On the number of zero order interpolants

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1. Introduction

Our motivation for determining the set of all interpolants of arbitrarily given sentences φ and ψ is twofold, both originating in computer science.

Firstly, according to the well-known method of Floyd—Hoare in the theory of program verification, a program (or more precisely, a program schema) must be associated by so called assertions, which are, actually, first order open formulae. This association can be partially mechanized; the difficulty arises in associating assertions to loops. If φ is the assertion immediately before the loop and ψ is the one immediately after it, then the assertion associated to the loop is not so easy to look for. One possible escape is provided by the theory of interpolation: the assertion to be associated to the loop must be an interpolant of φ and ψ. The celebrated model theoretic result of W. Craig states the existence of an interpolant if φ and ψ are first order sentences and φ is a logical consequence of ψ. In the above mentioned problem, however, one needs more than one (possibly, all of the) interpolants to support the choice of the loop-assertion, on the one hand, and then, obviously, he must generalize to open formulae. At the first stage of this process, we aim the investigation of the set of all interpolants of any two first order sentences φ and ψ. Our method is traditional: we reduce φ and ψ into the zero order language, where matters are very much smoother. Thus, algorithmic generation of the set of all zero order interpolants of any two zero order sentences, the topic of the present paper, is a part of our treatment of the first order case.

Our second motivation can be paraphrased as follows: on the zero order level, an interpolant of φ and ψ can be considered as a generalization (or a relativization) of the well-known concept of "implicant". Indeed, taking φ as the false formula, the set of interpolants of φ and ψ concides with the set of implicants of ψ. This observation provides us with the possibility to consider "implicants of ψ relative to φ", which, in turn, may yield to a better understanding of synthesis problems of truth-functions and automata.

These considerations, however, will remain in the background in the present paper and will be published elsewhere. Our purpose here is much simpler: to investigate the case of zero order sentences and to present an algorithm which returns the set of all interpolants of arbitrarily given zero order sentences.

The method employed here is based on the isomorphism between the zero order Lindenbaum—Tarski algebra and the Boolean algebra of truth-functions associated to the equivalence classes of zero order sentences.

By an interpolant of φ and ψ, we mean a zero order sentence χ which is an interpolant in the sense of Craig [1] and χ is equivalent neither to φ nor to ψ; i.e. χ is proper. According to this strengthening, Craig's Theorem on the existence of (proper) interpolants no longer holds without additional assumptions: it may well happen, that for fixed φ and ψ, no proper interpolant exists: i.e. any interpolant (which exists in the sense of Craig) is equivalent to either φ or ψ.

To study the Boolean algebra of truth functions, we shall use trees. To every truth function, we associate a binary tree, the "valuation tree" of the function at hand. The valuation tree associated to a function is a compressed form of the truth-table of that function. Being so, the tree contains every information (up to logical equivalence) about the function [2]; and since interpolants are defined by means of logical consequence, the trees associated to (the arbitrarily given) φ and ψ contain every information about the set of their interpolants. On the other hand, the "geometrical content" of trees gives us the possibility of expressing semantical properties of functions, and in particular, of interpolants in a simple and "visualizable" way. Additionally, an easy method is imposed to calculate the exact number as well as the number and length of maximal chains of equivalence classes of interpolants. The conditions under which proper interpolants exist are formulated in terms of trees; they have, however, a natural and easily comprehensible meaning for sentences, too.

The organization of the paper is as follows. In the next section, we concretize our terminology and notations. In Section 3 we give conditions which are equivalent to the existence of proper interpolants. The method developed there will be applied to obtain our main results in Section 4 on the number of interpolants and chains of interpolants, respectively. We conclude a next to trivial consequence on the algebraic structure of interpolants in Section 5. Finally, we reformulate our results for sentences in terms of model theory, in Section 6.

2. Preliminaries

Throughout the paper we keep fixed a countably infinite set S, which will play the role of sentence symbols when we are dealing with formulae, while in case of truth functions, S will be considered as a set of variables.

2.1. Let F be the set of zero order sentences over S. Let $\equiv$ denote the logical equivalence relation on F. Clearly, $\equiv$ is an equivalence relation indeed; let us denote by $[\varphi]$ the equivalence class containing φ ($\varphi\in F$). It is well-known, that $\mathcal{F}=\langle F/\equiv, \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra, the so called Lindenbaum—Tarski algebra of F, [1], where 0 denotes the class of unsatisfiable elements of F while 1 stands for the class of valid ones; the operations being defined in the natural way: $\neg[\varphi]=[\neg\varphi]$, $[\varphi]\wedge[\psi]=[\varphi\wedge\psi]$, $[\varphi]\vee[\psi]=[\varphi\vee\psi]$.

2.2. Let $B=\bigcup_{n\in\omega} B_n$, where $B_n=\{f\mid f\colon 2^n\to 2;\ 2=\{0,1\}\}$, the set of Boolean functions of finite number of variables taken from S. By an assignment we mean an element of the set ${}^{\omega}2=\{\langle\xi_0,\xi_1,\ldots\rangle\mid\xi_i\in\{0,1\}$ for $i\in\omega\}$. The value of $f\in B$

