On involutorial automata and involutorial events

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1. Basic notions and facts

By an *automaton* we mean a triple $\mathscr{A}=(A, X, \delta)$, where A and X are finite nonempty sets, the set of *states* and the set of *inputs* of $\mathscr{A}$, and $\delta\colon A\times X\to A$ is a function, the *transition function* of $\mathscr{A}$.

Let X^* be the free monoid generated by X with identity element e. Then δ is assumed to be extended to $A\times X^*$ in the usual way. A finite nonempty set is called an *alphabet*. Each subset $E\subseteq X^*$ is called an *event* or a *language* over the alphabet X. Let us now define the notion of an involutorial automaton.

1.1. Definition. An automaton $\mathscr{A}=(A, X, \delta)$ is called *involutorial* iff $\delta(a, xx)=a$ holds for all $a\in A$, $x\in X$.

The following lemma is a trivial but useful one.

1.2. Lemma. Let $\mathscr{A}=(A, X, \delta)$ be cyclic and involutorial. Then $\mathscr{A}$ is strongly connected.

(For the automata-theoretic notions not defined in this paper see [6] and [3].)

Let $\mathscr{A}=(A, X, \delta)$ be an arbitrary automaton and define the congruence ϱ on X^* by:

$$\forall w_1, w_2\in X^*\colon (w_1, w_2)\in\varrho \quad\text{iff}\quad \delta(a, w_1) = \delta(a, w_2) \quad\text{for all}\quad a\in A.$$

The quotient $S(\mathscr{A})=X^*/\varrho$ is called the *characteristic semigroup* of $\mathscr{A}$. Let us use the notation $[w]_\varrho$ for the set $\{w'\mid(w', w)\in\varrho\}$. Concerning characteristic semigroups of involutorial automata, we can state:

1.3. Theorem. Let $\mathscr{A}=(A, X, \delta)$ be an involutorial automaton. Then $S(\mathscr{A})$ is an involutorial generated group.

Proof. Let $x\in X$. Then $(xx, e)\in\varrho$, therefore $[x]_\varrho[x]_\varrho=[e]_\varrho$, hence $[x]_\varrho$ is an involutorial element of $S(\mathscr{A})$. Now let $[w]_\varrho\in S(\mathscr{A})$ with $w=x_1\ldots x_n$. Denote $x_n\ldots\ldots x_1$ by w^R. Then $[w]_\varrho[w^R]_\varrho=[w^R]_\varrho[w]_\varrho=[e]_\varrho$.

With the aid of the following well known lemma, automata, which are involutorial and commutative, can be characterized.

1.4. Lemma. A group, in which each element is involutorial, is an abelian group.

1.5. Theorem. An automaton $\mathscr{A}=(A, X, \delta)$ is involutorial and commutative iff $\delta(a, ww)=a$ holds for all $a\in A$ and $w\in X^*$.

1.6. Example. An important example of an involutorial automaton is the *T-Flip-Flop (trigger)* $\mathscr{T}=(\{0, 1\}, \{0, 1\}, \delta)$ with $\delta(z, 0)=z$ and $\delta(z, 1)=\bar{z}$ for all $z\in\{0, 1\}$. Here $\bar{0}=1$ and $\bar{1}=0$. $\mathscr{T}$ is involutorial and commutative.

If we generalize the notion of a trigger, we arrive at the following: Let X be an alphabet and let $y\in X$. Set $\mathscr{T}_y^X=(\{0, 1\}, X, \delta_y^X)$ with

$$\delta_y^X(z, x) = z \quad \text{for all} \quad z\in\{0, 1\} \quad \text{and} \quad x\in X \quad \text{with} \quad x\neq y \quad \text{and}$$

$$\delta_y^X(z, y) = \bar{z} \quad \text{for all} \quad z\in\{0, 1\}.$$

We will call these automata generalized triggers.

If we now specialize the proof of Theorem 1 in [5] to the involutorial and commutative case, we obtain:

1.7. Theorem [Gécseg]. Every commutative involutorial automaton is the homomorphic image of a subdirect product of finitely many generalized triggers.

We can characterize commutative involutorial automata from another point of view. It is easy to see that every commutative involutorial automaton is a finite direct sum of cyclic commutative involutorial automata. From the basis theorem for abelian groups (see e.g. [12], p. 121) and Fleck's result ([4], Theorem 6), we obtain the following results, which were suggested by B. Imreh.

1.8. Theorem. Every cyclic commutative ivolutorial automaton is a one-state automaton or a direct product of finitely many two-state cyclic involutorial automata.

1.9. Corollary. Every cyclic commutative involutorial automaton has 2^n states, where n is a nonnegative integer.

2. The minimal involutorial congruence on X^*

Let X be an alphabet. A finite subset T of $X^*\times X^*$ is called a *Thue-system over* X^*. T defines a relation $\varrho_T\subseteq X^*\times X^*$ in the following way:

$$\forall v, w\in X^*\colon (v, w)\in\varrho_T \quad \text{iff} \quad v = v_1v_2v_3, \quad w = v_1w_2v_3 \quad \text{and} \quad (v_2, w_2)\in T \quad \text{or} \quad (w_2, v_2)\in T.$$

The congruence generated by the Thue-system T is the reflexive and transitive closure of ϱ_T.

Let us now consider the Thue-system $T=\{(aa, e)|a\in X\}$ over X^*. The relation ϱ_T will be denoted by $\xleftrightarrow{(i)}$ and the congruence generated by T will be denoted by $\xleftrightarrow{i}$ and called *minimal involutorial congruence on* X^*. Obviously, $X^*/\xleftrightarrow{i}$ is an involutorial generated group. In order to investigate the congruence $\xleftrightarrow{i}$, we will first establish some properties of the context-free language $L(G_i)$ generated by the context-free grammar $G_i=(\{S\}, X, P, S)$ with the set of productions $P=\{S\to e, S\to SS\}\cup\cup(S\to aSa|a\in X\}$.

(For the notions of formal language theory not defined in this paper see [1], [7], and [8].)

2.1. Lemma. Let G_i be given as above. Then we have for all $w_1, w_2, w, u_1, u_2 \in X^*$:
(1) If $w_1, w_2 \in L(G_i)$, then $w_1 w_2 \in L(G_i)$,
(2) If $w \in L(G_i)$, then $awa \in L(G_i)$ for all $a \in X$,
(3.a) If $w \in L(G_i)$, $w \neq e$, then $w = a w_1 a w_2$ with $w_1, w_2 \in L(G_i)$ and $a \in X$,
(3.b) If $w \in L(G_i)$, $w \neq e$, then $w = w_1 a w_2 a$ with $w_1, w_2 \in L(G_i)$ and $a \in X$,
(4) If $u_1 aa u_2 \in L(G_i)$ with $a \in X$, then $u_1 u_2 \in L(G_i)$ (involutorial cancellation),
(5) If $u_1 u_2 \in L(G_i)$ then $u_1 aa u_2 \in L(G_i)$ for all $a \in X$ (involutorial extension).

