On finite definite automata

By B. IMREH

In this paper we consider the isomorphic realizations of finite definite automata. First we present a characterization of finite subdirectly irreducible definite automata. Secondly we give necessary and sufficient conditions for a system of automata to be isomorphically complete for the class of all finite definite automata with respect to the α_i-products. It will turn out that every finite definite automaton can be embedded isomorphically into an α_0-product of reset automata with two states.

By automaton we always mean a finite automaton without output. Since an automaton can be considered a unoid the notions such as subautomaton, isomorphism, embedding, homomorphism, congruence relation can be introduced in a natural way. Further on we shall use the following notations: X^* denotes the free monoid generated by X, $|p|$ denotes the length of the word $p \in X^*$, if there is no danger of confusion then we use the convenient notation ax and ap for $\delta(a, x)$ and $\delta(a, p)$ respectively.

An automaton $\mathbf{A}=(X, A, \delta)$ is called *definite* if there exists a natural number n such that $|p| \geqq n$ implies $|Ap|=1$ for any $p \in X^*$ where $Ap=\{ap\colon a \in A\}$. Specially, if $n=1$ then $\mathbf{A}$ is called a *reset* automaton, furthermore if there exists a state $a_0 \in A$ such that $|p| \geqq n$ implies $Ap=\{a_0\}$ for any $p \in X^*$ then $\mathbf{A}$ is *nilpotent*.

In the paper [3] we gave a characterization of finite subdirectly irreducible nilpotent automata. Now we generalize this result for definite automata. Namely, it holds the following

Theorem 1. A definite automaton $\mathbf{A}=(X, A, \delta)$ $(|A|>2)$ is subdirectly irreducible if and only if $\mathbf{A}$ has two different states a_0, b_0 such that

(i) $a_0x=b_0x$ holds for any $x \in X$,

(ii) for any $a, b \in A$ if $a \neq b$ and $\{a, b\} \subsetneqq \{a_0, b_0\}$ then there exists an input sign $x \in X$ with $ax \neq bx$.

Proof. In order to prove the necessity assume that $\mathbf{A}$ is subdirectly irreducible. Consider the set B of all subsets of A with two elements and define the relation $\leqq$ on B in the following way: for any $\{a, b\}, \{c, d\} \in B$ $\{a, b\} \leqq \{c, d\}$ if and only if there exists a word $p \in X^*$ such that $\{a, b\}p=\{c, d\}$. Since $\mathbf{A}$ is definite the defined relation is antisymmetric and thus, it is a partial ordering on B. Now we shall show that there is a greatest element in B. Because of finiteness it is enough to show that there exists only one maximal element in B. Assume to the contrary that $\{a, b\}$ and $\{c, d\}$ $\{a, b\} \neq \{c, d\}$, $\{a, b\}$, $\{c, d\} \in B$ are maximal in B. Then $ax=bx$ and $cx=dx$ hold for any $x \in X$. Consider the following relations on A: for any $u, v \in A$

$u\varrho v$ if and only if $\{u, v\} \subseteq \{a, b\}$ or $u=v$,

$u\sigma v$ if and only if $\{u, v\} \subseteq \{c, d\}$ or $u=v$.

It is obvious that ϱ and σ are nontrivial congruence relations of **A** and $\varrho \cap \sigma = \Delta_A$ where Δ_A denotes the equality relation on **A**. This yields that **A** is subdirectly reducible which contradicts our assumption on **A**. Therefore, B has a greatest element. Let $\{a_0, b_0\}$ denote this one. Then it is obvious that conditions (i) and (ii) are satisfied.

To prove the sufficiency assume that (i) and (ii) are satisfied by a definite automaton $\mathbf{A}=(X, A, \delta)$. Consider again the set B with its ordering defined above. From conditions (i) and (ii) it follows that $\{a_0, b_0\}$ is the greatest element of B. Take the following relation on A: for any $u, v \in A$

$u\theta v$ if and only if $\{u\ v\} \subseteq \{a_0, b_0\}$ or $u=v$.