under an assignment $\xi \in {}^\omega 2$ (in notation: $f(\xi)$) is obtained firstly, by substituting for all $i \in \omega$, the i-th component ξ_i of ξ for the i-th variable s_i $(\in S)$ everywhere in f provided s_i occurs in f (otherwise the i-th component of ξ has no effect on the value of f) and secondly, by calculating that value. We say, that f and $g\,(\in B)$ are equivalent, in notation: $f \sim g$, iff $f(\xi)=g(\xi)$ for all $\xi \in {}^\omega 2$. It follows, that $\sim$ is an equivalence relation over B; the equivalence classes are denoted as those in F: i.e. for $f \in B$, the equivalence class containing f is denoted by $[f]$. We shall use the symbols 0 and 1 in B, too: $0=\{f \mid f(\xi)=0$ for all $\xi \in {}^\omega 2\}$ and $1=\{f \mid f(\xi)=1$ for all $\xi \in {}^\omega 2\}$. For $g, f \in B$, we can define the operations $+, \cdot$, and $^-$(bar) as follows: for $\xi \in {}^\omega 2, f(\xi)+g(\xi)=\max\{f(\xi), g(\xi)\}$, $f(\xi)\cdot g(\xi)= =\min\{f(\xi), g(\xi)\}$ and $\bar{f}(\xi)=1-g(\xi)$, respectively. Since $\sim$ is compatible with these operations, we can carry them over classes in $B/\sim$: $\overline{[f]}=[\bar{f}]$, $[f]\cdot[g]=[f\cdot g]$, $[f]+[g]=[f+g]$. What is obtained is the well-known Boolean algebra $\mathscr{B}= =\langle B/\sim, \cdot, +, ^-, 0, 1\rangle$. Obviously, $\mathscr{F}$ is isomorphic to $\mathscr{B}$. For the sake of simplicity, from now on, when we speak about functions, we shall tacitly mean the equivalence classes they do represent, and we shall omit brackets in notations, i.e. $f \in \mathscr{B}$ is always to be understood as $[f] \in B/\sim$. Legality of this seemingly abuse of terminology will be justified in Section 5, Theorem 14.

2.3. By a full binary tree of level n $(n \in \omega)$ we mean an ordered pair $T=\langle V, E\rangle$ where V, the set of vertices is defined by

$$V=\bigcup_{j=0}^{n} \bigcup_{k=1}^{2^j} \{V_{jk}\}$$

and E, the set of edges is

$$E=\{\langle V_{jk}, V_{tl}\rangle \mid t=j+1,\ l=2\cdot k-\beta \text{ where } 0 \leqq j \leqq n-1,\ 1 \leqq k \leqq 2^j,\ \beta \in \{0, 1\}\}.$$

In particular, if $n=0$, then $V=\{V_{01}\}$, $E=\emptyset$, i.e. the full binary tree of level 0 is a point. The indices j, k of a vertex $V_{jk} \in V$ mean that V_{jk} is the k-th point of T on the j-th level. We shall label the edge $\langle V_{jk}, V_{(j+1)(2k-\beta)}\rangle$ by s_{j+1}^{β}. Note, that the label s_{j+1}^{β} does not depend on k.

Let $T=\langle V, E\rangle$ be a full binary tree of level n. By a path p in T we mean a sequence of vertices $V_{0k_0}, V_{1k_1}, \ldots, V_{nk_n}$ such that $k_0=1$ and for all $j\,(0 \leqq j \leqq n-1)$, $\langle V_{jk_j}, V_{(j+1)k_{j+1}}\rangle \in E$. The set of paths in T will be denoted by P_T. Clearly, $\operatorname{card} P_T=2^n$. If $P \subseteqq P_T$, then P determines in the natural way a subtree of T. If we write "T_1 is a tree of level n", then we always mean, that T_1 is determined by a subset of paths P of a full binary tree T of level n. Similarly, "T_1 is a subtree of T_2" is to be understood, as both, T_1 and T_2 are determined by subsets P_1 and P_2 of a full binary tree T such that $P_1 \subseteqq P_2$ (i.e. T_1, T_2 and T are of the same level). The set of all subtrees of a full binary tree T will be denoted by $\operatorname{Sub} T$, and in each element of $\operatorname{Sub} T$, the vertices will be indexed by the same indices as they were in T. If $T_1 \in \operatorname{Sub} T$ and $T_1 \neq T$, then we write $T_1 \subset T$. Similar notation applies to arbitrary binary tree. Obviously, if T is a full binary tree of level n, then $\operatorname{card}(\operatorname{Sub} T)=2^{2^n}$.

Let $T=\langle V, E\rangle$ be a full binary tree of level n and $\langle V_{0k_0}, \ldots, V_{jk_j}, \ldots, V_{nk_n}\rangle$ be a path of T. By $\operatorname{FBT}(V_{jk})$ we mean a subtree of T, the vertices of which is determined by the set

$$\{V_{0k_0}, \ldots, V_{jk_j}\} \cup \bigcup_{t=j+1}^{n} \bigcup_{r=(k_j-1)2^{t-j}+1}^{k_j 2^{t-j}} \{V_{tr}\}$$

and the set of edges is defined in the natural way; in other words, $\mathrm{FBT}(V_{jk})$ is determined by those paths of T, the initial segment of which is $\langle V_{0k_0}, \ldots, V_{jk_j}\rangle$ and are continued in all possible ways allowed by T.

2.4. Let $n\in\omega$ and T be a full binary tree of level n. We can define a mapping $\tau_1\colon B_n/\sim \to \mathrm{Sub}\, T$ by the following recurrence. Let $s_i^{\alpha_i}=s_i$ if $\alpha_i=1$ and otherwise $s_i^{\alpha_i}=\bar{s}_i$.

(i) $\tau_1(0)=\emptyset$, $\tau_1(1)=T$.

(ii) If $f=s_1^{\alpha_1}\cdot \ldots \cdot s_n^{\alpha_n}\in B_n$, then let $p=\langle V_{01}, \ldots, V_{nk_n}\rangle$ be that path of T for which $\langle V_{jk_j}, V_{(j+1)(k_j+1)}\rangle$ is labelled by $s_{j+1}^{\alpha_{j+1}}$ for all $j\,(0\le j\le n-1)$ and define $\tau_1(f)=p$.

(iii) Let $g=f_1+f_2+\ldots+f_m$ where each f_j is of the form $s_{j_1}^{\alpha_{j_1}}\cdot \ldots \cdot s_{j_n}^{\alpha_{j_n}}$ and define

$$\tau_1(g) = \bigcup_{j=1}^{m} \tau_1(f_j).$$

Since the cardinalities of $B_n/\sim$ and $\mathrm{Sub}\, T$ are equal, and every $g\in B_n$ has a form, determined uniquely up to the ordering of the variables, required by the clauses of the recursion, it follows that τ_1 is one-one and onto.