Proof. (1) and (2) are trivial. To prove (3.a), consider a leftmost derivation $S \overset{*}{\Rightarrow} S^k \Rightarrow aSaS^{k-1} \overset{*}{\Rightarrow} \dots \overset{*}{\Rightarrow} a w_1 a w_2 = w$ of w, and to prove (3.b), consider a rightmost derivation $S \overset{*}{\Rightarrow} S^k \Rightarrow S^{k-1} aSa \overset{*}{\Rightarrow} \dots \overset{*}{\Rightarrow} w_1 a w_2 a = w$ of w. Now let us consider (4). Let $S \Rightarrow v_1 \Rightarrow v_2 \Rightarrow \dots \Rightarrow v_n = u_1 aa u_2$ be a derivation of $u_1 aa u_2$. We will prove the assertion by induction on n. The assertion holds for $n=1$. Let $S \Rightarrow v_1 \Rightarrow v_2 \Rightarrow \dots$ $\dots \Rightarrow v_{n+1} = u_1 aa u_2$ be a derivation of $u_1 aa u_2$ of length $n+1$. We have to consider the following cases:

(a) Let $S \Rightarrow v_1 = SS$. Then there exist derivations $S \overset{*}{\Rightarrow} r_1$ and $S \overset{*}{\Rightarrow} r_2$ of length n_1 and n_2 with $n_1, n_2 \leqq n$ and $r_1 r_2 = u_1 aa u_2$. If aa occurs in r_1 or r_2, then the assertion follows from the induction hypothesis. It remains to consider the case $r_1 = u_1 a$ and $r_2 = a u_2$. Here the application of (3.a), (3.b) and (1), (2) yields the assertion.

(b) Let $S \Rightarrow v_1 = bSb$ with $b \in X$. Then there exists a derivation $S \overset{*}{\Rightarrow} w$ of length $n_1 \leqq n$ and $u_1 aa u_2 = bwb$ holds. The case $w = e$ is trivial, therefore let us assume $w \neq e$. If aa occurs in w, then the assertion follows from the induction hypothesis and (2). If aa are the first two letters of $u_1 aa u_2$ $(a=b, u_1=e)$, then a is the leftmost letter of w. According to (3.a) we have $w = a w_1 a w_2$ with $w_1, w_2 \in L(G_i)$, and therefore $u_1 u_2 = w_1 a w_2 a \in L(G_i)$. The case that aa are the last two letters of $u_1 aa u_2$ $(a=b, u_2=e)$ is proved similarly.

Now let us prove (5) by induction on the length of $u_1 u_2$. The case $|u_1 u_2| = 0$ is trivial, therefore let us assume $|u_1 u_2| > 0$.

According to (3.a) we have $u_1 u_2 = b w_1 b w_2$ with $b \in X$ and $w_1, w_2 \in L(G_i)$. If $|u_1| \geqq |b w_1 b|$, then $u_1 = b w_1 b r$ and $w_2 = r u_2$. From the induction hypothesis we conclude $raau_2 \in L(G_i)$, and therefore $u_1 aa u_2 = b w_1 b r aa u_2 \in L(G_i)$ for all $a \in X$.

If $|u_1| < |b w_1 b|$, then $b w_1 b = u_1 r_2 b$. The case $|u_1| = 0$ is trivial. Therefore assume $u_1 = b r_1$. Then $w_1 = r_1 r_2$, and with the aid of the induction hypothesis we conclude $r_1 aa r_2 \in L(G_i)$, and hence we have $u_1 aa u_2 = b r_1 aa r_2 b w_2 \in L(G_i)$ for all $a \in X$. This completes the proof of the lemma.

Now we can prove the following theorem.

2.2. Theorem. Let G_i be the context-free grammar given above and let $w = x_1 x_2 \dots x_k \in X^*$ with $x_j \in X$ for $j \in [1:k]$, $x_{j-1} \neq x_j$ for $j \in [2:k]$, and $k \geqq 0$. If

(I) $w = w_0 \overset{(i)}{\longleftrightarrow} w_1 \overset{(i)}{\longleftrightarrow} w_2 \overset{(i)}{\longleftrightarrow} \dots \overset{(i)}{\longleftrightarrow} w_n = w'$ holds, where $n \geqq 0$, then we have $w' = \alpha_0 x_1 \alpha_1 x_2 \alpha_2 \dots \alpha_{k-1} x_k \alpha_k$ with $\alpha_i \in L(G_i)$ for $i \in [0:k]$.

Proof. The assertion holds for $n=0$ and $n=1$. Now let $j \in [0:n]$ be maximal with the following property:

(II) There are words $\alpha_0^j, \alpha_1^j, \ldots, \alpha_k^j \in L(G_i)$ such that

$$w_j = \alpha_0^j x_1 \alpha_1^j x_2 \alpha_2^j \ldots \alpha_{k-1}^j x_k \alpha_k^j \quad \text{holds.}$$

We will show $j=n$. Assume $j<n$. Let us consider the following two cases:

(1) w_{j+1} can be derived from w_j by insertion of aa, $a \in X$ (involutorial extension). Then, according to (5) of 2.1. Lemma, (II) holds for $j+1$.

(2) w_{j+1} can be derived from w_j by deletion of aa, $a \in X$ (involutorial cancellation). If this aa can be chosen in an α_l^j, $l \in [0: k]$, then, according to (4) of 2.1. Lemma, (II) holds for $j+1$. It remains to consider $aa = x_l x_l$, $l \in [1: k]$, and

$$\alpha_{l-1}^j x_l \alpha_l^j = \overline{\alpha_{l-1}^j} x_l x_l \alpha_l^j \quad \text{or} \tag{2.1}$$

$$\alpha_{l-1}^j x_l \alpha_l^j = \alpha_{l-1}^j x_l x_l \overline{\alpha_l^j}, \tag{2.2}$$

and w_{j+1} can be derived from w_j by cancellation of the occurrences of $x_l x_l$ in the right sides of (2.1) or (2.2).

(2.1) According to (3.b) of 2.1. Lemma, divide α_{l-1}^j in $\alpha_{l-1}^j = w_1 x_l w_2 x_l$ with $w_1, w_2 \in L(G_i)$. Then we have $\overline{\alpha_{l-1}^j} \alpha_l^j = w_1 x_l w_2 \alpha_l^j$, and therefore (II) holds for $j+1$.