By (i) we obtain that θ is congruence relation. Next we shall show that θ is the smallest nontrivial congruence of **A**. Indeed, let ϱ denote an arbitrary nontrivial congruence relation of **A**. Then there exist $a, b \in A$ such that $a \neq b$ and $a\varrho b$, furthermore, there is a word $p \in X^*$ such that $\{a, b\}p = \{a_0, b_0\}$ since $\{a_0, b_0\}$ is the greatest element in B. But then $a_0 \varrho b_0$ also holds which implies $\theta \leqq \varrho$. On the other hand, it is known that the existence of the smallest nontrivial congruence implies the subdirect irreducibility which completes the proof of Theorem 1.

Remark 1. Using Theorem 1 we obtain a simple algorithm to decide whether a definite automaton is subdirectly irreducible. Namely, if the automaton is given by a transition table of which rows correspond to each input sign and columns correspond to each state then we have the following criterion.

A definite automaton is subdirectly irreducible if and only if there exist two equal columns in its table and leaving one of them, the columns of the remained table are pairwise different.

Remark 2. In [5] M. Katsura introduced the following family of strongly connected definite automata. Let $X=\{x, y\}$ and denote by $1<p_1<p_2<\dots$ the sequence of all prime numbers. Then any natural number $n>1$ can be written uniquely as $p_1^{e_1}\dots p_{r-1}^{e_{r-1}} p_r^{e_r}$ where $e_{r-1} \geqq 1$ and $e_r=0$. Let $\mathbf{A}_n=(X, A_n, \delta_n)$ where $A_n=\{a_0\} \cup \cup \{a_{ij}\colon 1 \leqq i \leqq r;\ 0 \leqq j \leqq e_i\}$ and

$$\delta_n(a_{ij}, x) = \begin{cases} a_{ij+1} & \text{if } 1 \leqq i \leqq r-1 \text{ and } 0 \leqq j \leqq e_i - 1, \\ a_0 & \text{otherwise,} \end{cases}$$

$$\delta_n(a_{ij}, y) = \begin{cases} a_{i+1,0} & \text{if } j=0 \text{ and } 1 \leqq i \leqq r-1, \\ a_{r0} & \text{if } j=0 \text{ and } i=r, \\ a_{10} & \text{otherwise,} \end{cases}$$

$$\delta_n(a_0, x) = a_0,\ \delta_n(a_0, y) = a_{10}.$$

From the definition of $\mathbf{A}_n$ it follows that if $n=p_1^{e_1}\dots p_r^{e_r}$ and $e_i \geqq 1$, $e_j \geqq 1$ hold for some $1 \leqq i \neq j \leqq r$ then $\{a_0, a_{ie_i}, a_{je_j}\}x = a_0$ and $\{a_0, a_{ie_i}, a_{je_j}\}y = a_{10}$ and thus, by Theorem 1, we obtain that $\mathbf{A}_n$ is subdirectly reducible. On the other hand, if $n=p_1^0\dots p_{i-1}^0 p_i^{e_i} p_{i+1}^0 (e_i \geqq 1)$ is a prime-power then $\{a_0, a_{ie_i}\}x=a_0$, $\{a_0, a_{ie_i}\}y=a_{10}$ and, by a simple computation, it can be seen that $|\{u, v\}x|=2$ or $|\{u, v\}y|=2$ holds for any elements $u, v \in A$ such that $u \neq v$ and $\{u, v\} \nsubseteq \{a_0, a_{ie_i}\}$. Therefore, by Theorem 1, we get that $\mathbf{A}_n$ is subdirectly irreducible. Summarizing, we have the following statement.

For any natural number $n>1$ the automaton $\mathbf{A}_n$ is subdirectly irreducible if and only if n is prime-power.