Let us define $\tau_0\colon B_n/\sim \to \mathrm{Sub}\, T$ by $\tau_0(f)=\overline{\tau_1(f)}$ where $\overline{\tau_1(f)}$ denotes a subtree of T determined by all paths of T which is not contained in $\tau_1(f)$; i.e. by the complement of $\tau_1(f)$ with respect to P_T. We have immediately,

Lemma 1. For all $f\in B_n/\sim$

(i) $\tau_0(f)=\tau_1(\bar{f})$,

(ii) $\tau_1(f)=\overline{\tau_1(\bar{f})}$.

Lemma 2 [4, Theorem 1]. Let $T_1\in\mathrm{Sub}\, T$ and assume, that T_1 is determined by the set of paths $\{p_1, \ldots, p_r\}$ and let $s_{1j}^{\alpha_{1j}}, \ldots, s_{nj}^{\alpha_{nj}}$ be the labels associated to the edges in p_j. Then,

$$\tau_1\left(\sum_{k=1}^{r}\left(\prod_{j=1}^{n} s_{jk}^{\alpha_{jk}}\right)\right) = T_1.$$

We call $\tau_1^{-1}(T_1)=\sum_{k=1}^{r}\left(\prod_{j=1}^{n} s_{jk}^{\alpha_{jk}}\right)$ the function to which T_1 is associated. Using Lemma 1 above, the dual of this assertion is easily obtained. In the sequel when speaking about associating a tree T to a function $f\in\mathscr{B}$ it will always mean the tree assigned by τ_1. (The duals of the assertions will not be mentioned because of being obtainable immediately.)

2.5. Let $f\in\mathscr{B}$. We say, that f does not depend on the variable $s_j\in S$, in other words, s_j is dummy for f, iff s_j occurs in f and for all $\xi, \xi'\in {}^{\omega}2$ for which $\xi'_j=1-\xi_j$ and $\xi'_k=\xi_k$ if $k\ne j$ we have $f(\xi)=f(\xi')$. It is easy to construct an algorithmic function δ, such that for all $f\in\mathscr{B}$, $\delta(f)$ is the set of variables occuring in f which are not dummy for f. Clearly, dummy variables do not effect the values of functions and thus they can freely be omitted or introduced when necessary. Let $p_1=\langle V_{0k_0}, \ldots, V_{jk_j}, V_{(j+1)k_{j+1}}, \ldots, V_{nk_n}\rangle$ and $p_2=\langle V_{0k_0}, \ldots, V_{jk_j}, V_{(j+1)l_{j+1}}, \ldots, V_{n,l_n}\rangle$ be two paths in a full binary tree T. We say, that p_1 and p_2 are amicable paths w.r.t. j iff all pairs of edges of the form $\langle V_{rk_r}, V_{(r+1)k_{r+1}}\rangle$ and $\langle V_{rl_r}, V_{(r+1)l_{r+1}}\rangle$

are labelled by the same label (which, of course depends on r) provided $r \neq j$ and either $l_{j+1}=k_{j+1}+1$ or $k_{j+1}=l_{j+1}+1$.

A path $p=\langle V_{0k_0}, \dots, V_{j,k_j}, \dots, V_{n,k_n}\rangle$ goes through V_{r,k_r} iff for some $j\,(0 \leqq j \leqq n)$ $r=j$.

By definitions, we have

Lemma 3 [2, Special case of Theorem 15]. Let $f \in \mathscr{B}$ and assume that $T_1=\langle V_1, E_1\rangle$ is the tree associated to f. Then, for some $j\,(1 \leqq j \leqq n)$, s_j is dummy for f iff for all k such that $V_{(j-1)k} \in V_1$, all amicable paths w.r.t. $j-1$ going through $V_{(j-1)k}$ are paths of T_1.

2.6. Let $f, g \in \mathscr{B}$. We shall use the following notations: Δ_{fg} for $\delta(f) \cap \delta(g)$, the set of variables which are not dummy in both f and g. Let $\Phi_{fg}=\delta(f)-\Delta_{fg}$ and $\Gamma_{fg}=\delta(g)-\Delta_{fg}$, the sets of variables which are not dummy for f but do not occur in g and for g but do not occur in f, respectively. For the sake of convenience, we shall denote the elements of Δ_{fg} by $x_0, x_1, \dots$, the elements of Φ_{fg} by $y_0, y_1, \dots$ and the elements of Γ_{fg} by $z_0, z_1, \dots$ throughout the paper; e.g. any appearence of x_j will always be meant as an element of $\Delta_{fg} \cap S$ e.t.c. Moreover, we tacitly assume that an ordering is fixed on these sets.

Since for given $f, g \in \mathscr{B}$, the case when $\Delta_{fg}=\emptyset$ is of no interest from our point of view, i.e. from the point of view of interpolants, we shall suppose that $\Delta_{fg} \neq \emptyset$ and distinguish the following four cases:

Case 1: $\Phi_{fg}=\Gamma_{fg}=\emptyset$.
Case 2: $\Phi_{fg} \neq \emptyset$, $\Gamma_{fg}=\emptyset$.
Case 3: $\Phi_{fg}=\emptyset$, $\Gamma_{fg} \neq \emptyset$.
Case 4: $\Phi_{fg} \neq \emptyset$, $\Gamma_{fg} \neq \emptyset$.

Let $f, g \in \mathscr{B}$. We shall supply both f and g with all variables from $\Delta_{fg} \cup \Phi_{fg} \cup \Gamma_{fg}$. One can distinguish the functions obtained in this way by $\tilde{f}$ and $\tilde{g}$, however, such distinction is not necessary. Indeed, by definition, the variables of Φ_{fg} will be dummy for g (and that of Γ_{fg} for f), hence f and $\tilde{f}$ (similarly, g and $\tilde{g}$) do represent the same equivalence class, thus, by our agreement on terminology, we can choose $\tilde{f}$ as the representative of that class. In fact, we shall do, and simply write f for $\tilde{f}$ (g for $\tilde{g}$). We shall fix an ordering of the variables occuring in f and g as follows: all elements of Δ_{fg} precede all elements of Φ_{fg} which, in turn, precede all elements of Γ_{fg} while we keep the previously fixed orderings inside Δ_{fg}, Φ_{fg} and Γ_{fg}. By this fixing of ordering, the construction of trees associated to f and g will be definitive.