(2.2) According to (3.a) of 2.1. Lemma, divide α_l^j in $\alpha_l^j = x_l w_1 x_l w_2$ with $w_1, w_2 \in$ $\in L(G_i)$. Then we have $\alpha_{l-1}^j \overline{\alpha_l^j} = \alpha_{l-1}^j w_1 x_l w_2$, and therefore (II) holds for $j+1$.

We must have $j=n$, which proves the theorem.

2.3. Corollary. Under the assumptions of 2.2. Theorem either $|w'| > |w|$ or $w = w'$ holds.

2.4. Corollary. The context-free language $L(G_i)$ is the congruence class of e w.r.t. $\overset{i}{\longleftrightarrow}$.

Now let us consider the local regular language $X^* \setminus X^* V X^*$ with $V = \{xx | x \in X\}$. We have:

2.5. Corollary.

(1) If w is a word of minimal length in a congruence class of $\overset{i}{\longleftrightarrow}$, then $w \in X^* \setminus$ $\setminus X^* V X^*$.

(2) If $w \in X^* \setminus X^* V X^*$, then w is a word of minimal length in a congruence class of $\overset{i}{\longleftrightarrow}$.

(3) Two different words of $X^* \setminus X^* V X^*$ are in different congruence classes of $\overset{i}{\longleftrightarrow}$.

2.6. Corollary. Each congruence class of $\overset{i}{\longleftrightarrow}$ contains exactly one word of minimal length.

2.7. Corollary. If $X = \{x\}$ is a one element alphabet, then $\overset{i}{\longleftrightarrow}$ contains exactly two classes, namely $\{e, x^2, x^4, \ldots\}$ and $\{x, x^3, x^5, \ldots\}$. If $|X| \geqq 2$, then the index of $\overset{i}{\longleftrightarrow}$ is infinite.

We will denote the unique word of minimal length in the congruence class of w w.r.t. $\overset{i}{\longleftrightarrow}$ by $w_{i\,min}$. If $K \subseteq X^*$, then we denote the set $\{w_{i\,\min} | w \in K\}$ by $L_{min}(K)$.

The function ϱ: $X^* \to X^*$ with $\varrho(w) = w_{i\,\min}$ for $w \in X^*$ is a *Dyck-simplification* in the sense of [10], which implies the following theorem.

2.8. Theorem. (Sakarovitch [10]). If $R \subseteq X^*$ is regular, then $L_{\min}(R)$ is regular.

3. Involutorial events and involutorial closure

In this section we will introduce and investigate involutorial events and involutorial closure. Let us first define these notions.

3.1. Definition. Let X be an alphabet and let $E \subseteq X^*$ be an event (subset of X^*). The set $E^i = \{u' | u' \in X^*, \exists u \in E \text{ with } u \overset{i}{\longleftrightarrow} u'\}$ is called the *involutorial closure* of E. An event $E \subseteq X^*$ is called *involutorial* iff $E = E^i$. E is called *i-regular* iff E is involutorial and regular.

Obviously, for $E \subseteq X^*$ we have $E^i = L_{\min}(E)^i$, and for $E, K \subseteq X^*$ we have $E^i = K^i$ iff $L_{\min}(E) = L_{\min}(K)$.

Let $\mathscr{A} = (A, X, \delta, a_0, F)$ be a recognizer. Here (A, X, δ) is an automaton, $a_0 \in A$ is the initial state and $F \subseteq A$ is the set of final states of $\mathscr{A}$. $T(\mathscr{A}) = \{w | \delta(a_0, w) \in \in F\}$ is the event recognized by $\mathscr{A}$. $\mathscr{A}$ is called *involutorial* iff (A, X, δ) is an involutorial automaton.

Now let $E \subseteq X^*$ be an arbitrary event, and let $\equiv$ be the *Nerode right congruence associated with E*, i.e. right congruence defined by:

$$\forall v, w \in X^*: \; v \equiv w \quad \text{iff} \quad \forall u \in X^*: \; (vu \in E \quad \text{iff} \quad wu \in E).$$

E is regular iff $\equiv$ is of finite index (see e.g. [9])

We can now prove the following theorem.

3.2. Theorem. Let X be an alphabet and $E \subseteq X^*$ be an event. Then E is *i*-regular iff E is recognized by an involutorial recognizer $\mathscr{A}$.

Proof. If $E = T(\mathscr{A})$ for an involutorial recognizer $\mathscr{A}$, then obviously E is involutorial. Conversely, let E be *i*-regular. Since E is regular, the Nerode right congruence $\equiv$ of E is of finite index. Construct the recognizer $\mathscr{A} = (A, X, \delta, a_0, F)$ with $A = \{[w]_\equiv | w \in X^*\}$, $\delta([w]_\equiv, x) = [wx]_\equiv$, $a_0 = [e]_\equiv$, and $F = \{[w]_\equiv | w \in E\}$. We have $E = T(\mathscr{A})$. Since E is involutorial, the congruence $\overset{i}{\longleftrightarrow}$ is contained in $\equiv$. Since $wxx \overset{i}{\longleftrightarrow} w$ for all $w \in X^*$ and $x \in X$, we have $\delta([w]_\equiv, xx) = [wxx]_\equiv = [w]_\equiv$, i.e., $\mathscr{A}$ is involutorial.

The recognizer constructed in this proof is the *complete minimal (accessible and reduced) recognizer* (see [3]) which accepts the *i*-regular set E. Therefore we can state:

3.3. Corollary. If E is an *i*-regular set, then the complete minimal recognizer for E is involutorial.

3.4. Example. Let $\mathscr{A} = (A, \{x\}, \delta, a_0, \{a_2\})$ be a recognizer with

$$A = \{a_0, a_1, a_2\}, \quad \delta(a_0, x) = a_2, \quad \delta(a_2, x) = a_1, \quad \text{and} \quad \delta(a_1, x) = a_2.$$

Then $T(\mathscr{A}) = \{x, x^3, \ldots, x^{2n+1}, \ldots\}$ is involutorial, but $\mathscr{A}$ is not involutorial. The complete minimal recognizer for $T(\mathscr{A})$ is

$$\mathbf{A}_r = (\{a_0, a_1\}, \{x\}, \delta_r, a_0, \{a_1\}) \quad \text{with} \quad \delta_r(a_0, x) = a_1 \quad \text{and} \quad \delta_r(a_1, x) = a_0.$$

which is obviously involutorial.

The following two theorems state some closure and nonclosure properties of *i*-regular sets.

