verify this statement take an arbitrary isomorphically θ-complete system M. Since $I_\theta(M)$ contains the one-element algebras there is an $\mathfrak{A}=\langle A, G\rangle\in M$ and an $a\in A$ such that

(*) for all $r\in r(G)$ there is a $g\in G_r$ satisfying $g(a) = (a, \ldots, a)$

holds. Now take the system $M^*=\{\mathfrak{A}^*|\mathfrak{A}\in M\}$ where

$$\mathfrak{A}^* = \langle A\cup B, G\rangle, \quad B = \{a^*|a\in A \text{ and } a \text{ satisfies } (*)\}.$$

The operations of $\mathfrak{A}^*$ are defined for all $g\in G$, $a\in A$ and $a^*\in B$ in the following way: $g^{\mathfrak{A}^*}(a^*)=g^{\mathfrak{A}}(a)$ and

$$g^{\mathfrak{A}^*}(a) = \begin{cases} g^{\mathfrak{A}}(a) & \text{if } g^{\mathfrak{A}}(a) \neq (a, \ldots, a) \\ (a^*, \ldots, a^*) & \text{otherwise.} \end{cases}$$

Evidently, M^* cannot be isomorphically θ-complete but it will turn out to be homomorphically θ-complete.

Let $\mathfrak{C}=\langle C, H\rangle$ be an arbitrary algebra. Assume that $\mathfrak{C}$ is isomorphic to a subalgebra $\mathfrak{D}$ of the θ-product $\prod_{i=1}^{k} \mathfrak{A}_i[H, \psi]$ from M. Constructing the θ-product $\prod_{i=1}^{k} \mathfrak{A}_i^*[H, \psi^*]$ from M^* it is not difficult to prove that $\mathfrak{C}$ is a homomorphic image of a subalgebra $\mathfrak{D}^*$ of this product. Here ψ^* is defined by $\psi^*(\mathbf{a}, h)=\psi(\hat{\mathbf{a}}, h)$ where $\hat{\mathbf{a}}$ can be obtained 'by removing the stars', i.e., if $\pi_i(\mathbf{a})\in A$ then $\pi_i(\hat{\mathbf{a}})=$ $=\pi_i(\mathbf{a})$ else if $\pi_i(\mathbf{a})=a^*\in B$ then $\pi_i(\hat{\mathbf{a}})=a$ for all $i=1, \dots, k$.

For ordinary automata the notion of the completeness with respect to the automaton mappings have been introduced. (See, e.g. [4].) Now we shall define a similar concept concerning 'tree automaton mappings', i.e., top-down tree transformations.

In the sequel we shall use the general terms such as *top-down tree transducers* and *top-down tree transformations* induced by them, *deterministic, connected* or *minimal transducers* in their usual meaning (c.f. [1], [2] or [6]).

A top-down tree transducer $\mathscr{A}=\langle T_{F,X_n}, A, T_{G,Y_m}, A', \Sigma_{\mathscr{A}}\rangle$ is *uniform* if each rule $af\to p(a\in A,\ f\in F_l,\ l\in R(F),\ p\in T_{G,Y_m\cup A\Xi_l})$ can be written in the form $af\to q(a_1\xi_1, \dots, a_l\xi_l)$ for some $q\in T_{G,Y_m\cup\Xi_l}$. In this section by a transducer $\mathscr{A}$ we shall mean a deterministic uniform top-down tree transducer having exactly one rule $af\to p$ for every $(a,f)\in A\times F$. Moreover, all transducers are assumed to have the fixed input rank type R.

Let $\mathscr{A}=\langle T_{F,X_n}, A, T_{G,Y_m}, a', \Sigma_{\mathscr{A}}\rangle$ and $\mathscr{B}=\langle T_{F,X_n}, B, T_{G,Y_m}, b', \Sigma_{\mathscr{B}}\rangle$ be transducers and take a mapping $\varphi\colon A\to B$. If the following three conditions are satisfied for arbitrary $af\to q(a_1\xi_1, \dots, a_l\xi_l)$ and $ax_i\to t\in\Sigma_{\mathscr{A}}$ then φ is called a *homomorphism* of $\mathscr{A}$ into $\mathscr{B}$

(i) if $af\to q(a_1\xi_1, \dots, a_l\xi_l)\in\Sigma_{\mathscr{A}}$ then
$bf\to q(b_1\xi_1, \dots, b_l\xi_l)\in\Sigma_{\mathscr{B}}$ where
$b=\varphi(a),\ b_j=\varphi(a_j)\ (j=1, \dots, l)$,

(ii) if $ax_i\to t\in\Sigma_{\mathscr{A}}$ then
$bx_i\to t\in\Sigma_{\mathscr{B}}$ where
$b=\varphi(a)$,

(iii) $\varphi(a')=\varphi(b')$.

