A new method for the analysis and synthesis of finite Mealy-automata

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In this paper we deal with the problem of analysis and synthesis of finite Mealy-automata. As it is known, this problem has already been solved, namely, it is proved, that if X and Y are non-empty finite sets and $\alpha: X^* \to Y^*$ is an automaton mapping, then there is a finite Mealy-automaton inducing α if and only if all classes of the partition C_α of X^+ corresponding to α are regular languages (see [3]). For a given automaton mapping, a Mealy-automaton can be constructed inducing it and vice versa, but the known algorithms use, as an intermediate step, the notion of the acceptance of languages in automata without outputs and the synthesis algorithms give no reduced automaton, generally. In this paper we give a new proof of the previous theorem, which provides us more advantageus algorithms for both the analysis and the synthesis of finite Mealy-automata. In the latter case, our method supplies immediately the minimal Mealy-automaton inducing a given finite automaton mapping.

Preliminaries

Let X be a finite non-empty set. We shall denote the algebra of all languages over X by $\mathscr{L}(X)$ and the set of all matrices over $\mathscr{L}(X)$ by $M(X)$. A matrix $\mathbf{N} \in M(X)$ is said to be of type $m \times n$ if it has m rows and n columns. The language in the i-th row and in the j-th column of $\mathbf{N}$ will be denoted by $(\mathbf{N})_{ij}$. Based on the regular operations (addition, multiplication and iteration, denoted by $+$, $\cdot$ and $\{\}$, respectively) in $\mathscr{L}(X)$, we introduce the following operations on $M(X)$. If $L \in \mathscr{L}(X)$ and $\mathbf{N} \in M(X)$, then $L \cdot \mathbf{N}$ and $\mathbf{N} \cdot L$ are language matrices, defined by

$$(L \cdot \mathbf{N})_{ij} = L \cdot (\mathbf{N})_{ij} \quad \text{and} \quad (\mathbf{N} \cdot L)_{ij} = (\mathbf{N})_{ij} \cdot L,$$

respectively. Let $\mathbf{N}$ and $\mathbf{P}$ be two language matrices of the same type. Then the sum $\mathbf{N}+\mathbf{P}$ is the language matrix, given by

$$(\mathbf{N}+\mathbf{P})_{ij} = (\mathbf{N})_{ij} + (\mathbf{P})_{ij}.$$

If $\mathbf{N}$ is a language matrix of type $m \times n$ and $\mathbf{P}$ is another one of type $n \times p$, then we define the product $\mathbf{N}\cdot\mathbf{P}$ in the usual way of matrix products, i.e.,

$$(\mathbf{N}\cdot\mathbf{P})_{ij} = \sum_{k=1}^{n} (\mathbf{N})_{ik}\cdot(\mathbf{L})_{kj}.$$

Using the definition of the product we can form the powers of quadratic matrices as follows: let

$$\mathbf{N}^k = \mathbf{N}^{k-1}\cdot\mathbf{N} \quad (k = 1, 2, \ldots),$$

where $\mathbf{N}^0=\mathbf{E}$ means the unit language matrix, that is,

$$(\mathbf{E})_{ij} = \begin{cases} e & \text{if} \quad i = j, \\ \emptyset & \text{if} \quad i \neq j. \end{cases}$$

Finally, the iteration $\{\mathbf{N}\}$ of a quadratic matrix $\mathbf{N}$ is defined by

$$\{\mathbf{N}\} = \sum_{k=0}^{\infty} \mathbf{N}^k.$$

We note that we use the term language vector instead of language matrix if it has only one row or only one column. The set of all row language vectors over $\mathscr{L}(X)$ will be denoted by $V^r(X)$ and the set of all column language vectors over $\mathscr{L}(X)$ will be denoted by $V^c(X)$.