Let $n=\text{card}\,(\Delta_{fg} \cup \Phi_{fg} \cup \Gamma_{fg})$ and $i=\text{card}\,\Delta_{fg}$ (recall, that $\Delta_{fg} \neq \emptyset$, hence $1 \leqq i \leqq n$ follows) and consider a full binary tree T of level n. For f, let $T_f=\langle V_f, E_f\rangle$ be that subtree of T which is associated to f. We introduce the following notations:

$$\mathscr{V}(f) = \{V_{ik} | V_{ik} \in V_f,\ 1 \leqq k \leqq 2^i\},$$

$$\mathscr{U}(f) = \{V_{ik} | V_{ik} \in \mathscr{V}(f) \ \text{ and } \ FBT(V_{ik}) \notin \text{Sub}\, T_f,\ \ 1 \leqq k \leqq 2^i\}$$

$$\mathscr{W}(f) = \begin{cases} \mathscr{V}(f) - \mathscr{U}(f) & \text{provided } \ i \neq n, \\ \mathscr{V}(f) & \text{otherwise.} \end{cases}$$

In the rest of the paper we shall keep the reference of the (lower case) letter i fixed, namely, $i=\text{card}\,\Delta_{fg}$ and every occurence of i not in English words will always refer to this cardinality.

3. Existence of interpolants

3.1. Let $f, g \in \mathscr{B}$. We write $f \leqq g$ iff $\Delta_{fg} \neq \emptyset$ and for all $\xi \in {}^{\omega}2, f(\xi)=1$ entails $g(\xi)=1$; and $f<g$ iff $f \leqq g$ but $f \neq g$. The following assertion is immediate by definitions.

Lemma 4. Let $f, g \in \mathscr{B}$ and assume that T_f and T_g are the trees associated to f and g, respectively. Then $f \leqq g$ iff $T_f \in \operatorname{Sub} T_g$; in particular, $f<g$ iff $T_f \subset T_g$.

From now on, we shall fix (arbitrarily) $f, g \in \mathscr{B}$ such that $f<g, f \neq 0, g \neq 1$. All assertions in the rest are valid under these assumptions only, but, for the sake of being short we shall omit them everywhere when stating lemmata or theorems formally. Accordingly, every formal assertion is to be read as "If $f, g \in \mathscr{B}, f<g, f \neq 0, g \neq 1$ then" followed by the assertion written as such. This remark applies also for definitions.

First we set $I_{fg}=\{h \mid h \in \mathscr{B}, f<h, h<g$ and $\delta(h) \subseteqq \Delta_{fg}\}$. We say, that $h \in \mathscr{B}$ is an interpolant of f and g iff $h \in I_{fg}$. By Lemma 4, we have

Corollary 5. Let $h \in \mathscr{B}$ and T_f, T_g, T_h be the trees associated to f, g, h, respectively. Then,
(1) $h \in I_{fg}$ implies $T_f \subset T_h \subset T_g$, and
(2) $T_f \subset T_h \subset T_g$ and $\mathscr{W}(h)=\mathscr{V}(h)$ together imply $h \in I_{fg}$.

The following two lemmata readily follow from definitions by Lemma 4 and Corollary 5.

Lemma 6. Let $h \in \mathscr{B}$ and $h \in I_{fg}$. Then,
(1) $\mathscr{V}(f) \subseteqq \mathscr{V}(h)$,
(2) $\mathscr{W}(f) \subset \mathscr{W}(h)$,
(3) $\mathscr{V}(h)=\mathscr{W}(h)$,
(4) $\mathscr{W}(h) \subseteqq \mathscr{W}(g)$, and
(5) $\mathscr{V}(h) \subset \mathscr{V}(g)$.

Lemma 7. Let $h \in \mathscr{B}$. If
(1) $\mathscr{W}(f) \subset \mathscr{W}(h)$,
(2) $\mathscr{V}(h)=\mathscr{W}(h)$, and
(3) $\mathscr{V}(h) \subset \mathscr{V}(g)$
are satisfied, then $h \in I_{fg}$.

3.2. Recall that $\Phi_{fg}=\Gamma_{fg}=\emptyset$ in Case 1; $\Phi_{fg} \neq \emptyset, \Gamma_{fg}=\emptyset$ in Case 2; $\Phi_{fg}=\emptyset$, $\Gamma_{fg} \neq \emptyset$ in Case 3; and $\Phi_{fg} \neq \emptyset, \Gamma_{fg} \neq \emptyset$ in Case 4.

Lemma 8.

(1)	$\mathscr{U}(f)=\mathscr{U}(g)=\emptyset$	in Case 1.
(2)	$\mathscr{U}(f) \neq \emptyset$	in Cases 2 and 4,
	$\mathscr{U}(f)=\emptyset$	in Case 3.
(3)	$\mathscr{U}(g) \neq \emptyset$	in Cases 3 and 4,
	$\mathscr{U}(g)=\emptyset$	in Case 2.
(4)	$\mathscr{W}(g)=\mathscr{V}(g)$	in Cases 1 and 2.
(5)	$\mathscr{W}(f)=\mathscr{V}(f)$	in Cases 1 and 3.
(6)	$\mathscr{W}(g)-\mathscr{V}(f) \neq \emptyset$	in Case 1.
(7)	$\mathscr{U}(g)-\mathscr{V}(f) \neq \emptyset$	in Cases 3 and 4.

Proof. All statements exept (7) in Case 4 readily follow from Lemma 3 by definitions.