If φ is surjective then $\mathscr{B}$ is a *homomorphic image* of $\mathscr{A}$.

The following result has been obtained in [2].

Proposition 2.2. If there is a homomorphism from $\mathscr{A}$ into $\mathscr{B}$ then $\tau_{\mathscr{A}}=\tau_{\mathscr{B}}$.

The n-ary F-automaton $\mathbf{A}=\langle\langle A, F\rangle, a', \mathbf{a}\rangle$ *belongs to the* transducer $\mathscr{A}=$ $=\langle T_{F,X_n}, A, T_{G,Y_m}, a', \Sigma_{\mathscr{A}}\rangle$ if

(i) for all $a\in A,\ k\in R(F)$ and $f\in F_k,\ f^{\mathfrak{A}}(a)=(a_1, \dots, a_k)$ implies
$af\to p(a_1\xi_1, \dots, a_k\xi_k)\in\Sigma_{\mathscr{A}}$ for some $p\in T_{G,Y_m\cup\Xi_k}$ and

(ii) for $1\leqq i\leqq n$, $a\in A^{(i)}$ iff $ax_i\to q\in\Sigma_{\mathscr{A}}$ for some $q\in T_{G,Y_m}$.

Aut $(\mathscr{A})$ denotes the class of all automata belonging to $\mathscr{A}$. Now we can introduce the class Alg $(\mathscr{A})$ of all algebras belonging to $\mathscr{A}$:

$\mathfrak{A}=\langle A, F\rangle\in$ Alg $(\mathscr{A})$ iff there is an automaton $\mathbf{A}=\langle\mathfrak{A}, a', \mathbf{a}\rangle\in$ Aut $(\mathscr{A})$.

A system $M\subseteq K(R)$ is *complete* with respect to the θ-product if for every tree transformation $\tau\colon T_{F,X_n}\to T_{G,Y_m}$ there is a transducer $\mathscr{A}$ and a θ-product $\mathfrak{A}$ of algebras from M such that $\tau=\tau_{\mathscr{A}}$ and $\mathfrak{A}\in$ Alg $(\mathscr{A})$ hold.

In the proof of the following theorem we need the concept of the paths of a tree p. For arbitrary type F, $n\in N$ and $p\in T_{F,X_n}$, path (p) stands for the smallest subset of $(F\times N)^*$ satisfying

(i) if $p=x_i$ $(1\leqq i\leqq n)$ then path (p) consist of the empty word e, and

(ii) if $p=f(p_1, \ldots, p_k)$, $k\in R$, $f\in F_k$, $p_1, \ldots, p_k\in T_{F,X_n}$ then

$$\text{path}(p)=\bigcup_{i=1}^{k}(f, i)\,\text{path}(p_i).$$

Moreover, for arbitrary set $T\subseteq T_{F,X_n}$ define $\text{path}(T)=\bigcup(\text{path}(t)|t\in T)$. The *realization of the path* $v\in\text{path}(T_{F,X_n})$ in the F-algebra $\mathfrak{A}=\langle A, F\rangle$ is the mapping $v^{\mathfrak{A}}: A\to A$ given by

(i) $ae^{\mathfrak{A}}=a$ for all $a\in A$ and

(ii) $av^{\mathfrak{A}}=c$ iff $v=u(f, i)$, $au^{\mathfrak{A}}=b$ and $\pi_i(f(b))=c$ holds for $u\in\text{path}(T_{F,X_n})$, $k\in R, f\in F_k$, $1\le i\le k$ and $b\in A$.

Theorem 2.3. With respect to arbitrary θ-product the homomorphic completeness, the completeness and the forest completeness are equivalent to each other.

Proof. (1) Let the system $M\subseteq K(R)$ be homomorphically complete with respect to the θ-product. Further, let τ be an arbitrary transformation induced by the connected transducer $\mathcal{A}=\langle T_{F,X_n}, A, T_{G,Y_m}, a', \Sigma_{\mathcal{A}}\rangle$ and let $\mathbf{A}=\langle\mathfrak{A}, a', \mathbf{a}\rangle\in\in\text{Aut}(\mathcal{A})$.