Let $\mathbf{N}\in M(X)$ be a quadratic matrix of type $n\times n$. Take a directed graph with n nodes, which are labelled by natural numbers $1, \ldots, n$ and there is an arrow from the node i to the node j if and only if $e\in(\mathbf{N})_{ij}$. This graph is called the *characteristic graph* (see [3]) of the matrix $\mathbf{N}$. If the characteristic graph of $\mathbf{N}$ has cycles and the node i belongs to a cycle, then the number i is said to be a *cyclic number* with respect to $\mathbf{N}$.

Now we consider matrix equations of form

$$\mathbf{N}\cdot\mathbf{Q}+\mathbf{P} = \mathbf{Q}, \tag{1}$$

where $\mathbf{N}$ and $\mathbf{P}$ are given language matrices and $\mathbf{N}$ is of type $n\times n$.

We shall use the following results which are generalizations of some results due to V. G. Bodnarčuk [2] (see also [3, 4, 5, 7]):

Statement 1 [6]. *If the characteristic graph of* $\mathbf{N}$ *has no cycle, then*

$$\mathbf{Q} = \{\mathbf{N}\}\cdot\mathbf{P}$$

is the unique solution of the equation (1). *In the opposite case, every solution of* (1) *has the form*

$$\mathbf{Q} = \{\mathbf{N}\}\cdot(\mathbf{P}+\mathbf{R}),$$

where $\mathbf{R}$ *is an arbitrary language matrix with the same type as* $\mathbf{P}$, *such that if* i $(1\leq i\leq n)$ *is not a cyclic number with respect to* $\mathbf{N}$, *then* $(\mathbf{R})_{ij}=\emptyset$ *for all* j.

Statement 2 [6]. *If the equation* (1) *has a unique solution, then it can be determined by subsequent elimination of unknown rows of the matrix* $\mathbf{Q}$.

Statement 3 [6]. *If every element* $(\mathbf{N})_{ij}$ *and* $(\mathbf{P})_{ij}$ *of the matrices* $\mathbf{N}$ *and* $\mathbf{P}$, *respectively, is regular and the characteristic graph of* $\mathbf{N}$ *has no cycle, then every element* $(\mathbf{Q})_{ij}$ *of the solution matrix* $\mathbf{Q}$ *of the matrix equation* (1) *is regular.*

Connections between language vectors and automaton mappings

It is known, that every automaton mapping $\alpha: X^* \to Y^*$, where X and Y are non-empty finite sets, determines a partition C_α of X^+, which consists of classes

$$L_y = \langle p \in X^+ | \overline{\alpha(p)} = y \rangle \quad (y \in Y).$$

Here $\bar{r}$ denotes the last letter of the non-empty word r. Conversely, every partition C of X^+ defines a unique automaton mapping $\alpha: X^* \to Y^*$ appart from the notation of elements of Y. This fact makes possible for us to establish a one-to-one correspondence between automaton mappings and certain language vectors.

In the following we use the term *l-vectors* instead of row language vectors and they will be denoted by $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$.

An *l*-vector $\mathbf{a} \in V^r(X)$ is said to be *complete* if the sum of its components a_i is the free semigroup X^+ and the intersection of any two components a_i and a_j $(i \neq j)$ of $\mathbf{a}$ is the empty language.

It is obvious that if $X = \langle x_1, \ldots, x_l \rangle$, $Y = \langle y_1, \ldots, y_m \rangle$ and $\alpha: X^* \to Y^*$ is an automaton mapping then we can correspond to α a complete *l*-vector $\mathbf{a}$ of m components, such that

$$a_i = \langle p \in X^+ | \overline{\alpha(p)} = y_i \rangle \quad (i = 1, \ldots, m).$$

Conversely, if $\mathbf{a} \in V^r(X)$ is a complete *l*-vector of m components then it determines an automaton mapping $\alpha: X^* \to Y^*$, such that $\alpha(e) = e$ and for $p = x_{i_1} x_{i_2} \ldots x_{i_k}$, $\alpha(p) = y_{j_1} y_{j_2} \ldots y_{j_k}$ if and only if $x_{i_1} \in a_{j_1}, x_{i_1} x_{i_2} \in a_{j_2}, \ldots, x_{i_1} x_{i_2} \ldots x_{i_k} \in a_{j_k}$.