Remark 3. A well-known special type of definite automata is the shift register. It was investigated in papers [4] and [6]. Now we consider the subdirect irreducibility of these automata. For this reason let $n \geqq 1$ be an arbitrary natural number and X a non-empty finite set with at least two elements. Then the automaton $\mathbf{A}(X, n)=(X, X^n, \delta_n)$ is called a *shift register* where $\delta_n(x_1\dots x_n, x)=x_2\dots x_n x$ for any $x_1\dots x_n \in X^n$ and $x \in X$. Observe that $\delta_n(ux_2\dots x_n, x)=\delta_n(vx_2\dots x_n, x)$ holds for any words $ux_2\dots x_n, vx_2\dots x_n \in X^n$ and $x \in X$. From this it follows that conditions of Theorem 1 are not satisfied. Thus we have the following assertion.

Every shift register with at least three elements is subdirectly reducible.

Next we shall study the α_i-products (see [1] or [2]) from the point of view of isomorphic completeness for the class of all definite automata. For this reason let i be a nonnegative integer, furthermore, let Σ be an arbitrary system of automata. Σ is called *isomorphically complete* for the class of all definite automata with respect to the α_i-product if any definite automaton can be embedded isomorphically into an α_i-product of automata from Σ. We are going to use the following obvious statement.

Lemma. If an automaton $\mathbf{A}$ can be embedded isomorphically into an α_0-product of automata $\mathbf{A}_t$ $(t=1, \dots, k)$ and for some $1 \leqq j \leqq k$ the automaton $\mathbf{A}_j$ can be embedded into an α_0-product of automata $\mathbf{B}_m$ $(m=1, \dots, s)$ then the automaton $\mathbf{A}$ can be embedded isomorphically into an α_0-product of automata $\mathbf{A}_1, \dots, \mathbf{A}_{j-1}, \mathbf{B}_1, \dots, \mathbf{B}_s, \mathbf{A}_{j+1}, \dots, \mathbf{A}_k$.

Concerning the isomorphic realization of definite automata with respect to the α_0-product we have the following result.

Theorem 2. A system Σ of automata is isomorphically complete for the class of all definite automata with respect to the α_0-product if and only if Σ contains an automaton which has two different states a, b and two input signs x, y such that $ax=bx=b$ and $ay=by=a$ hold.

Proof. In order to prove the necessity take the definite automaton $\mathbf{U}=(\{x, y\}, \{a, b\}, \delta)$ where $\delta(a, x)=\delta(b, x)=b$ and $\delta(a, y)=\delta(b, y)=a$. Because of the isomorphic completeness of Σ there exists an α_0-product $\prod_{j=1}^{k} \mathbf{A}_j(X, \varphi)$ of automata from Σ such that $\mathbf{U}$ can be embedded isomorphically into this product. Let $(a_1, \dots, a_k)$, $(a_1', \dots, a_k')$ denote the images of the elements a, b under a suitable isomorphism. Then among the sets $\{a_t, a_t'\}$ $(t=1, \dots, k)$ there should be at least one which has more then one element. Let r be the least index for which $a_r \neq a_r'$. It is obvious that the automaton $\mathbf{A}_r \in \Sigma$ satisfies the condition.

To prove the sufficiency assume that the automaton $\mathbf{B}\in\Sigma$ has the suitable states and input signs. To verify the completeness it is enough to show that any definite automaton can be embedded isomorphically into an α_0-product of reset automata with two states since any reset automaton with two states can be embedded isomorphically into an α_0-product of $\mathbf{B}$ with a single factor, and thus, by our Lemma, we obtain the completeness of Σ.