As M is homomorphically θ-complete there exist a θ-product $\mathfrak{C}=\langle C, F\rangle=$ $=\prod_{i=1}^{s}\mathfrak{C}_i[F, \psi]$ from M and a subalgebra $\bar{\mathfrak{C}}$ of $\mathfrak{C}$ such that $\mathfrak{A}$ is a homomorphic image of $\bar{\mathfrak{C}}$ under some homomorphism φ. Taking the subalgebra $\mathfrak{C}^*$ of $\bar{\mathfrak{C}}$ generated by a $c'\in\varphi^{-1}(a')$ it follows easily that $\mathbf{A}$ is a homomorphic image of the connected automaton $\mathbf{C}^*=\langle\mathfrak{C}^*, c', \mathbf{c}\rangle$ under φ. (The final state vector $\mathbf{c}$ of $\mathbf{C}^*$ can be given using the inverse of φ.) Let us consider the transducer $\mathcal{C}^*=\langle T_{F,X_n}, C^*, T_{G,Y_m}, c', \Sigma_{\mathcal{C}^*}\rangle$ satisfying

(i) $cf\to p(c_1\xi_1, \ldots, c_k\xi_k)\in\Sigma_{\mathcal{C}^*}$ iff $\varphi(c)=a$, $\varphi_i(c_i)=a_i$ $(i=1, \ldots, k)$ and $af\to p(a_1\xi_1, \ldots, a_k\xi_k)\in\Sigma_{\mathcal{A}}$,

(ii) $cx_i\to q\in\Sigma_{\mathcal{C}^*}$ iff $\varphi(c)=b$ and $bx_i\to q\in\Sigma_{\mathcal{A}}$

for all $f\in F$, $c\in C^*$ and $1\le i\le n$.

This construction ensures that $\varphi: \mathcal{C}^*\to\mathcal{A}$ is a homomorphism. Hence, by Proposition 2.2, $\tau_{\mathcal{C}^*}=\tau_{\mathcal{A}}$. But taking the transducer $\mathcal{C}=\langle T_{F,X_n}, C, T_{G,Y_m}, c', \Sigma_{\mathcal{C}}\rangle$ where

$$\Sigma_{\mathcal{C}}=\Sigma_{\mathcal{C}^*}\cup\big(cf\to q_{c,f}(c_1\xi_1, \ldots, c_k\xi_k)|c\notin C^*, f\in F, f^{\mathfrak{C}}(c)=(c_1, \ldots, c_k)\big),$$

and $q_{c,f}$ is an arbitrary tree from $T_{G,Y_m}\cup\Xi_k)$ it is obvious that $\tau_{\mathcal{C}}=\tau_{\mathcal{C}^*}=\tau$ and $\mathfrak{C}\in\text{Alg}(\mathcal{C})$ which proves the θ-completeness of M.

(2) Now let M be a θ-complete system. Take an arbitrary algebra $\mathfrak{A}=\langle A, F\rangle$ with $A=(a_0, a_1, \ldots, a_{n-1})$. Without loss of generality we may assume $n>1$. Choose a $j\in R$ and construct the algebra $\hat{\mathfrak{A}}=\langle A, G\rangle$ with $G_i=F_i$ if $i\ne j$ and $G_j=F_j\cup\{h\}$ where h is a new operational symbol. For all $g\in G$ and $a_i\in A$ realize g such that

$$g^{\hat{\mathfrak{A}}}(a_i)=\begin{cases}(\underbrace{a_{i+1(\text{mod}\,n)}, \ldots, a_{i+1(\text{mod}\,n)}}_{j\text{ times}}) & \text{if } g=h \text{ and}\\ g^{\mathfrak{A}}(a_i) & \text{otherwise.}\end{cases}$$

Define the associated transducer $\hat{\mathcal{A}}=(T_{G,X_1}, A, T_{G,Y_n}, a_0, \Sigma_{\hat{\mathcal{A}}})$ by the following rules for all $g\in G$ and $a_i\in A$

(i) $a_i g \to g(a_1\xi_1, \ldots, a_k\xi_k) \in \Sigma_{\hat{\mathscr{A}}}$ iff
$g^{\hat{\mathfrak{A}}}(a_i) = (a_1, \ldots, a_k)$ and
(ii) $a_i x_1 \to y_{i+1} \in \Sigma_{\hat{\mathscr{A}}}$.