An *l*-vector $\mathbf{a} \in V^r(X)$ is called *regular* if every component of $\mathbf{a}$ is a regular language.

A system $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$ of complete *l*-vectors from $V^r(X)$ is said to be *closed* if there exist functions

$$f: \langle 1, \ldots, n \rangle \times \langle 1, \ldots, l \rangle \to \langle 1, \ldots, n \rangle$$

and

$$\mathbf{b}: \langle 1, \ldots, n \rangle \to V^r(X),$$

such that

$$b(i)_k = \sum_{x_j \in a_{ik}} x_j$$

and

$$\mathbf{a}_i = \sum_{x_j \in X} x_j \mathbf{a}_{f(i,j)} + \mathbf{b}(i) \tag{2}$$

for all i $(=1, \ldots, n)$ holds.

We would like to direct attention to the fact, that a closed complete *l*-vector system can be considered as the rows of a solution language matrix of a matrix equation (1). Indeed, if we set

$$(\mathbf{N})_{ij} = \sum_{f(i,k)=j} x_k \quad (i, j = 1, \ldots, n)$$

and we put $\mathbf{a}_i$ and $\mathbf{b}(i)$ into the i-th row of $\mathbf{Q}$ and $\mathbf{P}$, respectively, then (2) gains the form (1). Therefore, by Statement 3, we have got immediately the following

Lemma 4. *If $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$ is a closed complete l-vector system then $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are regular.*

Theorem 5. *Let $\mathfrak{A}=(A, X, Y, \delta, \lambda)$ be a finite Mealy-automaton with state set $A=\langle a_1, \ldots, a_n\rangle$, input set $X=\langle x_1, \ldots, x_l\rangle$, output set $Y=\langle y_1, \ldots, y_m\rangle$, transition function $\delta: A\times X\to A$ and output function $\lambda: A\times X\to Y$. Let f and* $\mathbf{b}$ *be the functions*

$f: \langle 1, \ldots, n\rangle\times\langle 1, \ldots, l\rangle\to\langle 1, \ldots, n\rangle$, *defined by* $\delta(a_i, x_j)=a_{f(i,j)}$ *and*

$$\mathbf{b}: \langle 1, \ldots, n\rangle \to V^r(X), \quad \text{given by} \quad b(i)_k = \sum_{\lambda(a_i, x_j)=y_k} x_j.$$

Then the l-vectors $\mathbf{a}_i\in V^r(X)$ $(i=1, \ldots, n)$, *where*

$$a_{ik} = \langle p\in X^+ | \overline{\lambda(a_i, p)} = y_k\rangle \quad (i = 1, \ldots, n;\ k = 1, \ldots, m)$$

form a closed complete l-vector system, that is, satisfy the equalities (2).

Proof. It is obvious that $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are complete l-vectors. Thus we have to show that $\mathbf{a}_1, \ldots, \mathbf{a}_n$ satisfy the equalities (2). Let i $(1\leqq i\leqq n)$ and k $(1\leqq k\leqq m)$ be arbitrary index pair. We prove that

$$a_{ik} = \sum_{x_j\in X} x_j a_{f(i,j)k} + b(i)_k.$$

Let p be an arbitrary element of a_{ik}. We distinguish two cases.

Case 1. If $|p|=1$, i.e., $p=x_j$ for some j $(1\leqq j\leqq l)$ then $x_j\in a_{ik}$ implies $\lambda(a_i, x_j)=y_k$. Hence, by definition of $\mathbf{b}$, we have that $x_j\in b(i)_k$ and therefore $p\in \sum_{x_j\in X} x_j a_{f(i,j)k}+b(i)_k$.

