On the lattice of clones acting bicentrally

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1. Introduction

For a set F of operations on a set A the centralizer F^* of F is the set of operations on A commuting with every member of F. If $F=F^{**}$ then we say that F acts bicentrally. The sets of operations on A acting bicentrally forms a complete lattice $\mathscr{L}_A$ with respect to $\subseteqq$.

The sets of operations acting bicentrally were characterized in [5] and [11]. For $|A|=3$ the lattice $\mathscr{L}_A$ is completely described in [2] and [3]. The aim of this paper is to investigate the lattice $\mathscr{L}_A$. Among others we show that for any set A there exists a single operation f such that $\{f\}^{**}$ is the set of all operations of A (Theorem 5). Furthermore, it is proved that if $B\subseteqq A$ then $\mathscr{L}_B$ can be embedded into $\mathscr{L}_A$ (Corollary 7).

2. Preliminaries

Let A be an at least two element set which will be fixed in the sequel. The set of n-ary operations on A will be denoted by $O_A^{(n)}$ $(n\geqq 1)$. Furthermore, we set $O_A=\bigcup_{n=1}^{\infty} O_A^{(n)}$. A set $F\subseteqq O_A$ is said to be a *clone* if it contains all projections and is closed with respect to superpositions of operations. Denote by $[F]$ the clone generated by F. Let f and g be operations of arites n and m, respectively. If M is an $m\times n$ matrix of elements of A, we can apply f to each row of M to obtain a column vector consisting of m elements, which will be denoted by $f(M)$. Similarly, we can apply g to each column of M to obtain a row vector of n elements, which will be denoted by $(M)g$. We say that f and g *commute* if for every $m\times n$ matrix M over A, we have $(f(M))g=f((M)g)$.

By the *centralizer* of a set $F\subseteqq O_A$ we mean the set $F^*\subseteqq O_A$ consisting of all operations on A that commute with every member of F. It can be shown by a simple computation that $F^*=[F]^*=[F^*]$ for every $F\subseteqq O_A$. The mapping $F\to F^*$ defines a Galois-connection between the subsets of O_A. Indeed, $F_1\subseteqq F_2$ implies $F_1^*\supseteqq F_2^*$ and $F\subseteqq (F^*)^*=F^{**}$ for every $F_1, F_2, F\subseteqq O_A$. From this it follows that $F^*=F^{***}$ for every $F\subseteq O_A$. Thus the mapping $F\to F^{**}$ is a closure operator on the subsets of O_A. The set F^{**} is called the bicentralizer of F. If $F=F^{**}$ then we say that F acts bicentrally. The sets of operations on A acting bicentrally form a complete lattice with respect to $\subseteq$. Denote by $\mathscr{L}_A$ this lattice. In $\mathscr{L}_A$ we have $\bigwedge_{i\in I} F_i=\bigcap_{i\in I} F_i$, $\bigvee_{i\in I} F_i=(\bigcup_{i\in I} F_i)^{**}$ and $(\bigvee_{i\in I} F_i)^*=\bigwedge_{i\in I} F_i^*$, $(\bigwedge_{i\in I} F_i)^*=\bigvee_{i\in I} F_i^*$. It follows that the mapping $F\to F^*(F\in\mathscr{L}_A)$ is a dual automorphism of $\mathscr{L}_A$.

The set of all projections, and the set of all injective unary operations on A will be denoted by P_A and S_A, respectively. An operation $f\in F$ is said to be homogeneous if $f\in S_A^*$. The symbol H_A denotes the set of all homogeneous operations, i.e., $H_A=S_A^*$.

We say that an operation $f\in O_A$ is parametrically expressible or generated by a set $F\subseteq O_A$ if the predicate $f(x_1, ..., x_n)=y$ is equivalent to a predicate of the form

$$(\exists t_1)\dots(\exists t_l)\big((A_1=B_1)\wedge\dots\wedge(A_m=B_m)\big)$$

where A_i and B_i contain only operation symbols from F, variables $x_1, ..., x_n$, $y, t_1, ..., t_l$, commas and round brackets.