3.5. Theorem. Let X be an alphabet.

(1) The family of i-regular sets of X^* is a Boolean algebra of sets,

(2) If E is an i-regular set, then the transpose E^R of E is also an i-regular set. (Recall that the transpose of a word $w=a_1a_2\dots a_n$ is the word

$$w^R = a_n \dots a_2 a_1, \quad \text{and} \quad E^R = \{w^R | w\in E\}.)$$

Proof. (1): Let E, E_1, and E_2 be i-regular sets.

Then $(E_1\cup E_2)^i = E_1^i \cup E_2^i = E_1\cup E_2$. Let $\mathscr{A}=(A, X, \delta, a_0, F)$ be an involutorial recognizer with $E=T(\mathscr{A})$. Then $\bar{E}=T(\bar{\mathscr{A}})$, where $\bar{\mathscr{A}}=(A, X, \delta, a_0, A\setminus F)$ is an involutorial recognizer. The proof of (2) is trivial.

3.6. Theorem. The product of two i-regular sets and the iteration (star operation) of an i-regular set are not necessarily i-regular.

Proof. Consider $X=\{x\}$ and $E=\{x, x^3, \dots, x^{2n+1}, \dots\}$. Then E is i-regular. But $E^2=\{x^2, x^4, \dots, x^{2n}, \dots\}$ is not i-regular, since $(E^2)^i=\{e, x^2, x^4, \dots, x^{2n}, \dots\}$. This shows nonclosure under product. Now let $\mathscr{A}=(A, X, \delta, a_0, F)$ be the recognizer given by

$$A = \{a_0, a_1, a_2\}, \quad X = \{x, y\}, \quad F = \{a_2\}, \quad \text{and} \quad \delta(a_0, y) = \delta(a_1, x) = a_0,$$

$$\delta(a_0, x) = \delta(a_2, y) = a_1, \quad \text{and} \quad \delta(a_1, y) = \delta(a_2, x) = a_2.$$

$\mathscr{A}$ is an involutorial recognizer, therefore $T(\mathscr{A})$ is an i-regular set. If $w\in T(\mathscr{A})$, then $|w|\geqq 2$, and $x^2\notin T(\mathscr{A})$. This implies $x^2\notin T(\mathscr{A})^*$. But $e\in T(\mathscr{A})^*$ implies $x^2\in (T(\mathscr{A})^*)^i$, therefore $T(\mathscr{A})^*\neq (T(\mathscr{A})^*)^i$.

We will now consider the formation of the involutorial closure E^i of an event E. Let us first investigate those events E for which E^i is regular.

Remember that in contrast to the above mentioned (complete deterministic) recognizer, the transition function δ of an *incomplete (deterministic) recognizer* $\mathscr{A}= =(A, X, \delta, a_0, F)$ is a partial function from $A\times X$ to A. We assume that δ is extended to $A\times X^*$ in the usual manner. We will call an incomplete (deterministic) recognizer $\mathscr{A}$ a *trim* recognizer, if each state of $\mathscr{A}$ is accessible and coaccessible [3]. If $E\neq\emptyset$ is regular, then there is a trim recognizer $\mathscr{A}$ with $E=T(\mathscr{A})$. Now we can prove:

3.7. Theorem. For each alphabet X there is a nonregular event $E\subseteqq X^*$ such that E^i is regular.

Proof. Let $X=\{x\}$ be a one-element alphabet. Then $E=\{x^{2n^2}|n\geqq 1\}$ is not regular, but $E^i=\{x^{2n}|n\geqq 0\}$ is regular. Now consider the case $|X|\geqq 2$. $L_{\min}(X^*)$ is regular. Let $\mathscr{A}=(A, X, \delta, a_0, F)$ be a trim recognizer such that $T(\mathscr{A})=L_{\min}(X^*)$. According to the structure of the words of $L_{\min}(X^*)$ and the fact that $\mathscr{A}$ is trim, $a'= =\delta(a, x)$ for $a, a'\in A$, $x\in X$, implies $\delta(a', x)=\emptyset$. In particular we have $a\neq\delta(a, x)$ for all $a\in A$, $x\in X$. Now choose $x, y\in X$ with $x\neq y$. Since $x\in L_{\min}(X^*)$, there is a final state $a'=\delta(a_0, x)$.

Define an incomplete recognizer $\mathscr{B}=(B, X, \beta, a_0, F)$, whose state set is infinite, in the following way: Set $B=A\cup\{a_i|i\geqq 1\}\cup\{b_i|i\geqq 1\}$ (disjoint union) and

$$\beta(a, z) = \delta(a, z) \quad \text{for all} \quad a\in A, \quad a \neq a', \quad z\in X,$$

$$\beta(a', z) = \delta(a', z) \quad \text{for all} \quad z\in X, \quad z \neq x, \quad \text{and}$$

$$\beta(a', x) = a_1, \; \beta(a_1, x) = a'.$$

Furthermore set

$$\beta(a_i, y) = b_i, \quad \beta(b_i, x) = a_{i+1}, \quad \text{and}$$
$$\beta(b_i, y) = a_i, \quad \beta(a_{i+1}, x) = b_i \quad \text{for all} \quad i \geqq 1.$$

Then it is easy to see that $L_{\min}(X^*)=T(\mathscr{A})\subseteqq T(\mathscr{B})$. Set $E=T(\mathscr{B})$, then $E^i= =X^*$, i.e., E^i is regular. We will now show that E is not regular. Assume the contrary. Then there exists a recognizer $\mathscr{C}$ such that $E=T(\mathscr{C})$. Let n be the number of states of $\mathscr{C}$. If $vw\in T(\mathscr{C})$, $v, w\in X^*$, then there exists some $\theta\in X^*$, $|\theta|\leqq n$, such that $v\theta\in \in T(\mathscr{C})=E$.

Consider a word $x(xy)^p$ with $p>n$. Then $\beta(a_0, x(xy)^p(yx)^p)=a'\in F$, hence $x(xy)^p(yx)^p\in E$. But obviously, for all $\xi\in X^*$ with $|\xi|\leqq n$ we have $x(xy)^p\xi\notin T(\mathscr{B})=E$, which yields a contradiction. Hence E is not regular.

Events $E\neq\emptyset$ over an alphabet X with $|X|\geqq 2$, for which E^i is regular, possess an interesting property w.r.t. $L_{\min}(X^*)$.

Let us first give the following definition.

3.8. Definition. Let $E, F\subseteqq X^*$ be events with $E\subseteqq F$. E is said to be *dense in* F iff there exists a nonnegative integer k such that the following holds:

$\forall v\in F\,\exists w\in X^*$ with $|w|\leqq k$ such that $vw\in E$. (Note that $k=0$ implies $E=F$.)