Case 2. If $|p|\geqq 2$ then $p=x_j q$ $(1\leqq j\leqq l)$ and

$$y_k = \overline{\lambda(a_i, p)} = \overline{\lambda(a_i, x_j q)} = \overline{\lambda(\delta(a_i, x_j), q)} = \overline{\lambda(a_{f(i,j)}, q)},$$

that is, $q\in a_{f(i,j)k}$ and $p=x_j q\in x_j a_{f(i,j)k}\subseteq \sum_{x_j\in X} x_j a_{f(i,j)k}+b(i)_k$. Conversely, let p be an arbitrary element of $\sum_{x_j\in X} x_j a_{f(i,j)k}+b(i)_k$. If $p\in b(i)_k$, then $|p|=1$ and $\lambda(a_i, p)=y_k$ and therefore $p\in a_{ik}$. If $p\in x_j a_{f(i,j)k}$ for some j $(1\leqq j\leqq l)$, then $p=x_j q$ with $q\in a_{f(i,j)k}$. This implies that

$$y_k = \overline{\lambda(a_{f(i,j)}, p)} = \overline{\lambda(\delta(a_i, x_j), q} = \overline{\lambda(a_i, x_j q)} = \overline{\lambda(a_i, p)},$$

i.e., $p\in a_{ik}$. Thus we have shown that

$$a_{ik} = \sum_{x_j\in X} x_j a_{f(i,j)k} + b(i)_k$$

for all i $(1\leqq i\leqq n)$ and k $(1\leqq k\leqq m)$ holds. □

By Lemma 4 and Theorem 5 we immediately get

Corollary 6. *If $\mathfrak{A}=(A, X, Y, \delta, \lambda)$ is a finite Mealy-automaton then for all state $a\in A$ and output $y\in Y$ the language*

$$a_y = \langle p\in X^+ | \overline{\lambda(a; p)} = y\rangle$$

is regular.

Theorem 5 and Statement 2 provide us an algorithm to the analysis of finite Mealy-automata. To illustrate this, let us consider

Example 1. Let $\mathfrak{A}$ be the Mealy-automaton, given by the transition-output table:

$\mathfrak{A}$	a_1	a_2	a_3
x	(a_2, u)	(a_3, v)	(a_2, u)
y	(a_3, v)	(a_2, w)	(a_3, w)

Taking the ordering $u \prec v \prec w$ of the output letters, by Theorem 5 we have the following l-vector equations:

$$\mathbf{a}_1 = x\mathbf{a}_2 + y\mathbf{a}_3 + [x, y, \emptyset],$$

$$\mathbf{a}_2 = x\mathbf{a}_3 + y\mathbf{a}_2 + [\emptyset, x, y],$$

$$\mathbf{a}_3 = x\mathbf{a}_2 + y\mathbf{a}_3 + [x, \emptyset, y].$$

From the third equation we obtain that

$$\mathbf{a}_3 = \{y\}(x\mathbf{a}_2 + [x, \emptyset, y]).$$

Substituting this into the expressions of $\mathbf{a}_1$ and $\mathbf{a}_2$, we have, that

$$\mathbf{a}_1 = x\mathbf{a}_2 + y\{y\}(x\mathbf{a}_2 + [x, \emptyset, y]) + [x, y, \emptyset] =$$
$$= \{y\}x\mathbf{a}_2 + [\{y\}x, y, y^2\{y\}],$$

$$\mathbf{a}_2 = x\{y\}(x\mathbf{a}_2 + [x, \emptyset, y]) + y\mathbf{a}_2 + [\emptyset, x, y] =$$
$$= (y + x\{y\}x)\mathbf{a}_2 + [x\{y\}x, x, y + x\{y\}y].$$

Now we can already determine the l-vector $\mathbf{a}_2$:

$$\mathbf{a}_2 = \{y + x\{y\}x\}[x\{y\}x, x, y + x\{y\}y] =$$
$$= [\{y + x\{y\}x\}x\{y\}x, \{y + x\{y\}x\}x, \{y + x\{y\}x\}(y + x\{y\}y)]$$