We prove by induction on the number of states of the automaton. In the case $n\leqq 2$ our statement is trivial. Now let $n>2$ and suppose that the statement is valid for any $m<n$. Let $\mathbf{A}=(X, A, \delta_A)$ be an arbitrary definite automaton with n states. If $\mathbf{A}$ is subdirectly reducible then $\mathbf{A}$ can be embedded isomorphically into a direct product of definite automata with fewer states than n. Therefore, by our induction hypothesis and Lemma, the statement is valid. Now assume that $\mathbf{A}$ is subdirectly irreducible. Let $A=\{a_1, ..., a_n\}$. Then from Theorem 1 it follows that there exist $a_i, a_j\in A$ $(i\neq j)$ such that $a_ix=a_jx$ holds for any $x\in X$. Without loss of generality we may assume that $i=n-1$ and $j=n$. Define the relation ϱ on A as follows: for any $a_r, a_s\in A$

$$a_r\varrho a_s \text{ if and only if } \{a_r, a_s\}\subseteq\{a_{n-1}, a_n\} \text{ or } a_r=a_s.$$

Obviously ϱ is a congruence relation of $\mathbf{A}$. Then the quotient automaton $\mathbf{A}/\varrho$ is definite and $A/\varrho=\{\{a_1\},..., \{a_{n-2}\}, \{a_{n-1}, a_n\}\}$. Now let c be an arbitrary symbol $(c\notin A)$ and take the automaton $\mathbf{C}=(A/\varrho\times X, \{c, a_{n-1}, a_n\}, \delta)$ where

$$\delta(u, (\{a_i\}, x)) = \begin{cases}\delta_{\mathbf{A}}(a_i, x) & \text{if } \delta_{\mathbf{A}}(a_i, x)\in\{a_{n-1}, a_n\},\\ c & \text{otherwise,}\end{cases}$$

$$\delta(u, (\{a_{n-1}, a_n\}, x)) = \begin{cases}\delta_{\mathbf{A}}(a_n, x) & \text{if } \delta_{\mathbf{A}}(a_n, x)\in\{a_{n-1}, a_n\},\\ c & \text{otherwise,}\end{cases}$$

for any $u\in\{c, a_{n-1}, a_n\}$, $(\{a_i\}, x)\in\{\{a_1\}, ..., \{a_{n-2}\}\}\times X$, $(\{a_{n-1}, a_n\}, x)\in\{\{a_{n-1}, a_n\}\}\times X$. Now consider the α_0-product $\mathbf{A}/\varrho\times\mathbf{C}(X, \varphi)$ where $\varphi_1(x)=x$ for any $x\in X$ and $\varphi_2(\{a_i\}, x)=(\{a_i\}, x)$, $\varphi_2(\{a_{n-1}, a_n\}, x)=(\{a_{n-1}, a_n\}, x)$ for any $x\in X$ and $1\leqq i\leqq n-2$. Then it is not difficult to see that the correspondence

$$\nu(a_i) = \begin{cases}(\{a_i\}, c) & \text{if } 1\leqq i\leqq n-2,\\ (\{a_{n-1}, a_n\}, a_i) & \text{if } i\in\{n-1, n-2\},\end{cases}$$

is an isomorphism of $\mathbf{A}$ into the α_0-product $\mathbf{A}/\varrho\times\mathbf{C}(X, \varphi)$. On the other hand, observe that $\mathbf{C}$ is a reset automaton and it is a well-known fact that any reset automaton can be embedded isomorphically into a direct product of reset automata with two states. Therefore, by our induction hypothesis and Lemma, we have a required decomposition of $\mathbf{A}$. This completes the proof of Theorem 2.

Regarding α_i-products with $i\geqq 1$ we have the following statement.

Theorem 3. A system Σ of automata is isomorphically complete for the class of all definite automata with respect to the α_i-product $(i\geqq 1)$ if and only if Σ contains an automaton which has two different states a, b and four input signs v, x, y, z (need not be different) such that $av=bx=b$ and $by=az=a$ hold.

Proof. The necessity of the condition is obvious. To prove the sufficiency, by Theorem 2, it is enough to show that an α_0-product of α_1-products with single factors is an α_1-product which follows from the definition of the α_i-products.

DEPT. OF COMPUTER SCIENCE
A. JÓZSEF UNIVERSITY
ARADI VÉRTANÚK TERE 1.
SZEGED, HUNGARY
H—6720

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