Now we can prove:

3.9. Theorem. Let X be an alphabet with $|X|\geqq 2$ and let $E\subseteqq X^*$ with $E\neq\emptyset$ such that E^i is regular. Then $L_{\min}(E)$ is dense in $L_{\min}(X^*)$.

Proof. Let $\mathscr{A}=(A, X, \delta, a_0, F)$ with $|A|=n$ be an involutorial recognizer such that $T(\mathscr{A})=E^i$. Since $E^i\neq\emptyset$ we have $F\neq\emptyset$, and furthermore, according to 1.2. Lemma, we can assume that (A, X, δ) is strongly connected. We have to show that there exists a nonnegative integer k such that, for all $v\in L_{\min}(X^*)$ there exists $w\in X^*$, $|w|\leqq k$, with $vw\in L_{\min}(E)$. First assume $v=v'x\in L_{\min}(X^*)$, where $x\in X$. Choose $y\in X$ with $x\neq y$ and set $u=(yx)^n$. Since $|A|=n$, $F\neq\emptyset$, and (A, X, δ) is strongly connected, there exists $u'\in X^*$ with $|u'|\leqq n$ such that $\delta(a_0, vuu')=\delta(a_0, v'x(yx)^n u')\in \in F$. Set $\alpha=v'x(yx)^n u'$, then $\alpha_{I\min}=v'xw$ with $|w|\leqq 3n=k$. Since $\mathscr{A}$ is an involutorial recognizer, we have $\delta(a_0, vw)=\delta(a_0, vuu')\in F$, i.e. $vw\in E^i$. Furthermore $vw\in E^i\cap L_{\min}(X^*)=L_{\min}(E)$. The case $v=e$ is trivial.

3.10. Corollary. Let X be an alphabet with $|X|\geqq 2$ and let $E\subseteqq X^*$ be a nonempty event. If $L_{\min}(E)$ is finite, then E^i is not regular. In particular, if E is finite, E^i is not regular.

The converse of 3.9. Theorem does not hold. Namely, we have:

3.11. Theorem. The involutorial closure E^i of an event E such that $L_{\min}(E)$ is dense in $L_{\min}(X^*)$ need not be regular.

Proof. Let X be an alphabet with $|X|\geqq 2$. Set $E=L_{\min}(X^*)\setminus\{x\}$, where $x\in X$. Obviously, $L_{\min}(E)$ is dense in $L_{\min}(X^*)$ and $\{x\}^i=X^*\setminus E^i$. Assume that the theorem does not hold. Then, E^i becomes regular. Since X^* and E^i are regular, so is $\{x\}^i$. This contradicts 3.10. Corollary. Hence, the theorem has to hold.

3.10. Corollary shows the existence of involutorial events, which are not regular. This situation is impossible for events which are involutorial and commutative.

An event $E \subseteqq X^*$ is called *commutative*, if E is a union of congruence classes of a congruence $\varkappa$, which is defined by:

$$\forall v, w \in X^*: (v, w) \in \varkappa \quad \text{iff} \quad v = y_1 \ldots y_n, \quad w = y_{i_1} \ldots y_{i_m},$$

where $i_1, \ldots, i_m$ is a permutation of $1, \ldots, m$ and $y_i \in X$ for $i \in [1: m]$ with $m \geqq 0$.

It can easily be shown that the congruence $\varkappa + \overset{i}{\longleftrightarrow}$ (the sum of the congruences $\varkappa$ and $\overset{i}{\longleftrightarrow}$) is of finite index. Therefore we have:

3.12. Theorem. If $E \subseteqq X^*$ is involutorial and commutative, then E is regular.

Let us now consider those events E for which E^i is context-free. We will prove a theorem characterizing these events, part of which can be viewed as a special case of a theorem due to Sakarovitch [11].

3.13. Theorem. Let $E \subseteqq X^*$ be an event. Then E^i is context-free iff $L_{\min}(E)$ is context-free.

Proof. Let $L_{\min}(E)$ be a context-free language and $G=(V, X, S, P)$ be a context-free grammar in Greibach normal form with $L(G)=L_{\min}(E)$. Each production of G is of the form $A \to a\alpha$ with $a \in X, \alpha \in V^*$. If $e \in L_{\min}(E)$, then there is a production $S \to e$, and S does not occur on the right-hand side of any production.

Construct a context-free grammar $G_1=(V_1, X, S, P_1)$ with $V_1 = V \cup \{S_a | a \in \in X\} \cup \{S_1\}$ (disjoint union) and furthermore (1) if $e \notin L(G)$ then

$$P_1 = \{A \to S_a \alpha | A \to a\alpha \in P\} \cup \{S_a \to S_1 a S_1 | a \in X\} \cup \bar{P} \quad \text{with}$$

$$\bar{P} = \{S_1 \to e, \quad S_1 \to S_1 S_1\} \cup \{S_1 \to a S_1 a | a \in X\},$$

(2) if $e \in L(G)$ then extend P_1 of (1) with the production $S \to S_1$.

Since $L_{\min}(E) \subseteqq L_{\min}(X^*)$, $L(G_1) = L_{\min}(E)^i = E^i$ can easily be shown with the aid of 2.2. Theorem. Hence E^i is a context-free language.

Conversely, let E^i be a context-free language. Since the intersection of a context-free language with a regular set is context-free, $L_{\min}(E) = E^i \cap L_{\min}(X^*)$ is context-free.

In analogy to regular involutorial closures, we can state:

3.14. Theorem. For each alphabet X there is a non context-free event $E \subseteqq X^*$ such that E^i is context-free.

Proof. Let $x \in X$. It is easy to see that $E = \{x^{n!} | n \geqq 1\}$ is not context-free. Since $L_{\min}(E) = \{e, x\}$ is regular, E^i is context-free.

On the other hand, we have:

3.15. Theorem. The involutorial closure of a context-free event need not be context-free.

Proof. We have to show the existence of a context-free event E such that $L_{\min}(E)$ is not context-free. Consider the context-free event $E_1 = \{(ca)^n cb (ac)^{2n} | n \geqq 1\}$ over the alphabet $X = \{a, b, c\}$, and set $E = E_1^+$.

Then E is context-free and each word $w \in E$ is of the form

$$w = (ca)^{i_1} cb (ac)^{2i_1} (ca)^{i_2} cb (ac)^{2i_2} \ldots (ca)^{i_k} cb (ac)^{2i_k}$$

with $i_j \geqq 1$ for $j \in [1: k]$ and $k \geqq 1$.

Assume that $L_{\min}(E)$ is context-free. Then $E_2 = L_{\min}(E) \cap (ca)(cb)^+(ac)^+$ has to be context-free. Each word $v \in E_2$ is of the form $v = (ca)(cb)^k(ac)^{2^k}$ with $k \geqq 1$. Consider the homomorphism $h: X^* \to a^*$ given by $h(a) = a$ and $h(b) = h(c) = e$.