Then

$$\mathbf{a}_1 = [\{y\}x\{y + x\{y\}x\}x\{y\}x, \{y\}x\{y + x\{y\}x\}x, \{y\}x\{y + x\{y\}x\}(y + x\{y\}y)] +$$
$$+ [\{y\}x, y, y^2\{y\}] =$$
$$= [\{y\}x(e + \{y + x\{y\}x\}x\{y\}x), y + \{y\}x\{y + x\{y\}x\}x, y^2\{y\} +$$
$$+ \{y\}x\{y + x\{y\}x\}(y + x\{y\}y)]$$

and

$$\mathbf{a}_3 = \{y\}x\mathbf{a}_2 + [\{y\}x, \emptyset, \{y\}y] =$$
$$= [\{y\}x(e + \{y + x\{y\}x\}x\{y\}x), \{y\}x\{y + x\{y\}x\}x, y\{y\} +$$
$$+ \{y\}x\{y + x\{y\}x\}(y + x\{y\}y)].$$

Now we define a new operation on the set of l-vectors. It is well-known that if L is a language from $\mathscr{L}(X)$ and $p \in X^*$ then the left-side derivation of L with respect to p is the language ${}_lD_p(L)=\langle q\in X^* \mid pq\in L\rangle$. We modify this concept as follows: by the *left-side e-free derivation* of L with respect to p we mean the language ${}_lD_p^-(L)=\langle q\in X^+ \mid pq\in L\rangle$. It is obvious that

$$ {}_lD_p^-(L) = \begin{cases} {}_lD_p(L) & \text{if} \quad p\notin L, \\ {}_lD_p(L)-\langle e\rangle & \text{if} \quad p\in L. \end{cases} $$

We extend this operation to l-vectors, that is, if $\mathbf{a}=[a_1, \ldots, a_m]$ then we define the left-side e-free derivation of $\mathbf{a}$ with respect to p, by

$$ {}_lD_p^-(\mathbf{a}) = [{}_lD_p^-(a_1), \ldots, {}_lD_p^-(a_m)]. $$

Lemma 7. *If* $\mathbf{a}=[a_1, \ldots, a_m]$ *is a complete l-vector then for all* $p\in X^*$, ${}_lD_p^-(\mathbf{a})$ *is a complete l-vector as well.*

Proof. It is easily seen that $a_i\cap a_j=\emptyset$ implies that ${}_lD_p^-(a_i)\cap{}_lD_p^-(a_j)=\emptyset$. On the other hand, if q is an arbitrary element of X^+ then $pq\in X^+$. Consequently, there exist a unique component a_i of $\mathbf{a}$, such that $pq\in a_i$ because of the completeness of $\mathbf{a}$. Hence we obtain that $q\in{}_lD_p^-(a_i)$. Since $e\notin{}_lD_p^-(a_i)$ for all i $(1\leqq i\leqq m)$ holds, it follows that $X^+=\sum_{i=1}^{m}{}_lD_p^-(a_i)$. □

Lemma 8. *If* $\mathbf{a}=[a_1, \ldots, a_m]$ *is an arbitrary l-vector in* $V^r(X)$ *then*

$$ \mathbf{a} = \sum_{x_j\in X} x_j\,{}_lD_{x_j}^-(\mathbf{a})+\mathbf{b}, $$

where $\mathbf{b}=[b_1, \ldots, b_m]$ *is an l-vector for which* $b_i=\sum_{x_j\in a_i} x_j$ $(i=1, \ldots, m)$.

Proof. Let a_i $(1\leqq i\leqq m)$ be an arbitrary component of $\mathbf{a}$. By the definition of $\mathbf{b}$ it is trivial that any word of length one from a_i is in b_i and there is no other element of b_i. On the other hand, the word $p\in a_i$, for which $|p|\geqq 2$, is in $x_j\,{}_lD_{x_j}^-(a_i)$ if and only if the first letter of p is x_j. □

A closed complete l-vector system $\langle \mathbf{a}_1, \ldots, \mathbf{a}_m\rangle$ is said to be *reduced* if $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are pairwise different l-vectors.