For proving (7) in Case 4, let us suppose, that $\mathcal{U}(g)-\mathcal{V}(f)=\emptyset$ and let $V_{ij}\in\mathcal{U}(g)$. We have either $V_{ij}\in\mathcal{U}(f)$ or $V_{ij}\in\mathcal{W}(f)$, immediately. Let us suppose first, that $V_{ij}\in\mathcal{U}(f)$ and let $k=\text{card}\,\Phi_{fg}$. Since f does not depend on elements of Γ_{fg}, there exists an l $(1\leqq l\leqq 2^{i+k})$, by Lemma 3, such that $\text{FBT}(V_{(i+k)l})\in\text{Sub}\,T_f$ (where T_f is the tree associated to f). On the other hand, since g does depend on elements of Γ_{fg}, it is impossible, again by Lemma 3, that the same is true for T_g (T_g being associated to g); i.e. there exist some vertices in $\text{FBT}(V_{(i+k)l})$ which are not contained in T_g. It follows, that $T_f\not\subset T_g$, a contradiction to Lemma 4. If $V_{ij}\in\mathcal{W}(f)$ then, using a similar argument, the assertion follows.

The next theorem gives necessary and sufficiant conditions under which proper interpolants exist.

Theorem 9. $I_{fg}\neq\emptyset$ iff $\text{card}\,(\mathcal{W}(g)-\mathcal{V}(f))\geqq\alpha$, where $\alpha=2$ in Case 1, $\alpha=1$ in Cases 2 and 3 and $\alpha=0$ in Case 4.

Proof. Let T_f and T_g be the trees associated to f and g, respectively.

For Cases 2 and 4, let T_1 be the tree obtained from T_f by adjoining FBT (V_{ij}) for all $V_{ij}\in\mathcal{U}(f)$ to T_f. By Lemma 8 (2), we have $\mathcal{U}(f)\neq\emptyset$ and hence, $T_f\subset T_1$ in both cases. In Case 4, $T_1\subset T_g$ follows from Lemma 8 (7). In Case 2, $\mathcal{W}(g)-\mathcal{V}(f)\neq\emptyset$ by assumption, thus $T_1\subset T_g$. Let h be the function to which T_1 is associated. By the construction of T_1, we have $\mathcal{W}(h)=\mathcal{V}(h)$, hence $h\in I_{fg}$, by Corollary 5 (2).

For Cases 1 and 3, let T_1 be constructed from T_f by adding to T_f the tree $\text{FBT}(V_{ij})$ for some $V_{ij}\in\mathcal{W}(g)-\mathcal{V}(f)$. Since $\mathcal{W}(g)-\mathcal{V}(f)$ is not empty by assumption, we have immediately, that $T_f\subset T_1$ (recall, that $\text{FBT}(V_{ij})$ is the path ending in V_{ij} in Case 1). In Case 3, $T_1\subset T_g$ is obtained by Lemma 8 (7), while in Case 1, this proper inclussion is entailed by the assumption, namely, by the fact, that $\mathcal{W}(g)-\mathcal{V}(f)-\{V_{ij}\}\neq\emptyset$ (where V_{ij} is the vertex used in the construction of T_1). Again, denoting by h the function to which T_1 is associated, $h\in I_{fg}$ follows from Corollary 5 (2) since $\mathcal{W}(h)=\mathcal{V}(h)$.

To prove the converse, let $I_{fg}\neq\emptyset$ and assume that $h\in I_{fg}$.

Case 1. $\text{card}\,(\mathcal{V}(g)-\mathcal{V}(h))\geqq 1$ and $\text{card}\,(\mathcal{V}(h)-\mathcal{V}(f))\geqq 1$ thus $\text{card}\,(\mathcal{W}(g)-\mathcal{V}(f))\geqq 2$ by Lemma 8 (4).

Case 2. $\mathcal{V}(f)\subseteqq\mathcal{V}(h)=\mathcal{W}(h)$ by Lemma 6 (1 and 3); $\mathcal{V}(h)\subset\mathcal{V}(g)$ by Lemma 6 (5) and $\mathcal{V}(g)=\mathcal{W}(g)$ by Lemma 8 (4). Summarizing up, $\mathcal{V}(f)\subset\mathcal{W}(g)$ and hence $\text{card}\,(\mathcal{W}(g)-\mathcal{V}(f))\geqq 1$.

Case 3. $\mathcal{V}(f)=\mathcal{W}(f)$ by Lemma 8 (5), $\mathcal{W}(f)\subset\mathcal{W}(h)=\mathcal{V}(h)$ by Lemma 6 (2 and 3) and finally, $\mathcal{W}(h)\subseteqq\mathcal{W}(g)$ by Lemma 6 (4). We have then $\mathcal{V}(f)\subset\mathcal{W}(g)$ which implies $\text{card}\,(\mathcal{W}(g)-\mathcal{V}(f))\geqq 1$.

3.3. We present here some counterexamples thus illustrating the very nature of proper interpolants.

Let the following functions be given: $f_1=x_1\cdot x_2$, $g_1=x_1\cdot x_2+\bar{x}_1\cdot\bar{x}_2$; $f_2=x_1\cdot x_2\cdot y_1$, $g_2=x_1\cdot x_2$; and $f_3=x_1\cdot x_2$, $g_3=x_1\cdot x_2+\bar{x}_1\cdot\bar{x}_2\cdot z_1$. The trees associated to these functions are indicated in bold line by Figs 1, 2 and 3, respectively.