For $3\leqq n\leqq|A|$ denote by l_n the n-ary near-projection, i.e. the n-ary operation defined as follows:

$$l_n(x_1, ..., x_n)=\begin{cases}x_1 & \text{if } \ x_i\neq x_j, \quad 1\leqq i<j\leqq n,\\ x_n & \text{otherwise.}\end{cases}$$

We need the ternary dual discriminator-function d which is defined in the following way:

$$d(x, y, z)=\begin{cases}x & \text{if } \ y\neq z,\\ z & \text{if } \ y=z.\end{cases}$$

If $f\in O_A$ and $B\subseteq A$ then f_B denotes the restriction of f to B.

3. Results

First we give two examples. For every subset $X\subseteq A$ let C_X be the set of all unary constant operations with value belonging to X. Furthermore, let I_X be the set of all operations $f\in O_A$ for which $f(x, ..., x)=x$ for every $x\in X$.

Example 1. For every subset $X\subseteq A$ we have $C_X^*=I_X$ and $I_X^*=[C_X]$. In particular, $P_A^*=O_A$ and $O_A^*=P_A$.

Proof. $C_X^*=I_X$ and $I_X^*\supseteq[C_X]$ are obvious. Now let $f\in I_X^*$ be an n-ary operation and suppose that $f\notin[C_X]$. Then f is neither a projection nor a constant operation with value belonging to X. Therefore there are elements $a_{i1}, ..., a_{in}\in A$, $i=1, ..., n+2$, such that $a_i=f(a_{i1}, ..., a_{in})\neq a_{ii}$, $i=1, ..., n$, and $(a_{n+1}, a_{n+2})=\big(f(a_{n+1,1}, ..., a_{n+1,n}), f(a_{n+2,1}, ..., a_{n+2,n})\big)\notin\{(x, x)|\, x\in X\}$. Let $M=(a_{ij})_{(n+2)\times n}$. Since $(a_1, ..., a_{n+2})\notin\{(x, ..., x)|x\in X\}$, and $(a_1, ..., a_{n+2})$ is distinct from each column of M, there exists an $(n+2)$-ary operation $g\in I_X$ such that $\big(f(M)\big)g=g(a_1, ..., a_{n+2})\neq f\big((M)g\big)$, showing that f and g do not commute and $f\notin I_X^*$. This contradiction shows that $I_X^*\subseteq[C_X]$. Hence $I_X^*=[C_X]$.

Finally if $X=\emptyset$ then we have $I_X=O_A$ and $[C_X]=[\emptyset]=P_A$. □

Example 2. If $|A|\geqq 3$ then $(S_A\cup C_A)^*=H_A$ and $H_A^*=[S_A\cup C_A]$.

Proof. It is well known that $H_A\subseteqq I_A$ if $|A|\geqq 3$ (see e.g. [1]). Therefore $(S_A\cup C_A)^*=S_A^*\cap C_A^*=H_A\cap I_A=H_A$. In [10] it is proved that $[S_A\cup C_A]$ acts bicentrally. Thus $H_A^*=((S_A\cup C_A)^*)^*=[S_A\cup C_A]^{**}=[S_A\cup C_A]$. □

For $|A|=2$, E. Post [8] described the lattice of clones over A. Using this result the lattice $\mathscr{L}_A$ can be determined by routine. Figure 1 is the diagram of $\mathscr{L}_A$ in case $|A|=2$. (We use the notation of [9]).