Lemma 9. *If* $\mathbf{a}$ *is a complete regular l-vector then there exists a unique reduced closed complete l-vector system containing* $\mathbf{a}$ *and it can be determined algorithmically.*

Proof. If $\mathbf{a}\in V^r(X)$ with $X=\langle x_1, \ldots, x_l\rangle$ then we extend the ordering of X, which is given by the indices of the elements in X onto X^* as follows: for arbitrary pair of words p and q let $p\prec q$ if either $|p|<|q|$ or $|p|=|q|$ and in the latter case p precedes q by the lexicographical ordering. Then we form the left-side e-free derivations of $\mathbf{a}$. Since $\mathbf{a}$ is a regular l-vector, it has only finite different left-side e-free derivations and they are regular as well. Therefore, there exist a system of words $p_1, \ldots, p_n$ in X^*, such that the following conditions hold:

(i) if $i\neq j$ $(1\leqq i, j\leqq n)$ then ${}_lD_{p_i}^-(\mathbf{a})\neq{}_lD_{p_j}^-(\mathbf{a})$,

(ii) for all $q\in X^*$ there exists a unique i $(1\leqq i\leqq n)$, such that ${}_lD_q^-(\mathbf{a})={}_lD_{p_i}^-(\mathbf{a})$,

(iii) if q is an arbitrary word in X^* for which ${}_lD_q^-(\mathbf{a})={}_lD_{p_i}^-(\mathbf{a})$ $(1\leqq i\leqq n)$ then $p_i\prec q$.

Let us assume that the elements of the system $\langle p_1, \ldots, p_n\rangle$ are indexed according to the ordering of X^*, that is, $p_1\prec p_2\prec\ldots\prec p_n$. Then $p_1=e$. Let $\mathbf{a}_i=$

$={}_lD^-_{p_i}(\mathbf{a})$ for all $i(=1, \ldots, n)$. The system $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$ consist of pairwise different complete l-vectors. We show that this system is closed as well. To prove this, we have to note that for all $q \in X^*$ and $\mathbf{a}_i \in \langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$ there exists p_j $(1 \leqq j \leqq n)$, such that ${}_lD^-_q(\mathbf{a}_i) = {}_lD^-_{p_j}(\mathbf{a}_1)$ because $\mathbf{a}_1 = {}_lD^-_{p_1}(\mathbf{a}) = {}_lD^-_e(\mathbf{a}) = \mathbf{a}$ and ${}_lD^-_q(\mathbf{a}_i) = {}_lD^-_q({}_lD^-_{p_i}(\mathbf{a}_1)) = {}_lD^-_{p_iq}(\mathbf{a}_1)$ and the system $\langle p_1, \ldots, p_n \rangle$ satisfy the condition (ii). We have to determine the functions $f: \langle 1, \ldots, n \rangle \times \langle 1, \ldots, l \rangle \to \langle 1, \ldots, n \rangle$ and $\mathbf{b}: \langle 1, \ldots, n \rangle \to V^r(X)$ yet. Let for all i $(1 \leqq i \leqq n)$ and j $(1 \leqq j \leqq l)$,

$$f(i, j) = k \Leftrightarrow {}_lD^-_{x_j}(\mathbf{a}_i) = {}_lD^-_{p_k}(\mathbf{a}_1) \quad (1 \leqq k \leqq n)$$

and

$$\mathbf{b}(i) = [b(i)_1, \ldots, b(i)_m], \quad \text{where} \quad b(i)_s = \sum_{x_j \in a_{is}} x_j \quad (s = 1, \ldots, m).$$

Finally, the fact that $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$ is the unique closed complete l-vector system, which contains the l-vector $\mathbf{a}(=\mathbf{a}_1)$ follows from Lemma 8. □

To illustrate the algorithm described above consider the

Example 2. Let $X = \langle x, y \rangle$ with the ordering $x \prec y$ and take

$$\mathbf{a} = [x\{x\}, y\{y\}, (x\{x\}y + y\{y\}x)\{x+y\}].$$

Let $\mathbf{a}_1 = \mathbf{a}$. Then

$$\mathbf{a}_1 = x[x\{x\}, \emptyset, \{x\}y\{x+y\}] + y[\emptyset, y\{y\}, \{y\}x\{x+y\}] + [x, y, \emptyset].$$