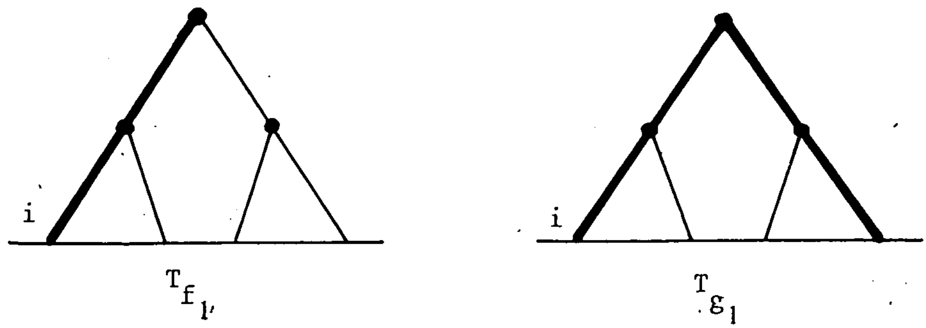


Fig. 1

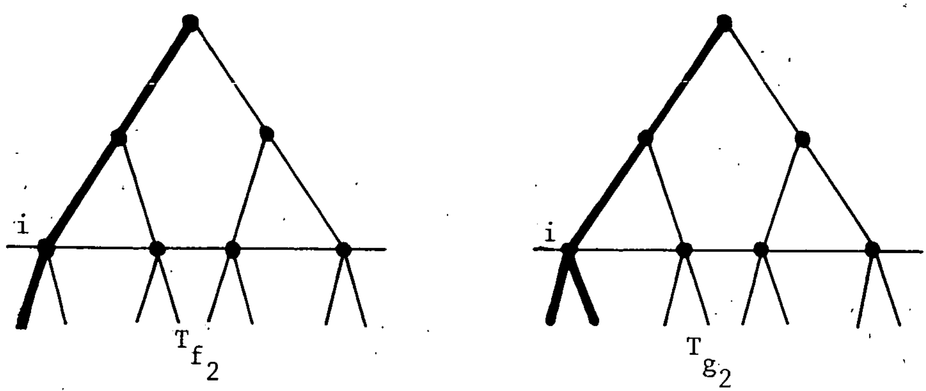


Fig. 2

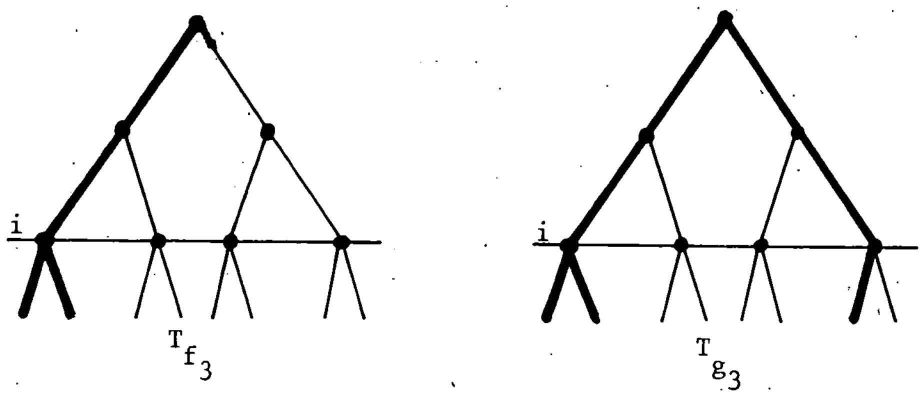


Fig. 3

It is clear, that $\Phi_{f_1g_1}=\Gamma_{f_1g_1}=\emptyset$; $\mathcal{V}(g_1)-\mathcal{V}(f_1)=\mathcal{W}(g_1)-\mathcal{V}(f_1)\neq\emptyset$ and card$(\mathcal{W}(g_1)-\mathcal{V}(f_1))=1$, nevertheless $I_{f_1g_1}=\emptyset$. Similarly, $\Phi_{f_2g_2}=\{y_1\}$, $\Gamma_{f_2g_2}=\emptyset$ and $\mathcal{V}(g_2)-\mathcal{V}(f_2)=\mathcal{W}(g_2)-\mathcal{V}(f_2)=\emptyset$ and $I_{f_2g_2}=\emptyset$. Finally, $\Phi_{f_3g_3}=\emptyset$, $\Gamma_{f_3g_3}=\{z_1\}$ and $\mathcal{W}(g_3)-\mathcal{V}(f_3)=\emptyset$, thus $I_{f_3g_3}=\emptyset$.

4. The number of interpolants

4.1.

Theorem 10. Let $m = \text{card}(\mathscr{W}(g) - \mathscr{V}(f))$. Then,

$$\text{card}(I_{fg}) = 2^m - \alpha$$

where $\alpha = 2$ in Case 1, $\alpha = 1$ in Cases 2 and 3 and $\alpha = 0$ in Case 4.

Proof. Let $M = \mathscr{W}(g) - \mathscr{V}(f)$. In all cases, if $M \neq \emptyset$ ($M = \emptyset$ can occur in Cases 2, 3 and 4 only, by Lemma 8 (6)), then the whole set I_{fg} can be constructed by the following recurrence.

Let us denote by T_1 the tree obtained by adjoining FBT (V_{ik}) to the tree T_f associated to f for all $V_{ik} \in \mathscr{U}(f)$. Obviously, $T_f \subseteqq T_1$.

Let T_2 be a tree such that $T_1 \subseteqq T_2 \subseteqq T_g$ and T_2 is associated to an interpolant h_2 of f and g (or, to f if $T_2 = T_1 = T_f$) and suppose, that $V_{ij} \in M$. Let T_3 be constructed from T_2 by adding FBT (V_{ij}) to T_2. Clearly, $T_f \subset T_3 \subseteqq T_g$ and, for the function h_3 to which T_3 is associated, $\mathscr{W}(h_3) = \mathscr{V}(h_3)$. It follows from Corollary 5, that $h_3 \in I_{fg}$ iff $T_3 \subset T_g$, and from Lemma 4, that $h_2 < h_3$. Let $M_1 = M - \{V_{ij}\}$ and repeat this procedure with $V_{il} \in M_1$ and with T_3 (in place of T_2) until M is emptied.

Summarizing up, starting from T_1 and taking in all possible ways one, two, ..., m distinct elements from M (provided $M \neq \emptyset$) and proceeding as described above we can produce a set of functions $I = \{t_1, ..., t_r\}$ and it follows from the construction, that $I \cup \{f, g\} = I_{fg} \cup \{f, g\}$, i.e. any function which can be constructed in this way is either an element of I_{fg} or of $\{f, g\}$. Since one, two, ..., m distinct elements can be chosen from M in $\binom{m}{1}, \binom{m}{2}, ..., \binom{m}{m}$ possible ways, and $\sum_{j=1}^{m} \binom{m}{j} = 2^m - 1$, we have $\text{card}\, I = 2^m$.