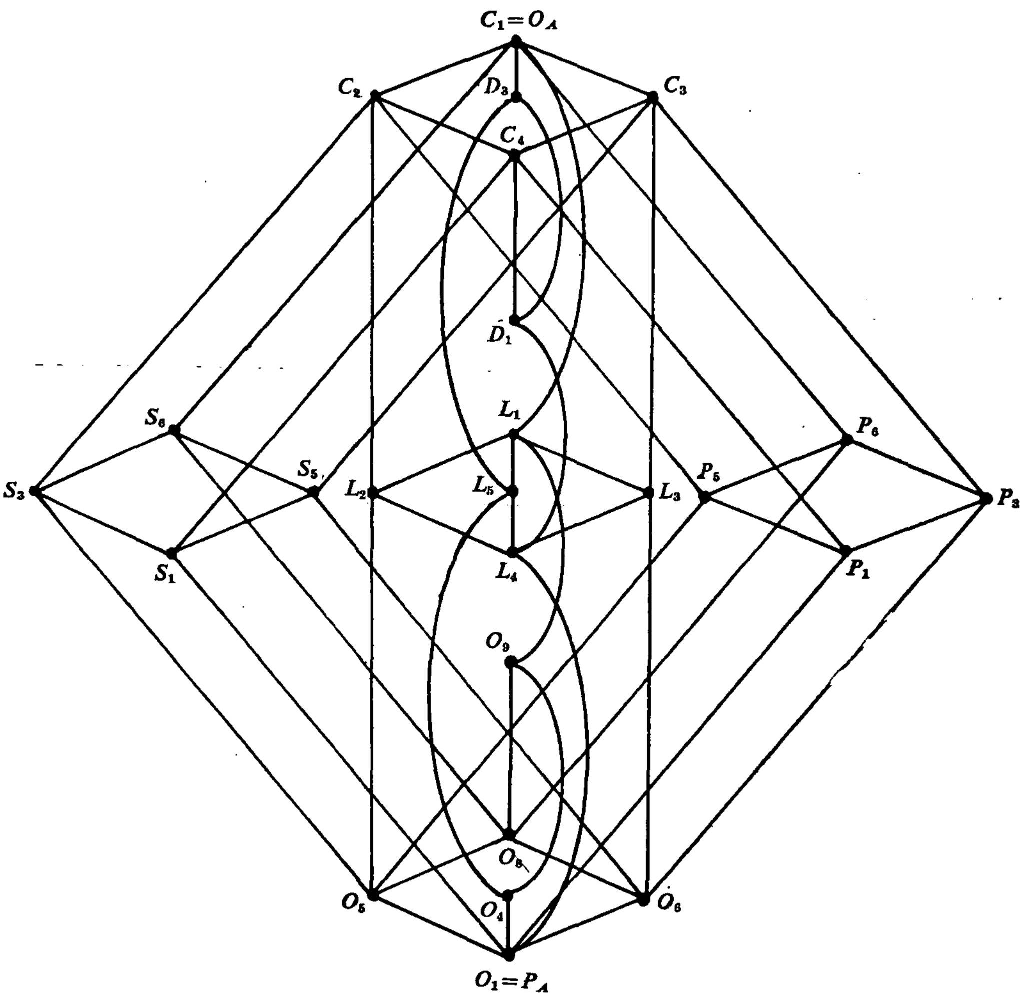

Fig. 1.

Considering the diagram we can observe the following facts: if $|A|=2$ then $\mathcal{L}_A$ has 25 elements, six atoms $(O_4, O_5, O_6, S_1, P_1, L_4)$, and six dual atoms $(D_3, C_2, C_3, S_6, P_6, L_1)$. Remark that the dual automorphism $F\to F^*$ coincides with the reflection of the diagram with respect to the axis S_3-P_3.

For $|A|=3$, $\mathcal{L}_A$ is a finite lattice of power 2986 and it has 44 atoms and dual atoms (see [2], [3] and [4]).

In general we have the following.

Theorem 3. If A is a finite set, then the closure operator $F\to F^{**}$ is algebraic, and $\mathcal{L}_A$ is an atomic and dually atomic algebraic lattice. If A is infinite, then the closure operator $F\to F^{**}$ is not algebraic.

Proof. First let A be a finite set. A. V. Kuznecov showed in [5] that $F=F^{**}$ if and only if F contains every operation parametrically generated by F. From this it follows that the closure operator $F\to F^{**}$ is algebraic. Thus $\mathcal{L}_A$ is an algebraic lattice. It is well-known that there are finite sets $F\subseteq O_A$ such that $F^{**}=O_A$ (see e.g. [4]). Therefore $\mathcal{L}_A$ is dually atomic. Since $\mathcal{L}_A$ is dually isomorphic to itself, it is atomic, too.

A. F. Danil'cenko proved in [4] that if $|A|\geqq 3$ then every dual atom of $\mathcal{L}_A$ is of the form $\{f\}^*$ where $f\in O_A$ is an at most $|A|$-ary operation. From this it follows that $\mathcal{L}_A$ has finitely many dual atoms and atoms (the numbers of atoms and dual atoms are equal).