The θ-completeness of M ensures that there exists a θ-product $\mathfrak{B} = \prod_{i=1}^{s} \mathfrak{B}_i[G, \psi]$ from M and a transducer $\mathscr{B} = \langle T_{G,X_1}, B, T_{G,Y_n}, b_0, \Sigma_{\mathscr{B}} \rangle$ equivalent to $\hat{\mathscr{A}}$ such that $\mathfrak{B} \in \mathrm{Alg}(\mathscr{B})$. Take the connected subtransducer $\mathscr{B}^* = \langle T_{G,X_1}, B^*, T_{G,Y_n}, b_0, \Sigma_{\mathscr{B}^*} \rangle$ of $\mathscr{B}$ and the corresponding connected subalgebra $\mathfrak{B}^* = \langle B^*, G \rangle$ of $\mathfrak{B}$. Now we are going to prove that $\hat{\mathfrak{A}}$ is a homomorphic image of this $\mathfrak{B}^*$.

To this end define the correspondence $\varphi: B^* \to A$ by $\varphi(b_0 u^{\mathfrak{B}^*}) = a_0 u^{\hat{\mathfrak{A}}}$ for every $u \in \mathrm{path}(T_{G,X_1})$. Since $\mathfrak{B}^*$ and $\hat{\mathfrak{A}}$ are connected φ is defined for all $b \in B^*$ and $\varphi(B^*) = A$. We claim that φ is a well defined mapping, i.e. $b = b_0 u^{\mathfrak{B}^*} = b_0 v^{\mathfrak{B}^*}$ implies $a_0 u^{\hat{\mathfrak{A}}} = a_0 v^{\hat{\mathfrak{A}}}$ for all $u, v \in \mathrm{path}(T_{G,X_1})$. Assume to the contrary that there are $u, v \in \mathrm{path}(T_{G,X_1})$ such that $b = b_0 u^{\mathfrak{B}^*} = b_0 v^{\mathfrak{B}^*}$ and $a_i = a u^{\hat{\mathfrak{A}}} \neq a_0 v^{\hat{\mathfrak{A}}} = a_j$.

The realization of h ensures the existence of trees $p, q \in T_{G,X_1}$ with the following properties

(i) $u \in \mathrm{path}(p)$, $v \in \mathrm{path}(q)$,
(ii) if $z \in \mathrm{path}(p)$ then $a_0 z^{\hat{\mathfrak{A}}} = a_i$ and
if $w \in \mathrm{path}(q)$ then $a_0 w^{\hat{\mathfrak{A}}} = a_j$.

Then we have

(iii) $\mathrm{fr}(\tau_{\hat{\mathscr{A}}}(p)) \in \{y_{i+1}\}^*$, $\mathrm{fr}(\tau_{\hat{\mathscr{A}}}(q)) \in \{y_{j+1}\}^*$.

Taking two arbitrary trees $\bar{p}, \bar{q} \in T_{G,X_1}$ with $u \in \mathrm{path}(\bar{p})$ and $v \in \mathrm{path}(\bar{q})$ we can construct the trees p, q satisfying (ii) by substituting the leaves of $\bar{p}$ and $\bar{q}$ by suitable trees from $T_{\{h\},X_1}$.

From the equivalence of $\hat{\mathscr{A}}$ and $\mathscr{B}^*$ it follows easily that the transducer $\mathscr{B}^*$ is nondeleting. This, by property (iii) means that during the translation of p in $\mathscr{B}^*$ we have to apply some production $bx_1 \to t$ where $\mathrm{fr}(t) \in \{y_{i+1}\}^*$. On the other hand, the translation of q requires a production $bx_1 \to \bar{t}$ with $\mathrm{fr}(\bar{t}) \in \{y_{j+1}\}^*$. Hence, by the assumption $a_i \neq a_j$ the contradiction $bx_1 \to t$, $bx_1 \to \bar{t} \in \Sigma_{\mathfrak{B}^*}$ and $t \neq \bar{t}$ follows.

At last, by the definition of φ

$$f(\varphi(b)) = f(a_0 v^{\hat{\mathfrak{A}}}) = (a_0 v(f,1)^{\hat{\mathfrak{A}}}, \ldots, a_0 v(f,k)^{\hat{\mathfrak{A}}}) =$$
$$= (\varphi(b_0 v(f,1)^{\mathfrak{B}^*}), \ldots, \varphi(b_0 v(f,k)^{\mathfrak{B}^*})) = \varphi(f(b)) \quad \text{holds}$$

for all $v \in \mathrm{path}(T_{G,X_1})$, $b = b_0 v^{\mathfrak{B}^*} \in B^*$, $k \in R$ and $f \in F_k$ proving that φ is a homomorphism. Now it is evident that $\mathfrak{A}$ is a homomorphic image under φ of the subalgebra $\bar{\mathfrak{B}}^* = \langle B^*, F \rangle$ of the θ-product $\bar{\mathfrak{B}} = \prod_{i=1}^{s} \mathfrak{B}_i[F, \bar{\psi}]$ where $\bar{\psi}$ is the restriction of ψ to $\prod_{i=1}^{s} B_i \times F$.