It remains to investigate whether f and g do or do not appear in I. This will be done case by case.

Case 1. We have $\mathscr{U}(f) = \emptyset$ by Lemma 8 (1), hence $T_1 = T_f$, i.e. $f \in I$. On the other hand, taking all elements from M, we obviously obtain a tree identical to T_g, thus $g \in I$. All the other elements of I are proper interpolants, indeed, that is $I_{fg} = I - \{f, g\}$. It follows, that $\text{card}(I_{fg}) = 2^m - 2$.

Case 2. By Lemma 8 (2), $\mathscr{U}(f) \neq \emptyset$ which entails $T_f \subset T_1$, i.e. the function to which T_1 is associated is in I_{fg} (cf. the proof of Theorem 9). Taking all elements from M in the procedure above, we arrive to T_g by Lemma 8 (3), hence $g \in I$. We have $I_{fg} = I - \{g\}$, hence $\text{card}(I_{fg}) = 2^m - 1$.

Case 3. $\mathscr{U}(f) = \emptyset$, by Lemma 8 (2), which implies $T_1 = T_f$ and thus $f \in I$. Let T_r be the tree obtained by the procedure using all elements of M. Then by Lemma 8 (3) $T_r \subset T_g$. That is $g \notin I, I_{fg} = I - \{f\}$ and we have $\text{card}(I_{fg}) = 2^m - 1$.

Case 4. Since $\mathscr{U}(f) \neq \emptyset$ by Lemma 8 (2), we have $T_f \subset T_1$, i.e. $f \notin I$. On the other hand, taking all elements in M and constructing the tree T_r by the procedure, by Lemma 8 (3), $T_r \subset T_g$ holds. We obtain, that $g \notin I$ and so $I_{fg} = I$.

4.2. By a chain of interpolants we mean a finite sequence of distinct functions $h_0, \ldots, h_t$ such that the following clauses are satisfied:

(1) $h_0 = f, h_t = g$,

(2) $h_j \in I_{h_{j-1} h_{j+1}}$ for $1 < j < t$.

A chain $h_0, \ldots, h_t$ of interpolants is maximal iff for every j $(0 \leqq j < t)$, $I_{h_j h_{j+1}} = \emptyset$.

Corollary 11. Every maximal chain of interpolants of f and g has length $\mathrm{card}\,(\mathscr{W}(g) - \mathscr{V}(f)) + \beta$ where $\beta = 1$ in Case 1, $\beta = 2$ in Cases 2 and 3 and $\beta = 3$ in Case 4.

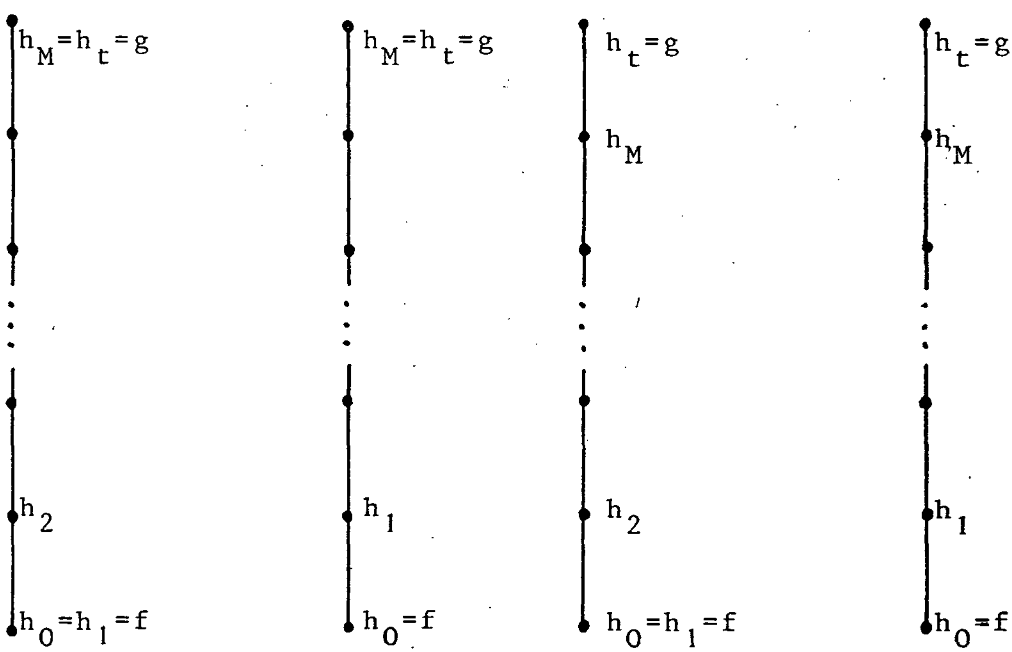
Proof. Immediate by the proof of Theorem 10.

Fig. 4 below indicates all of the four cases. h_1 stands for the function obtained from T_1 in the proof of the previous theorem, and h_M denotes the function which is constructed by using the whole set M.

Corollary 12. The set of all maximal chains of interpolants of f and g has cardinality given by

$$(\mathrm{card}\,(\mathscr{W}(g) - \mathscr{V}(f)))!.$$

Proof. The second (in Cases 1, 3) or the third member (in Case 2, 4) of a particular maximal chain is obtained by using exactly one element from the set $M = \mathscr{W}(g) - \mathscr{V}(f)$; this element can be taken in $\mathrm{card}\, M$ different ways. The next member of the chain can be taken in $\mathrm{card}\, M - 1$ different ways, and so on. The assertion follows by induction.



Case 1 Case 2 Case 3 Case 4

Fig. 4

5. The algebra of interpolants

Theorem 13. The algebra $\mathcal{I}=(I_{fg}\cup\{f,g\},\cdot,+,f,g)$ is a distributive sublattice of the Boolean algebra $\mathcal{B}=(B/\sim,\cdot,+,^{-},0,1)$ with zero element f and unit element g.