Let $\mathbf{a}_2 = {}_lD^-_x(\mathbf{a}_1) = [x\{x\}, \emptyset, \{x\}y\{x+y\}]$ and $\mathbf{a}_3 = {}_lD^-_y(\mathbf{a}_1) = [\emptyset, y\{y\}, \{y\}x\{x+y\}]$. Then

$$\mathbf{a}_2 = x[x\{x\}, \emptyset, \{x\}y\{x+y\}] + y[\emptyset, \emptyset, (x+y)\{x+y\}] + [x, \emptyset, y].$$

Let $\mathbf{a}_4 = {}_lD^-_y(\mathbf{a}_2) = {}_lD^-_{xy}(\mathbf{a}_1) = [\emptyset, \emptyset, (x+y)\{x+y\}]$.

$$\mathbf{a}_3 = x[\emptyset, \emptyset, (x+y)\{x+y\}] + y[\emptyset, y\{y\}, \{y\}x\{x+y\}] + [\emptyset, y, x]$$

and

$$\mathbf{a}_4 = x[\emptyset, \emptyset, (x+y)\{x+y\}] + y[\emptyset, \emptyset, (x+y)\{x+y\}] + [\emptyset, \emptyset, x+y].$$

It can be seen that ${}_lD^-_{xx}(\mathbf{a}_1) = {}_lD^-_x(\mathbf{a}_1)$, ${}_lD^-_{yy}(\mathbf{a}_1) = {}_lD^-_y(\mathbf{a}_1)$ and ${}_lD^-_{yx}(\mathbf{a}_1) = {}_lD^-_{xyx}(\mathbf{a}_1) = {}_lD^-_{xyy}(\mathbf{a}_1) = {}_lD^-_{xy}(\mathbf{a}_1)$, that is, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{a}_4$ are the all different left-side e-free derivations of $\mathbf{a}$. Finally, the functions $f: \langle 1, 2, 3, 4 \rangle \times \langle 1, 2 \rangle \to \langle 1, 2, 3, 4 \rangle$ and $\mathbf{b}: \langle 1, 2, 3, 4 \rangle \to V^r(X)$ are derived from the previous computations: $f(1, 1) = 2$, $f(1, 2) = 3$, $f(2, 1) = 2$, $f(2, 2) = 4$, $f(3, 1) = 4$, $f(3, 2) = 3$ and $f(4, 1) = f(4, 2) = 4$, furthermore $\mathbf{b}(1) = [x, y, \emptyset]$, $\mathbf{b}(2) = [x, \emptyset, y]$, $\mathbf{b}(3) = [\emptyset, y, x]$ and $\mathbf{b}(4) = [\emptyset, \emptyset, x+y]$.

Theorem 10. *Let $X = \langle x_1, \ldots, x_l \rangle$, $Y = \langle y_1, \ldots, y_m \rangle$ and let $\alpha: X^* \to Y^*$ be an automaton mapping. Let $\mathbf{a}_1$ be the complete l-vector corresponding to α. If $\mathbf{a}_1$ is a regular l-vector and $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$ is the reduced closed complete l-vector system containing $\mathbf{a}_1$ then α can be induced by the reduced initially connected Mealy-automaton $\mathfrak{A} = (\langle a_1, \ldots, a_n \rangle, a_1, X, Y, \delta, \lambda)$, where $\delta(a_i, x_j) = a_{f(i,j)}$ and $\lambda(a_i, x_j) = y_k$ if and only if $x_j \in b(i)_k$ $(i = 1, \ldots, n; j = 1, \ldots, l; 1 \leqq k \leqq m)$ and the functions f and $\mathbf{b}$ are determined by the system $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$.*