(3) It is quite obvious that every homomorphically θ-complete system M is forest complete with respect to the θ-product as well (cf. Proposition 1.1).

(4) At last, assume that M is a forest complete system with respect to the θ-product. Take an arbitrary algebra $\mathfrak{A} = \langle A, F \rangle$ with $A = \{a_0, \ldots, a_{n-1}\}$. Choosing

a new operational symbol h and proceeding in the same way as in case (2) construct the algebra $\hat{\mathfrak{A}}=\langle A, G\rangle$. The definition of h ensures that the automaton $\hat{\mathbf{A}}=\langle\hat{\mathfrak{A}}, a_0, \{a_0\}\rangle$ is connected and normalized. Moreover, by the proof of Theorem 8 in [5] $\hat{\mathbf{A}}$ is a minimal automaton since it has no two different equivalent states.

The forest completeness of M implies the existence of an automaton $\mathbf{C}=\langle\mathfrak{C}, c', \mathbf{c}\rangle$ equivalent to $\hat{\mathbf{A}}$ where $\mathfrak{C}=\prod_{i=1}^{t}\mathfrak{C}_i[G, \psi]$ is a θ-product from M. As the realization of h results that every connected automaton equivalent to $\hat{\mathbf{A}}$ is normalized and, even more, it has no 0-states, the connected subautomaton $\mathbf{C}^*=\langle\mathfrak{C}^*, c', \mathbf{c}^*\rangle$ of $\mathbf{C}$ is normalized, too. Therefore, by Proposition 1.2 the minimal automaton $\hat{\mathbf{A}}$ is a homomorphic image of $\mathbf{C}^*$. Now it is trivial that omitting h the algebra $\mathfrak{A}=\langle A, F\rangle$ is a homomorphic image of the subalgebra $\bar{\mathfrak{C}}^*=\langle C^*, F\rangle$ of the θ-product $\bar{\mathfrak{C}}=\prod_{i=1}^{t}\mathfrak{C}_i[F, \bar{\psi}]$, where $\bar{\psi}$ is the restriction of the feedback function ψ to $\prod_{i=1}^{t}C_i\times F$. $\square$

3. Complete systems with respect to some special types of products

In this section we shall investigate the isomorphically θ-complete systems if $\theta=Q$, α_0 and G, and derive some properties of the homomorphically G-complete systems as well.

For the sake of brevity let us introduce the relation $\mathfrak{A}\prec_\theta\mathfrak{B}$ iff $\mathfrak{A}$ can be isomorphically embedded into a θ-product of $\mathfrak{B}$ with a single factor. When $\theta=Q$ we have

Theorem 3.1. A system $K\subseteq K(R)$ is isomorphically complete with respect to the quasidirect product iff for every simple algebra $\mathfrak{A}$ there is a $\mathfrak{B}\in K$ such that $\mathfrak{A}\prec_Q\mathfrak{B}$ holds.

Proof. The sufficiency of the condition can easily be derived from the transitivity of the relation $\prec_Q$ and from the following assumption

(∗) For an arbitrary algebra $\mathfrak{C}$ a simple algebra $\mathfrak{A}$ satisfying $\mathfrak{C}\prec_Q\mathfrak{A}$ can be constructed.

To verify (∗), take the algebra $\mathfrak{C}=\langle C, F\rangle$, $C=\{c_0, ..., c_k\}$. We define the algebra $\mathfrak{A}=\langle A, G\rangle$ as follows. The base set of $\mathfrak{A}$ is the disjoint union $A=C\cup\{c_{k+1},...,c_{p-1}\}$, where p is an arbitrary prime number with $p-1>k$. Suppose that $j\in R$. In this case let $G_i=F_i$ for all $i\neq j$ and $G_j=F_j\cup\{h\}$ where h is a new operational symbol. The realization of the operations in $\mathfrak{A}$ is given by

$$g^{\mathfrak{A}}(c_i)=\begin{cases} g^{\mathfrak{C}}(c_i) & \text{if } g\in F \text{ and } 0\leqq i\leqq k,\\ \underbrace{(c_i, ..., c_i)}_{n\text{ times}} & \text{if } n\in R,\ g\in F_n \text{ and } k<i\leqq p-1,\\ \underbrace{(c_{i+1(\mathrm{mod}\,p)}, ..., c_{i+1(\mathrm{mod}\,p)})}_{j\text{ times}} & \text{if } g=h \text{ and } 0\leqq i\leqq p-1.\end{cases}$$

The introduction of the new operation h guarantees the simplicity of $\mathfrak{A}$. $\mathfrak{C} \prec_Q \mathfrak{A}$ follows evidently by considering the product $\mathfrak{A}[F, \psi]$ with the feedback function $\psi(f)=f$ for every $f \in F$.