Now let A be an infinite set and let $x_1, x_2, \ldots\in A$ be pairwise distinct elements. Put $X_i=\{x_i, x_{i+1}, \ldots\}$, $i=1,2,\ldots$. Then, by Example 1, $I_{X_i}\in\mathcal{L}_A$, $i=1,2,\ldots$ and clearly $I_{X_1}\subseteq I_{X_2}\subseteq\ldots$. Furthermore $\bigcup_{i=1}^{\infty} I_{X_i}\neq O_A$ and $(\bigcup_{i=1}^{\infty} I_{X_i})^{**}=(\bigcap_{i=1}^{\infty} I_{X_i}^*)^*=$ $=(\bigcap_{i=1}^{\infty}[C_{X_i}])^* P_A^*=O_A$. It follows that the closure operator $F\to F^{**}$ is not algebraic. □

Theorem 4. If $|A|\geqq 5$, then H_A is an atom and $[S_A\cup C_A]$ is a dual atom in $\mathcal{L}_A$.

Proof. First we show that if d is the ternary dual discriminator and l_n $(3\leqq n\leqq |A|)$ is a near-projection then $\{d\}^*=\{l_n\}^*=[S_A\cup C_A]$. The inclusions $\{d\}^*\supseteq[S_A\cup C_A]$ and $\{l_n\}^*\supseteq[S_A\cup C_A]$ are obvious. Let $f\in O_A\setminus[S_A\cup C_A]$ be an m-ary operation. If f depends on one variable only then we can assume without loss of generality that f is a unary operation. Since f is non-injective and non-constant, there are pairwise distinct elements $a, b, c\in A$ such that $f(a)\neq f(b)=f(c)$. Furthermore choose elements $x_4, \ldots, x_n\in A$ such that $a, b, c, x_4, \ldots, x_n$ are pairwise distinct. Then $f(d(a,b,c))=f(a)\neq f(c)=d(f(a), f(b), f(c))$ and $f(l_n(a, b, x_4, \ldots$ $\ldots, x_n, c))=f(a)\neq f(c)=l_n(f(a), f(b), f(x_4), \ldots, f(x_n), f(c))$ showing that f does not commute with d and l_n, i.e. $f\notin\{d\}^*$ and $f\notin\{l_n\}^*$. Now suppose that f depends on at least two variables, among others on the first. Therefore there are elements $a_2, \ldots, a_n\in A$ such that the unary operation $g(x)=f(x, a_2, \ldots, a_n)$ is not a constant. If f takes on at most $n-1$ elements from A then g is not injective. Therefore $g\notin\{d\}^*$ and $g\notin\{l_n\}^*$. From this it follows that $f\notin\{d\}^*$ and $f\notin\{l_n\}^*$. Finally suppose that f takes on at least $n\,(\geqq 3)$ values. Since f depends on at least two variables, there are elements $a_1, \ldots, a_m, b_1, \ldots, b_m, a, b, c\in A$ such that a, b and c are pairwise distinct and $a=f(a_1, \ldots, a_m)$, $b=f(b_1, a_2, \ldots, a_m)$, $c=f(a_1, b_2, \ldots, b_m)$

(see e.g. [6]). Then $d(f(a_1, b_2, \dots, b_m), f(b_1, a_2, \dots, a_m), f(a_1, \dots, a_m)) = d(c, b, a) = c \neq a = f(a_1, \dots, a_m) = f(d(a_1, b_1, a_1), d(b_2, a_2, a_2), \dots, d(b_m, a_m, a_m))$ showing that $f \notin \{d\}^*$. Finally, since f takes on at least n values, there are elements $x_{i1}, \dots, x_{im} \in A$, $i = 4, \dots, n$, such that $a, b, c, x_4, \dots, x_n$ are pairwise distinct elements where $x_i = f(x_{i1}, \dots, x_{im})$. Now consider the following $n \times m$ matrix M:

$$M = \begin{pmatrix} x_{41} \dots & x_{4m} \\ \vdots & \vdots \\ x_{n1} \dots & x_{nm} \\ a_1\, a_2 \dots & a_m \\ b_1\, a_2 \dots & a_m \\ a_1\, b_2 \dots & b_m \end{pmatrix}.$$

Then $(f(M))l_n = l_n(x_4, \dots, x_n, a, b, c) = x_4 \neq c = f(a_1, b_2, \dots, b_m) = f((M)l_n)$ showing that f and l_n do not commute. This completes the proof of the equalities $\{d\}^* = [S_A \cup C_A]$ and $\{l_n\}^* = [S_A \cup C_A]$.