Then $h(E_2) = \{a^{2^k+1} | k \geqq 1\}$ has to be a context-free language, which is a contradiction. Therefore, $L_{\min}(E)$ is not context-free.

4. The structure of some special recognizers

We will start this section with the investigation of the structure of any trim recognizer accepting $L_{\min}(E)$, where $E \subseteqq X^*$ ($|X| \geqq 2$) is a nonempty involutorial regular event.

Let us first introduce the following notation. If $\mathscr{A} = (A, X, \delta, a_0, F)$ is a recognizer and $a \in A$, then we set

$$I_a = \{x | x \in X \text{ and } \delta(a', x) = a \text{ for an } a' \in A\} \text{ and}$$

$$0_a = \{x | x \in X \text{ with } \delta(a, x) \neq \emptyset\}.$$

Now we can state:

4.1. Theorem. Let $X = \{x_1, \ldots, x_n\}$ be an alphabet with $n \geqq 2$, let $E \subseteqq X^*$ be a nonempty involutorial regular set, and let $\mathscr{A} = (A, X, \delta, a_0, F)$ be a trim recognizer with $T(\mathscr{A}) = L_{\min}(E)$. Then we have

(1) $\delta(a_0, x) \neq \emptyset$ for all $x \in X$. Furthermore, if $x, y \in X$ with $x \neq y$, then $\delta(a_0, x) \neq \delta(a_0, y)$ holds.

(2) The set I_{a_0} is empty. In particular $\delta(a_0, x) \neq a_0$ holds for all $x \in X$.

(3) Let $a \in A$ with $a \neq a_0$. Then $|I_a| = 1$, i.e. a can be reached by exactly one $x \in X$, and $I_a \cap 0_a = \emptyset$, $I_a \cup 0_a = X$, i.e. a can be leaved by exactly those $y \in X$ with $y \neq x$. Furthermore we have $\delta(a, x) \neq \delta(a, y)$ for all $x, y \in 0_a$ with $x \neq y$.

Proof. (1) Each letter x is an element of $L_{\min}(X^*)$, therefore, according to 3.9. Theorem, there exists $w \in X^*$ such that $xw \in L_{\min}(E)$, which implies $\delta(a_0, x) \neq \emptyset$. Now let $x, y \in X$ with $x \neq y$, and assume $\delta(a_0, x) = \delta(a_0, y)$. Since $xy \in L_{\min}(X^*)$, then, with the same argument, there exists a word $w \in X^*$ such that $xyw \in L_{\min}(E)$, i.e. $\delta(a_0, xyw) \in F$. Since $\delta(a_0, xy) = \delta(a_0, yy)$, we conclude $\delta(a_0, yyw) \in F$, i.e. $yyw \in L_{\min}(E)$, which is a contradiction. (2) Assume that there exist $x \in X$, $a \in A$ such that $\delta(a, x) = a_0$ holds. Since a is accessible, there exists $u \in X^*$ with $\delta(a_0, u) = a$. Hence $\delta(a_0, ux) = a_0$. With the aid of (1) we conclude $\delta(a_0, uxx) = \delta(a_0, x) \neq \emptyset$. Since $\delta(a_0, x)$ is coaccessible, there exists $v \in X^*$ such that $\delta(\delta(a_0, x), v) \in F$, i.e. $\delta(a_0, uxxv) \in F$. This means that $uxxv \in L_{\min}(E)$, which yields a contradiction. (3) Assume that there exist $x, y \in I_a$ with $x \neq y$. Then there are $a', a'' \in A$ with $\delta(a', x) = \delta(a'', y) = a$. Since $\mathscr{A}$ is trim, there are $v', v'' \in X^*$ with $\delta(a_0, v') = a'$ and $\delta(a_0, v'') = a''$, which implies $\delta(a_0, v'x) = \delta(a_0, v''y) = a$. Since $v''y \in L_{\min}(X^*)$ and $x \neq y$, we have $v''xy \in L_{\min}(X^*)$. There exists $w \in X^*$ with $v''yxw \in L_{\min}(E)$, i.e. $\delta(a_0, v''yxw) = \delta(a_0, v'xxw) \in F$. Therefore $v'xxw \in L_{\min}(E)$, which is a contradiction. Notice that, since $\mathscr{A}$ is trim, there exist $x \in X$, $a' \in A$ with $\delta(a', x) = a$. This, together with the foregoing, shows $|I_a| = 1$.

$I_a \cap 0_a = \emptyset$ follows from the fact that $\mathscr{A}$ is trim and $T(\mathscr{A}) = L_{\min}(E)$. Now let $I_a = \{x\}$ and $\delta(a', x) = a$. There exists $v \in X^*$ with $\delta(a_0, v) = a'$, and we have

$vx \in L_{\min}(X^*)$. Let $y \in X$ with $x \neq y$. Then $vxy \in L_{\min}(X^*)$ holds, too. There exists $w \in X^*$ such that $vxyw \in L_{\min}(E)$, i.e. $\delta(a_0, vxyw) \in F$, and therefore we have $\delta(a, y) \neq \emptyset$. The rest of the assertion can be proved in a way similar to the corresponding part of (1).

4.2. Example. For each alphabet X we will introduce an automaton, named $\mathscr{L}_X$, which accepts exactly $L_{\min}(X^*)$. To this end, set

$$\mathscr{L}_X = (L_X, X, \lambda_X, l_0, L_X) \quad \text{with} \quad L_X = \{l_0\} \cup \{l_x | x \in X\},$$

$$\lambda_X(l_0, x) = l_x \quad \text{for all} \quad x \in X, \quad \text{and} \quad \lambda_X(l_x, y) = l_y \quad \text{for all} \quad x, y \in X \quad \text{with} \quad x \neq y.$$

It is easy to see that $\mathscr{L}_X$ is trim and $T(\mathscr{L}_X) = L_{\min}(X^*)$ holds.

Recall that an incomplete recognizer is called *minimal*, if it is trim and reduced (see [3]). In the following we will construct for a given nonempty involutorial regular event $E \subseteq X^*$ ($|X| \geqq 2$) a minimal recognizer which accepts $L_{\min}(E)$. To this end, we first need two lemmas.

4.3. Lemma. Let $\mathscr{A} = (A, X, \delta, a_0, F)$ with $|X| \geqq 2$ and $F \neq \emptyset$ be a cyclic involutorial recognizer. Then for all $x, y \in X$ with $x \neq y$ and all $a \in A$ there exist $u, v \in X^*$ such that $uxyv \in L_{\min}(T(\mathscr{A}))$ and $\delta(a_0, ux) = a$ holds.