Conversely, let K be isomorphically Q-complete. Hence for arbitrary simple algebra $\mathfrak{A}=\langle A, F\rangle$ there is a Q-product $\mathfrak{B}=\prod_{i=1}^{k} \mathfrak{B}_i[F, \psi]$ from K such that $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{B}$. Let φ denote a suitable isomorphism. Now we can introduce the relations ϱ_i $(1 \leqq i \leqq k)$ on A in the following manner:

$$a \varrho_i b \quad \text{iff} \quad \pi_j(\varphi(a)) = \pi_j(\varphi(b)) \quad \text{for all} \quad 1 \leqq j \leqq i.$$

The fact that $\mathfrak{B}$ is a Q-product yields that all the ϱ_i are congruence relations and

$$\iota \supseteqq \varrho_1 \supseteqq \dots \supseteqq \varrho_k = \omega.$$

As $\mathfrak{A}$ is simple, all this relations are trivial, i.e., there is a natural number m $(1 \leqq m \leqq k)$ such that

$$\varrho_m = \omega$$

holds. Now we proceed to show that in this case $\mathfrak{A} \prec_Q \mathfrak{B}_m$. Take the Q-product $\mathfrak{C}=\langle B_m, F\rangle=\mathfrak{B}_m[F, \xi]$ with the feedback function $\xi(f)=\pi_m(\psi(f))$ for all $f \in F$. It can immediately be shown that the mapping $\eta \colon A \to B_m$ defined by $\eta(a)=\pi_m(\varphi(a))$ for all $a \in A$ is an isomorphic embedding of $\mathfrak{A}$ into $\mathfrak{B}_m$. The choice of m ensures the injectivity of η. On the other hand, η is the composition of the m^{th} projection with the isomorphism φ, hence η must be a homomorphism.

Corollary 3.2. There exists no minimal isomorphically Q-complete system of algebras.

Proof. Take an isomorphically Q-complete system $M \subseteqq K(R)$ and an arbitrary $\mathfrak{C}$ from M. We shall verify that the system $M_1=M-\{\mathfrak{C}\}$ satisfies the conditions of Theorem 3.1 as well. Let $\mathfrak{B}$ be a simple algebra. From the isomorphic completeness of M it follows that $\mathfrak{B} \prec_Q \mathfrak{A}$ holds for some $\mathfrak{A} \in M$. Now we claim that $\mathfrak{B} \prec_Q \overline{\mathfrak{A}}$ holds for some $\overline{\mathfrak{A}} \in M_1$, too. We distinguish the following two cases

(1) If $\mathfrak{A} \neq \mathfrak{C}$, then we put $\overline{\mathfrak{A}}=\mathfrak{A}$.

(2) In the case of $\mathfrak{A}=\mathfrak{C}$ we can, by assumption $(*)$, construct a simple algebra $\mathfrak{D}$ with $|\mathfrak{C}|<|\mathfrak{D}|$ and $\mathfrak{C} \prec_Q \mathfrak{D}$. But M is isomorphically Q-complete thus it contains an algebra $\mathfrak{E}$ satisfying $\mathfrak{D} \prec_Q \mathfrak{E}$. Of course, $\mathfrak{A} \neq \mathfrak{E}$ hence $\mathfrak{E} \in M_1$ and the transitivity of $\prec_Q$ implies $\mathfrak{B} \prec_Q \mathfrak{E}$. □

In the case of α_0-products we can state similar results.

Theorem 3.3. A system $K \subseteqq K(R)$ is isomorphically complete with respect to the α_0-product iff for every simple algebra $\mathfrak{A}$ there is a $\mathfrak{B} \in K$ satisfying $\mathfrak{A} \prec_{\alpha_0} \mathfrak{B}$.

Proof. The equivalence $\mathfrak{A} \prec_Q \mathfrak{B}$ iff $\mathfrak{A} \prec_{\alpha_0} \mathfrak{B}$ combined with Theorem 3.1 obviously implies the sufficiency of the condition.

The proof of the necessity can be performed as in Theorem 3.1 so it will be omitted. □

Inspecting Theorems 3.1, 3.2 and 3.3 we can infer that there exist no minimal isomorphically α_0-complete systems. Moreover, a system $K \subseteqq K(R)$ is isomorphically Q-complete iff it is isomorphically α_0-complete.