Proof. It is obvious that $\mathfrak{A}$ is well-defined. Since every l-vector $\mathbf{a}_i$ of the system $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$ is a left-side e-free derivation of $\mathbf{a}_1$ and $\mathbf{a}_i = {}_lD_p^-(\mathbf{a}_1)$ implies that $a_1 p = a_i$, it follows that $\mathfrak{A}$ is an initially connected Mealy-automaton. We have to prove that $\mathfrak{A}$ induces the mapping α. To verify this, it is sufficient to show that

$$a_{1s} = \langle p \in X^+ | \overline{\lambda(a_1, p)} = y_s \rangle$$

for all $s(=1, \ldots, m)$ holds. Instead of this equality, we prove more, namely that

$$a_{is} = \langle p \in X^+ | \overline{\lambda(a_i, p)} = y_s \rangle \tag{3}$$

for all $i(=1, \ldots, n)$ and $s(=1, \ldots, m)$ holds. If $|p|=1$, that is, $p = x_j$ for some $x_j \in X$ then by the definitions of the functions $\mathbf{b}$ and λ we obtain that

$$x_j \in a_{is} \Leftrightarrow x_j \in b(i)_s \Leftrightarrow \lambda(a_i, x_j) = y_s.$$

Let us assume that (3) have already been proved for all $p \in X^+$ of length less than or equal to r, for all i $(=1, \ldots, n)$ and s $(=1, \ldots, m)$. Now let $p \in X^+$, such that, $|p| = r+1$. Then $p = x_j q$ for some $x_j \in X$ and $|q| = r$. Thus taking into account that the system $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$ is closed and the previous hypothesis, we obtain that

$$p \in a_{is} = \sum_{x_t \in X} x_t a_{f(i,t)s} + b(i)_s \Leftrightarrow q \in a_{f(i,j)s} \Leftrightarrow$$

$$\Leftrightarrow \overline{\lambda(a_{f(i,j)}, q)} = y_s \Leftrightarrow \overline{\lambda(\delta(a_i, x_j), q)} = y_s \Leftrightarrow \overline{\lambda(a_i, p)} = y_s.$$

Therefore, (3) is true. But this means that $\mathbf{a}_i$ is just the l-vector corresponding to the automaton mapping induced by the state a_i of $\mathfrak{A}$ for all $i(=1, \ldots, n)$. Thus, the fact that the system $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$ is reduced implies that $\mathfrak{A}$ is reduced as well. □

To show how we can apply this result for the synthesis of finite Mealy-automata consider

Example 3. Let $X = \langle x, y \rangle$, $Y = \langle u, v, w \rangle$ and let $\alpha \colon X^* \to Y^*$ be the automaton mapping, given by

$$\alpha(e) = e,$$

$$\alpha(x^k) = u^k \quad (k \geqq 1),$$

$$\alpha(y^k) = v^k \quad (k \geqq 1),$$

$$\alpha(x^k y p) = u^k w^{m+1} \quad (k \geqq 1,\ m = |p|),$$

$$\alpha(y^k x p) = v^k w^{m+1} \quad (k \geqq 1,\ m = |p|).$$

Then the complete l-vector corresponding to α is just the regular l-vector $\mathbf{a}_1 = [x\{x\}, y\{y\}, (x\{x\}y + y\{y\}x)\{x+y\}]$ from Example 2. Thus the mapping α can be induced by the automaton $\mathfrak{A} = (\langle a_1, a_2, a_3, a_4 \rangle, a_1, X, Y, \delta, \lambda)$, where δ and λ is given by the transition-output table:

$\mathfrak{A}$	a_1	a_2	a_3	a_4
x	(a_2, u)	(a_2, u)	(a_4, w)	(a_4, w)
y	(a_3, v)	(a_4, w)	(a_3, v)	(a_4, w).

Summarizing the results of Theorems 5 and 10, we have got

Corollary 11. *If X and Y are finite non-empty sets and $\alpha: X^* \to Y^*$ is an automaton mapping then it can be induced by a finite Mealy-automaton if and only if the complete l-vector corresponding to α is regular.*

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