Now we are ready to prove the theorem. Since $H_A^* = [S_A \cup C_A]$, it is enough to show that H_A is an atom in $\mathcal{L}_A$, i.e. for any nontrivial operation $f \in H_A$ we have $\{f\}^{**} = H_A$ or equivalently $\{f\}^* = [S_A \cup C_A]$. In [1] and [7] it is shown that if $|A| \geqq 5$ then every non-trivial clone of homogeneous operations contains the dual discriminator or a near-projection. Therefore, if $f \in H_A$ is a non-trivial operation and $d \in [\{f\}]$ then $[S_A \cup C_A] \subseteq \{f\}^* = [\{f\}]^* \subseteq \{d\}^* = [S_A \cup C_A]$. If $l_n \in [\{f\}]$ for some $n \geqq 3$, then $[S_A \cup C_A] \subseteq \{f\}^* = [\{f\}]^* \subseteq \{l_n\}^* = [S_A \cup C_A]$. Hence $\{f\}^* = [S_A \cup C_A]$, which completes the proof. □

Theorem 5. There exists a function $f \in O_A$ such that $\{f\}^{**} = O_A$.

Proof. If A is a finite set then let $f \in O_A$ be a Sheffer function, i.e. an operation f for which $[\{f\}] = O_A$. Then $[\{f\}^{**} = [\{f\}]^{**} = O_A^{**} = O_A$.

Now let A be an infinite set. In [12] it is proved that there exists a binary rigid relation ϱ on A (ϱ is rigid if the identity operation is the only unary operation preserving ϱ). Choose a rigid relation ϱ and define a binary operation h as follows: $h(x, y) = x$ if $(x, y) \in \varrho$ and $h(x, y) = y$ if $(x, y) \notin \varrho$. We show that $\{h\}^* \cap S_A = \{id_A\}$. Indeed, let $t \in S_A$ and $t \neq id_A$. Then there is a pair $(x, y) \in \varrho$ such that $(t(x), t(y)) \notin \varrho$. Clearly $x \neq y$, since otherwise the unary constant operation $A \to \{x\}$ preserves ϱ. It follows that $t(h(x, y)) = t(x) \neq t(y) = h(t(x), t(y))$ and $t \notin \{h\}^*$.

Let $g \in O_A$ be a fixed point free permutation whose cycles are all infinite. Furthermore, let $a, b \in A$ with $a \neq b$.

Now we are ready to define an operation f such that $\{f\}^{**} = O_A$. Let

$$f(x, y, z, u) = \begin{cases} g(x) & \text{if } x = y = z = u, \\ d(y, z, u) & \text{if } y = g(x), \\ h(z, u) & \text{if } x = g(y), \\ a & \text{if } y = g(g(x)), \\ b & \text{if } x = g(g(y)), \\ x & \text{otherwise.} \end{cases}$$

Denote by c_a and c_b the unary constant operations with values a and b, respectively. Then $g, d, h, c_a, c_b \in [\{f\}]$ since $f(x, x, x, x) = g(x)$, $f(x, g(x), y, z) = d(x, y, z)$, $f(g(x), x, x, y) = h(x, y)$, $f(x, g(g(x)), x, x) = c_a(x)$ and $f(g(g(x)),$

$x, x, x)=c_b(x)$. If $t\in\{f\}^*$ then $t\in\{d, h, c_a, c_b\}^*$. Since $t\in\{d\}^*$, by Theorem 4, $t\in[S_A\cup C_A]$. We can suppose that t is unary. If $t\in S_A$ then $t\in\{h\}^*$ implies $t=id_A$. If $t\in C_A$, i.e. t is a constant operation with value x_0 then we have that $a=c_a(t(a))=$ $=t(c_a(a))=x_0=t(c_b(a))=c_b(t(a))=b$ which is a contradiction. Thus we have $\{f\}^*=P_A$ and $\{f\}^{**}=P_A^*=O_A$. □

Let $B\subseteq A(B\neq\emptyset)$ and let s be a mapping from A onto B such that $s(b)=b$ for every $b\in B$. For any operation $f\in O_B^{(n)}, n\geqq 1$, let us define an operation $f^S\in O_A$ as follows: $f^S(a_1, ..., a_n)=f(s(a_1), ..., s(a_n))$ for any $a_1, ..., a_n\in A$. For any $F\subseteq O_B$ let $F^S=P_A\cup\{f^S | f\in F\}$.