For isomorphic G-completeness we have

Theorem 3.4. A system $K \subseteq K(R)$ is isomorphically complete with respect to the general product iff K contains an algebra $\mathfrak{A} = \langle A, F \rangle$ having two distinct elements a_1 and a_2 such that for arbitrary $r \in R$, $a \in \{a_1, a_2\}$ and $\mathbf{a} \in \{a_1, a_2\}^r$ there exists an $f \in F_r$ satisfying $f^{\mathfrak{A}}(a) = \mathbf{a}$.

Proof. Suppose that K is isomorphically G-complete. Let $\mathfrak{B} = \langle \{b_1, b_2\}, G \rangle$ be an algebra such that for all $r \in R$, $b \in B$ and $\mathbf{b} \in \{b_1, b_2\}^r$ there exists a $g \in G_r$ with $g^{\mathfrak{B}}(b) = \mathbf{b}$. Because of the G-completeness of K there is a G-product $\mathfrak{A} = \prod_{i=1}^{k} \mathfrak{A}_i[G, \psi]$ from K and a subalgebra $\bar{\mathfrak{A}}$ of $\mathfrak{A}$ satisfying $\varphi(\mathfrak{B}) = \bar{\mathfrak{A}}$ under a suitable isomorphism φ. Let $(a_1', \ldots, a_k')$ and $(a_1'', \ldots, a_k'')$ be the φ-image of b_1 and b_2, respectively. Because of $b_1 \neq b_2$ an index j satisfying $a_j' \neq a_j''$ can be selected. We shall prove that in this case the algebra $\mathfrak{A}_j$ fulfils the conditions of the Theorem. To this end take an $r \in R$, $a \in \{a_j', a_j''\}$, and $\mathbf{a} \in \{a_j', a_j''\}^r$. The algebra $\mathfrak{B}$ satisfies the requirements of the Theorem as well. Hence it can be given a $g \in G_r$, $b \in B$ and $\mathbf{b} \in B^r$ with the properties $g^{\mathfrak{B}}(b) = \mathbf{b}$, $\pi_j(\varphi(b)) = a$ and $(\pi_j(\varphi(b_1)), \ldots, \pi_j(\varphi(b_r))) = \mathbf{a}$. From these equalities we can conclude that the operation $f = \psi^{(j)}(\varphi(b), g) \in F_r^j$ satisfies $f^{\mathfrak{A}_j}(a) = \mathbf{a}$.

Now assume that the elements a_1, a_2 of the algebra $\mathfrak{A} = \langle A, F \rangle \in K$ meet the requirements of the Theorem. Take an arbitrary algebra $\mathfrak{B} = \langle B, G \rangle$. Choose an injective mapping $\varphi\colon B \to \{a_1, a_2\}^k$ for a suitable $k \in N$, and construct the G-product $\mathfrak{C} = \prod_{i=1}^{k} \mathfrak{A}_i[G, \psi]$ where $\mathfrak{A}_i = \mathfrak{A}$ $(i = 1, \ldots, k)$. For all $\mathbf{c} = (c_1, \ldots, c_k) \in A^k$, $r \in R$ and $g \in G_r$ let $\psi(\mathbf{c}, g) = (f^1, \ldots, f^k)$ be defined for all $i = 1, \ldots, k$ by

$$f^i = \begin{cases} \text{an } f \in F_r \text{ satisfying } f^{\mathfrak{A}}(c_i) = (c_i^1, \ldots, c_i^r), \\ \text{if } \mathbf{c} = \varphi(b), \quad g^{\mathfrak{B}}(b) = (b_1, \ldots, b_r) \text{ and } \varphi(b_j) = (c_1^j, \ldots, c_k^j) \\ \text{for } j = 1, \ldots, r, \\ \text{an arbitrary element from } F_r \text{ otherwise.} \end{cases}$$

By virtue of this definition of the feedback function ψ an easy computation shows that φ is an isomorphic embedding of $\mathfrak{B}$ into the product $\mathfrak{C}$. □

By Theorem 3.4, there exists an algorithm to decide for arbitrary finite $K \subseteq K(R)$ whether K is isomorphically complete with respect to the general product.

Turning to the problem of homomorphic G-completeness we give a rephrased version of the known result from [8] for the case $R = \{1\}$ i.e., for unoids.