Proof. Since $\mathscr{A}$ is cyclic and involutorial, $\mathscr{A}$ is strongly connected (see 1.2. Lemma). Let $a \in A$, choose $m > |A|$ and set $c = \delta(a, (xy)^m)$. Since $\mathscr{A}$ is involutorial, we have $a = \delta(c, (yx)^m)$. Since $\mathscr{A}$ is strongly connected, there exists $u' \in X^*$ with $|u'| \leqq m$ and $\delta(a_0, u') = c$. Therefore we have $\delta(a_0, u'(yx)^m) = a$. Set $\bar{u} = u'(yx)^m$. Then, from $|u'| \leqq m$, we conclude $\bar{u}_{l\,\min} = ux$. Since $\mathscr{A}$ is involutorial, we have $\delta(a_0, ux) = a$. In a similar way we can show that there exists $\bar{v}$ with $\bar{v}_{l\,\min} = yv$ and $\delta(a, yv) \in F$. Then $\delta(a_0, uxyv) \in F$, and since $x \neq y$, we have $uxyv \in L_{\min}(T(\mathscr{A}))$.

The following lemma states a property of accessibility and coaccessibility in $\mathscr{A} \times \mathscr{L}_X$, where $\mathscr{A}$ is a complete minimal involutorial recognizer which satisfies the conditions of the foregoing lemma, and $\mathscr{L}_X$ is the trim recognizer of 4.2. Example.

4.4. Lemma. Let $\mathscr{A} = (A, X, \delta, a_0, F)$ be a complete minimal involutorial recognizer with $|X| \geqq 2$ and $F \neq \emptyset$. Then, in the product automaton $\mathscr{A} \times \mathscr{L}_X$, the following holds:

(1) (a, l_0) is not accessible for all $a \in A$ with $a \neq a_0$,

(2) All the other states of $\mathscr{A} \times \mathscr{L}_X$ are accessible and coaccessible.

Proof. (1) is trivial according to the fact that $\mathscr{L}_X$ is a trim recognizer accepting $L_{\min}(X^*)$ and (2) of 4.1. Theorem. To prove (2), we have to consider the set of states $\{(a_0, l_0)\} \cup \{(a, l_x) | a \in A, x \in X\}$. It is easy to see that (a_0, l_0) is accessible and coaccessible. Consider a state (a, l_x), $a \in A$, $x \in X$. By the foregoing lemma, for all $y \in X$ with $x \neq y$ there exist $u, v \in X^*$ with $uxyv \in L_{\min}(T(\mathscr{A}))$ and $\delta(a_0, ux) = a$. This implies $(\delta \times \lambda_X)((a_0, l_0), ux) = (\delta(a_0, ux), \lambda_X(l_0, ux)) = (a, l_x)$, i.e. (a, l_x) is accessible, and $(\delta \times \lambda_X)((a, l_x), yv) \in F \times L_X$, i.e. (a, l_x) is coaccessible.

4.5. Theorem. Let $E \subseteq X^*$, $|X| \geqq 2$, be a nonempty involutorial regular event and $\mathscr{A} = (A, X, \delta, a_0, F)$ be a complete minimal recognizer with $T(\mathscr{A}) = E$. Then $(\mathscr{A} \times \mathscr{L}_X)^t$ is a minimal recognizer with $T((\mathscr{A} \times \mathscr{L}_X)^t) = L_{\min}(E)$. (Here $(\mathscr{A} \times \mathscr{L}_X)^t$ denotes the trim recognizer associated with $\mathscr{A} \times \mathscr{L}_X$ (see [3]).)

Proof. According to 3.3. Corollary, $\mathscr{A}$ is an involutorial recognizer, and since $E\neq\emptyset$, we have $F\neq\emptyset$. Furthermore, $T((\mathscr{A}\times\mathscr{L}_X)^{\iota})=T(\mathscr{A}\times\mathscr{L}_X)=T(\mathscr{A})\cap T(\mathscr{L}_X)=$ $=E\cap L_{\min}(X^*)=L_{\min}(E)$. From the foregoing lemma we know that $\{(a_0, l_0)\}\cup$ $\cup\{(a, l_x)|a\in A,\ x\in X\}$ is the set of states of $(\mathscr{A}\times\mathscr{L}_X)^{\iota}$. If we show that $(\mathscr{A}\times\mathscr{L}_X)^{\iota}$ is reduced, then the theorem is proved.

Let us show that (a_0, l_0) is not equivalent to any state (a, l_y), $a\in A$, $a\neq a_0$, and $y\in X$. With the aid of 4.3. Lemma, we can find $v\in X^*$ such that $\delta(a_0, vy)\in F$ and $yv\in L_{\min}(X^*)$. Therefore $(\delta\times\lambda_X)((a_0, l_0), yv)\in F\times L_X$, but $(\delta\times\lambda_X)((a, l_y), yv)$ is not defined. Now choose $x, y\in X$ with $x\neq y$ and consider two states (a, l_x) and (a', l_y) with $a, a'\in A$. Again, with the aid of 4.3. Lemma, we can find a word $v\in X^*$ with $\delta(a, yv)\in F$ and $yv\in L_{\min}(X^*)$: Therefore, $(\delta\times\lambda_X)((a, l_x), yv)\in F\times L_X$, but $(\delta\times\lambda_X)((a', l_y), yv)$ is not defined. It remains to show that (a, l_x) and (a', l_x) are not equivalent for all $x\in X$ and $a, a'\in A$ with $a\neq a'$. To this end choose $y\in X$ with $y\neq x$ and $m>|A|^2$. Set $b=\delta(a, (yx)^m)$ and $b'=\delta(a', (yx)^m)$. Since an involutorial recognizer is obviously a permutation recognizer, we have $b\neq b'$. Since $\mathscr{A}$ is reduced, there exists $u\in X^*$ such that $\delta(b, u)\in F$ and $\delta(b', u)\notin F$ (or vice versa). Here we can assume that $|u|\leqq|A|^2$ holds. Hence we have $\delta(a, (yx)^m u)\in F$ and $\delta(a', (yx)^m u)\notin F$ (or vice versa). Set $w=(yx)^m u$. Since $|u|<m$, we have $w_{\iota\min}=yv$ for a suitable $v\in X^*$. Consequently, we have $\delta(a, yv)\in F$ and $\delta(a', yv)\notin F$ (or vice versa). Since $y\neq x$, we have $(\delta\times\lambda_X)((a, l_x), yv)\in F\times L_X$ and $(\delta\times\lambda_X)((a', l_x), yv)\notin F\times L_X$ (or vice versa). This ends the proof of the theorem.