Proof. The only thing to be proved is that $I_{fg}\cup\{f,g\}$ is closed under $+$ and $\cdot$. This is, however, obvious from the construction outlined in the proof of Theorem 10.

It is relatively easy to show using Lemma 1 and the construction of I_{fg}, that $I_{fg}\cup\{f,g\}$ is not closed under negation: the algebra $\mathcal{I}$ is not a Boolean algebra.

Theorem 14. Let $h_0, h_1\in I_{fg}\cup\{f,g\}$. Then in the Boolean algebra $\mathcal{B}$, the two equivalence classes $[h_0]$ and $[h_1]$ are identical iff their representatives h_0 and h_1 are such; i.e. $[h_0]=[h_1]$ iff $h_0=h_1$.

Proof. Obvious, by Lemma 2 and the construction of I_{fg}.

6. Conclusions

By using the isomorphism between $\mathcal{F}$ and $\mathcal{B}$, to every $\varphi\in\mathcal{F}$, there corresponds a class in $\mathcal{B}$ denoted by f_φ, and conversely, for $f\in\mathcal{B}$ one can associate an element φ_f in $\mathcal{F}$.

By a zero order model A we simply mean a subset of the set of sentence symbols S. Observe, that every assignment $\xi\in{}^{\omega}2$ represents a zero order model in the sense of [1]: let $A_\xi=\{s_i|s_i\in S$ and $\xi_i=1\}$ where ξ_i is the i-th component of ξ. The converse is also valid: every model $A\subset S$ can be associated by an assignment ξ_A defined by

$$\xi_i=\begin{cases}1 & \text{iff } s_i\in A,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, for every φ and A, we have $A\models\varphi$ iff $f_\varphi(\xi_A)=1$. We set $A_\varphi=\{A|A\subset S, A\models\varphi\}$.

Let ξ and ζ be two assignments and $\varphi\in\mathcal{F}$. We say that the models A_ξ and A_ζ are φ-equivalent with respect to a subset Δ of $\delta(f)$, the set of nondummy variables of f, iff $A_\xi\cap\Delta=A_\zeta\cap\Delta$. This is indeed an equivalence relation and being so we can set for $A\subset S$, $\varphi\in\mathcal{F}$ and $\Delta\subset\delta(f)$:

$[A]_\varphi^\Delta=\{B|B\subset S$ and B is φ-equivalent to A with respect to $\Delta\}$.

Notice, that by choosing $\Delta=\delta(f)$, the class $[A]_\varphi^{\delta(f_\varphi)}$ is represented by one path in the tree T_{f_φ} associated to f_φ.

Let $f, g\in\mathcal{B}$ and consider the sets of variables, Δ_{fg}, Φ_{fg} and Γ_{fg}. Then, clearly, $\text{card}\,\mathcal{V}(f)=\text{card}\,(\{[A]_{\varphi_f}^{\Delta_{fg}}\})$ and similarly, $\text{card}\,\mathcal{V}(g)=\text{card}\,(\{[A]_{\varphi_g}^{\Delta_{fg}}\})$, that is, $\mathcal{V}(f)$ and $\mathcal{V}(g)$ identify all φ_f-equivalent and φ_g-equivalent classes of models with respect to the common set of nondummy variables of f and g, Δ_{fg}, respectively.

By definition, $\mathcal{U}(f)\cap\mathcal{W}(f)=\emptyset$, $\mathcal{U}(f)\cup\mathcal{W}(f)=\mathcal{V}(f)$ and similar equations hold for g. If for some k, $V_{ik}\in\mathcal{W}(g)$, then FBT(V_{ik}) is a subtree of the tree T_g associated to g, and all paths of FBT(V_{ik}) represent the same φ_g-equivalence class of models with respect to the set of all nondummy variables of g, $\delta(g)$, while if $V_{ik} \notin \mathcal{W}(g)$ and hence $V_{ik} \in \mathcal{U}(g)$, then the paths of T_g going through V_{ik} will represent different classes (with respect to $\delta(g)$). We say that A is a respectable model (for φ_g) iff

$$[A]_{\varphi_g}^{\Delta_{fg}} = [A]_{\varphi_g}^{\delta(g)}.$$

Since interpolants of f and g (hence of φ_f and φ_g) can depend on the variables of Δ_{fg} only, respectable models for φ_g are exactly the ones which are of interest from the point of view of interpolants.

The φ_g-equivalence classes of respectable models for φ_g, however, are identified by elements of $\mathcal{W}(g)$, according to the remark above.

Let us introduce the following notations:

$A_{\varphi_g}^{\text{resp}} = \{[A]_{\varphi_g}^{\Delta_{fg}} \mid A$ is a respectable model for $g\}$ and $A_{\varphi_f} = \{[A]_{\varphi_f}^{\delta(f)} \mid A \in A_{\varphi_g}\}$. Then, the set $\mathcal{W}(g) - \mathcal{V}(f)$, playing a central role in our investigations, identifies those respectable model classes for φ_g which are not models of φ_f, and $\text{card}(\mathcal{W}(g) - \mathcal{V}(f)) = \text{card}(A_{\varphi_g}^{\text{resp}} - A_{\varphi_f})$ hence a reformulation of Theorem 9 in model theoretic terms can be easily obtained.

Summing up the results of the paper, for any two zero order formulae φ and ψ such that $\varphi \models \psi$, $\not\models \neg\varphi$, $\not\models \psi$, we can decide whether does or does not exist a proper interpolant for φ and ψ and if the answer is affirmative, we can give the number of equivalence classes of proper interpolants immediately, or we can construct the whole lattice of equivalence classes of interpolants when necessary. The method developed in the paper is much more effective (even if it is considered as inefficient in the more strict sense of [5]) than the one presented in [3].

Abstract

The number of equivalence classes of interpolants for arbitrarily given two zero order sentences are calculated using tree-theoretic arguments. As a by-product, the number of maximal chains and the algebraic structure of equivalence classes of interpolants are determined.

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