Proposition 3.5. A system $K \subseteq K(\{1\})$ is homomorphically complete with respect to the general product iff K contains a unoid $\mathfrak{A} = \langle A, F \rangle$ having an element a, two operational symbols f_1, f_2 and two polynomials p_1, p_2 satisfying

$$a_1 = f_1(a) \neq f_2(a) = a_2$$

and

$$p_1(a_1) = p_2(a_2) = a.$$

This proposition implies that every minimal homomorphically G-complete system in $K(\{1\})$ is a singleton. The subsequent constructions show that this situation is bounded to the case $R = \{1\}$.

Let $R=\{r_1, \dots, r_l\} \neq \{1\}$ be an arbitrary rank type. For every $r \in R$ and $1 \leqq j \leqq r$ take a two-element algebra $\mathfrak{A}_{rj} = \langle \{a^1_{rj}, a^2_{rj}\}, F^{rj} \rangle$ having for every $(m, n) \in \in \{1, 2\}^2$ exactly one r-ary operational symbol $f_{mn} \in F^{rj}_r$ such that

$$(*) \qquad f_{mn}(a^k_{rj}) = \begin{cases} (a^m_{rj}, \dots, \underbrace{a^n_{rj}}_{j^{\text{th}}}, \dots, a^m_{rj}) & \text{if } k = m \\ (a^k_{rj}, \dots, a^k_{rj}) & \text{otherwise} \end{cases}$$

and

$$(**) \qquad g(a^m_{rj}) = (a^m_{rj}, \dots, a^m_{rj}) \quad \text{for all } a^m_{rj} \in A_{rj}, \quad g \in F^{rj} \quad \text{and} \quad g \notin F^{rj}_r$$

hold.

Lemma 3.6. The system $K = \{\mathfrak{A}_{rj} | r \in R, \ 1 \leqq j \leqq r\}$ is homomorphically G-complete and minimal.

Proof. Let $\mathfrak{C} = \langle \{1, 2\}, F \rangle \in K(R)$ be an algebra satisfying

$$F = \bigcup(\{f_{st} | s \in \{1, 2\}, \mathbf{t} \in \{1, 2\}^{r_i}\} | r_i \in R)$$

and

$$f_{st}(k) = \begin{cases} \mathbf{t} & \text{if } s = k \\ (\underbrace{s, \dots, s}_{r_i \text{ times}}) & \text{otherwise} \end{cases}$$

for all $r_i \in R$ and $f_{st} \in F_{r_i}$.

Since the system $\{\mathfrak{C}\}$ is isomorphically G-complete to prove our lemma it is enough to show $\mathfrak{C} \in H_G(K)$. To this end take the G-product $\mathfrak{A} = \langle A, F \rangle = \prod_{\substack{r \in R \\ 1 \leqq j \leqq r}} \mathfrak{A}_{rj}[F, \psi]$. If $\mathbf{a} \in A$ then let $\nu(\mathbf{a})$ denote the sum of the upper indices occurring in $\mathbf{a}$. For all $r_i \in R$, $f_{st} \in F_{r_i}$, $\mathfrak{A}_{rj} \in K$ and $\mathbf{a} \in A$, ψ corresponds to f_{st} the operation $f_{sv} \in F^{rj}_{r_i}$ where

$$v = \begin{cases} \pi_j(\mathbf{t}) & \text{if } r_i = r \text{ and } (-1)^{\nu(\mathbf{a})} = (-1)^s, \\ s & \text{otherwise.} \end{cases}$$

Define the mapping $\varphi \colon A \to \{1, 2\}$ in the following way:

$$\varphi(a) = \begin{cases} 1 & \text{if } \nu(a) \text{ is an odd number,} \\ 2 & \text{otherwise.} \end{cases}$$

Now, using the previous definition of ψ, it can be proved that $\varphi \colon \mathfrak{A} \to \mathfrak{C}$ is a homomorphism.

Take an arbitrary algebra $\mathfrak{A}_{ri} \in M$. By virtue of the construction of M it is evident that $\mathfrak{A}_{sj} \in M$ and $(s, j) \neq (r, i)$ implies for all $f \in F^{sj}_r$ and $a \in A_{sj}$, $\pi_i(f(a)) = a$. From this assumption it follows directly that the system $K - \{\mathfrak{A}_{ri}\}$ is not homomorphically G-complete. □

By similar methods one can prove

Theorem 3.7. Let s be an arbitrary natural number satisfying $1 \leqq s \leqq \sum_{r_i \in R} r_i$. Then a minimal homomorphically G-complete system K consisting of s algebras can effectively be constructed.

Finally we would like to remark that in the proof of Lemma 3.1 in [12] there is a mistake. Its correction can be found in [6].

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