The proof of the following corollary now is trivial.

4.6. Corollary. Let $\mathscr{A}=(A, X, \delta, a_0, F)$ be a complete minimal involutorial recognizer with $|X|\geqq 2$ and $F\neq\emptyset$. Let $\mathscr{A}'=(A', X, \delta', a_0', F')$ be a minimal recognizer such that $T(\mathscr{A}')=L_{\min}(T(\mathscr{A}))$ holds. Then we have $|A'|=|A|\,|X|+1$.

5. Decidability results

In this section we will first investigate the decidability of the question "$T(\mathscr{A})^{\iota}=$ $=T(\mathscr{B})^{\iota}$", where $\mathscr{A}$ and $\mathscr{B}$ are two given recognizers.

To treat this problem we first need, for a given alphabet X and a language $L\subseteq X^*$, the operator λ_L mapping subsets of X^* to subsets of X^* (see e.g. [1]). λ_L is defined as follows: Let $w, w'\in X^*$. Then $w'\in\lambda_L(w)$ iff $w=w_0x_1w_1x_2\ldots w_{r-1}x_rw_r$ with $w_i\in L$ for $i\in[0:r]$, $x_j\in X$ for $j\in[1:r]$, $r\geqq 0$, and $w'=x_1x_2\ldots x_r$.

It is known that, if $R\subseteq X^*$ is a regular event, then $\lambda_L(R)$ is regular for arbitrary languages $L\subseteq X^*$. Taking into consideration of this fact given in [1], p. 60, it can be seen that, if L is a given context-free language and R is a given regular language, one can effectively construct a recognizer accepting $\lambda_L(R)$. If we choose $L=\{e\}^{\iota}$, then, according to 2.2. Theorem and 2.5. Corollary, we have $w_{\iota\min}\in\lambda_L(w)$ for all $w\in X^*$.

Now we can prove:

5.1. Theorem. Let two recognizers $\mathscr{A}$ and $\mathscr{B}$ be given. Then $T(\mathscr{A})^{\iota}=T(\mathscr{B})^{\iota}$ is decidable.

Proof. $T(\mathscr{A})^{\iota}=T(\mathscr{B})^{\iota}$ is equivalent to $L_{\min}(T(\mathscr{A}))=L_{\min}(T(\mathscr{B}))$. From $L_{\min}(T(\mathscr{A}))=\lambda_L(T(\mathscr{A}))\cap L_{\min}(X^*)$ with $L=\{e\}^{\iota}$ and similarly $L_{\min}(T(\mathscr{B}))=$

$=\lambda_L(T(\mathscr{B}))\cap L_{\min}(X^*)$ and the fact that we can construct recognizers for $\lambda_L(T(\mathscr{A}))$, $\lambda_L(T(\mathscr{B}))$, and $L_{\min}(X^*)$ the theorem follows.

We already mentioned that the involutorial closure of a regular set is a deterministic context-free language. If we specialize the proof of Theorem 3.3 in [2], then we can construct for a given regular set E, a deterministic pushdown acceptor for E^i. Since it is decidable whether the language accepted by a deterministic pushdown automaton is regular (see e.g. [8], p. 246), we conclude:

5.2. Theorem [Book]. Let a recognizer $\mathscr{A}=(A, X, \delta, a_0, F)$ be given. Then it is decidable whether $T(\mathscr{A})^i$ is regular or not.

Using our structure results in section 4, we are able to give an algorithm for this problem, which does not use deterministic pushdown automata, but finite automata only.

5.3. Algorithm.
Input: A recognizer $\mathscr{A}=(A, X, \delta, a_0, F)$.
Output: "YES", if $T(\mathscr{A})^i$ is regular, "NO" otherwise.
Method:

(1) If $T(\mathscr{A})=\emptyset$ or $|X|=1$, go to (5)

(2) Now we have $|X|\geqq 2$ and $F\neq\emptyset$.
Construct a minimal recognizer $\mathscr{A}'=(A', X, \delta', a_0', F')$ with $T(\mathscr{A}')=$ $=L_{\min}(T(\mathscr{A}))=\lambda_L(T(\mathscr{A}))\cap L_{\min}(X^*)$, where $L=\{e\}^i$.

(3) Construct the set $\mathfrak{C}=\left\{\mathscr{C}\,|\,\mathscr{C}\text{ is a complete minimal involutorial recognizer with } \frac{|A'|-1}{|X|}\text{ states}\right\}$.

(4) For all $\mathscr{C}\in\mathfrak{C}$ decide whether $T(\mathscr{A})^i=T(\mathscr{C})^i$ holds. If this is the case for a $\mathscr{C}\in\mathfrak{C}$, go to (5), otherwise go to (6).

(5) Output "YES".

(6) Output "NO".

5.4. Theorem. The output of 5.3. Algorithm is "YES" iff $T(\mathscr{A})^i$ is regular.

Proof. The case $T(\mathscr{A})=\emptyset$ or $|X|=1$ are trivial. Therefore we can assume that $T(\mathscr{A})\neq\emptyset$, which implies $F\neq\emptyset$, and $|X|\geqq 2$ holds. If the output is "YES" then there exists a $\mathscr{C}\in\mathfrak{C}$ with $T(\mathscr{A})^i=T(\mathscr{C})^i=T(\mathscr{C})$, i.e. $T(\mathscr{A})^i$ is regular.

Conversely, assume that $T(\mathscr{A})^i$ is regular. Consider the complete minimal involutorial recognizer $\bar{\mathscr{A}}=(\bar{A}, X, \bar{\delta}, \bar{a}_0, \bar{F})$ with $T(\bar{\mathscr{A}})=T(\mathscr{A})^i$. According to 4.6. Corollary we have $|\bar{A}|=\dfrac{|A'|-1}{|X|}$, i.e. $\bar{\mathscr{A}}\in\mathfrak{C}$, and therefore the output is "YES".

Abstract

In this paper we will study a special class of automata and events or languages, called involutorial. An automaton with input alphabet X is involutorial iff the double input of one input sign $x\in X$ induces the identity mapping of the state set, an event over an alphabet X is involutorial iff it is saturated w.r.t. to a special (the minimal) involutorial congruence on X^*. This congruence is investigated in section 2. In section 3 we will treat involutorial events and the involutorial closure of arbitrary events. In particular we will study those events, whose involutorial closure is regular or context-free. In section 4 the structure of some special recognizers is determined, and in section 5 we shall give with the aid of these results an algorithm based on finite automata, to decide for a given regular event, whether the involutorial closure is regular or not.

Acknowledgement

The authors would like to thank Prof. Dr. R. Parchmann for valuable discussions and suggestions.

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