Theorem 6. Let $F\subseteq O_B$ such that $id_B\in F$. Then $(F^S)^{**}=(F^{**})^S$. In particular, if $F=F^{**}$ then $F^S=(F^S)^{**}$.

Proof. We shall prove the theorem through some statements:

(1) $s\in F^S$ and $s\in(F^S)^*$.

Since $id_B\in F$, we have $s=id_B^S\in F^S$. Let $g\in F$. If $g\in P_A$ then, clearly, s commutes with g. If $g=f^S$ for some $f\in F$, then for any $a_1, ..., a_n\in A$ we have $s(g(a_1, ..., a_n))=s(f^S(a_1, ..., a_n))=s(f(s(a_1), ..., s(a_n))=f(s(s(a_1)), ..., s(s(a_n)))=$ $=g(s(a_1), ..., s(a_n))$. Hence s commutes with g and $s\in(F^S)^*$.

(2) If $g\in(F^S)^*$ then g preserves B.

Indeed, if g is n-ary and $b_1, ..., b_n\in B$ then $g(b_1, ..., b_n)=g(s(b_1), ..., s(b_n))=$ $=s(g(b_1, ..., b_n))\in B$.

(3) $g\in(F^S)^*$ if and only if $g_B\in F^*$ and g commutes with s.

First suppose that $g\in(F^S)^*$. Then g commutes with s, since $s\in F^S$. If $f\in F$, then g commutes with f^S. By (2), we have $g_B\in O_B$, and clearly the restriction of f^S to B coincides with f. These facts imply that g_B commutes with f. Hence $g_B\in F^*$. Now suppose that $g_B\in F^*$, g commutes with s, and $f^S\in F^S(f\in F)$. Let g and f be m-ary and n-ary, respectively, and choose arbitrary elements $a_{i1}, ..., a_{im}\in A$, $i=1, ..., n$. Then

$$\begin{aligned}f^S(g(a_{11}, ..., a_{1m}), ..., g(a_{n1}, ..., a_{nm})) &= f(s(g(a_{11}, ..., a_{1m})), ..., s(g(a_{n1}, ..., a_{nm}))) =\\ &= f(g_B(s(a_{11}), ..., s(a_{1m})), ..., g_B(s(a_{n1}), ..., s(a_{nm}))) =\\ &= g_B(f(s(a_{11}), ..., s(a_{n1}))), ..., f(s(a_{1m}), ..., s(a_{nm})) =\\ &= g(f^S(a_{11}, ..., a_{n1}), ..., f^S(a_{1m}, ..., a_{nm})).\end{aligned}$$

Hence g commutes with f^S and $g\in(F^S)^*$.

(4) If $f\in F^*$ then $f^S\in(F^S)^*$.

Clearly, the restriction f_B^S to B coincides with f, and f^S commutes with s. Therefore, by (3), we have $f^S\in(F^S)^*$.

(5) If $g\in(F^S)^{**}$ then $g\in P_A$ or g maps into B.

Suppose $g\in(F^S)^{**}\setminus P_A$ is an n-ary operation which takes on a value from $A\setminus B$. Since g is not a projection, for every $i\in\{1, ..., n\}$ there are $a_{i1}, ..., a_{in}\in A$ such that $a_i=g(a_{i1}, ..., a_{in})\neq a_{ii}$. Furthermore let $a_{n+1,1}, ..., a_{n+1,n}\in A$ such that $g(a_{n+1,1}, ..., a_{n+1,n})=a_{n+1}\notin B$. Let us define an $(n+1)$-ary operation $h\in O_A$ as follows:

$$h(x_1, ..., x_{n+1}) = \begin{cases} s(a_{n+1}) & \text{if } (x_1, ..., x_{n+1}) = (a_1, ..., a_{n+1}),\\ x_{n+1} & \text{otherwise.}\end{cases}$$

Then h commutes with s, and h_B, being a projection, belongs to F^*. Therefore, by (3), $h \in (F^S)^*$. Now $g(h(a_{11}, \ldots, a_{n+1,1}), \ldots, h(a_{1n}, \ldots, a_{n+1,n})) = g(a_{n+1,1}, \ldots, a_{n+1,n}) = a_{n+1} \neq s(a_{n+1}) = h(a_1, \ldots, a_{n+1}) = h(g(a_{11}, \ldots, a_{1n}), \ldots, g(a_{n+1,1}, \ldots, a_{n+1,n}))$. It follows that g does not commute with h, which is a contradiction.

(6) If $g \in (F^S)^{**}$ then g preserves B.

This follows from (5)

(7) If $g \in (F^S)^{**}$ then $g_B \in F^{**}$.

Let $g \in (F^S)^{**}$ and let f be an arbitrary operation in F^*. Then, by (4), we have that g commutes with f^S. Taking into consideration (6), this implies that g_B $(\in O_B)$ commutes with f (the restriction of f^S to B). It follows that $g_B \in F^{**}$.

Now we are ready to prove the theorem. First let $g \in (F^S)^{**}$. If $g \in P_A$ then clearly $g \in (F^{**})^S$. Suppose that $g \notin P_A$ and let $g_B = f$. Taking into consideration (5), (1) and (7), we have that g maps into B, g commutes with s, and $f \in F^{**}$. Thus if g is n-ary then for any $a_1, \ldots, a_n \in A$ we have $g(a_1, \ldots, a_n) = s(g(a_1, \ldots, a_n)) = g(s(a_1), \ldots, s(a_n)) = f(s(a_1), \ldots, s(a_n))$ showing that $g = f^S$ and $g \in (F^{**})^S$. Finally let $g \in (F^{**})^S$. If $g \in P_A$ then $g \in (F^S)^{**}$. If $g \notin P_A$ then there is an $f \in F^{**}$ such that $g = f^S$. Take an arbitrary operation h from $(F^S)^*$. Then, by (3), h commutes with s and $h_B \in F^*$. It follows that h_B commutes with f $(h_B \in F^* = (F^{**})^*)$. Let g and h be m-ary and n-ary, respectively, and choose arbitrary elements $a_{i1}, \ldots, a_{im} \in A$, $i = 1, \ldots, n$. Now

$$h(g(a_{11}, \ldots, a_{1m}), \ldots, g(a_{n1}, \ldots, a_{nm})) = h_B(f(s(a_{11}), \ldots, s(a_{1m})), f(s(a_{n1}), \ldots, s(a_{nm}))) =$$

$$= f(h_B(s(a_{11}), \ldots, s(a_{n1})), \ldots, h_B(s(a_{1m}), \ldots, s(a_{nm}))) =$$

$$= f(h(s(a_{11}), \ldots, s(a_{n1})), \ldots, h(s(a_{1m}), \ldots, s(a_{nm}))) =$$

$$= f(s(h(a_{11}, \ldots, a_{n1})), \ldots, s(h(a_{1m}, \ldots, a_{nm}))) = g(h(a_{11}, \ldots, a_{n1}), \ldots, h(a_{1m}, \ldots, a_{nm})).$$

It follows that g commutes with h and $g \in (F^S)^{**}$. □

Corollary 7. The mapping $F \to F^S$ from $\mathcal{L}_B$ into $\mathcal{L}_A$ is an isomorphism.

Proof. From Theorem 6 it follows that if $F \in \mathcal{L}_B$ then $F^S \in \mathcal{L}_A$. Observe that $(F_1 \cap F_2)^S = F_1^S \cap F_2^S$ and $(F_1 \cup F_2)^S = F_1^S \cup F_2^S$ for any $F_1, F_2 \in \mathcal{L}_B$. Therefore taking into consideration Theorem 6, for any $F_1, F_2 \in \mathcal{L}_B$ we have that $(F_1 \wedge F_2)^S = (F_1 \cap F_2)^S = F_1^S \cap F_2^S = F_1^S \wedge F_2^S$ and $(F_1 \vee F_2)^S = ((F_1 \cup F_2)^{**})^S = ((F_1 \cup F_2)^S)^{**} = (F_1^S \cup F_2^S)^{**} = F_1^S \vee F_2^S$. Finally, it is obvious that the mapping $F \to F^S$ is injective. □

Corollary 8. If $s \neq id_A$ then $[\{s\}]$ is an atom in $\mathcal{L}_A$.

Proof. Let $P_B \subseteq O_B$ be the set of projections on B. Then $P_B^S = [\{s\}]$ and therefore, by Theorem 6, $[\{s\}] \in \mathcal{L}_A$. It is trivial that $[\{s\}]$ is an atom in $\mathcal{L}_A$. □

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