Some results about keys of relational schemas

HO THUAN and LE VAN BAO

1. §.

In this section we recall some important notions and results in the theory of relational data base needed in subsequent sections.

In this paper, when we talk about a set of tuples the word relation is used while talking about structural description of sets of tuples we use the word relational schema [1]. With this approach a relation is an instance of a relational schema.

A *relation* on the set of attributes $\Omega=\{A_1, A_2, \ldots, A_n\}$ is a subset of the cartesion product $\mathrm{Dom}(A_1)\times\mathrm{Dom}(A_2)\times\ldots\times\mathrm{Dom}(A_n)$ where $\mathrm{Dom}(A_i)$ — the domain of A_i — is the set of possible values for that attribute. The elements of the relation are called *tuples* and will be denoted by $\langle t\rangle$.

A constraint involving the set of attributes $\{A_1, A_2, \ldots, A_n\}$ is a predicate on the collection of all relations on this set. A relation $R(A_1, A_2, \ldots, A_n)$ obeys the constraint if the value of the predicate for R is "true".

We shall restrict ourselves to the case of functional dependencies.

A *functional dependency* (abbr. FD) is a sentence denoted $\sigma\colon X\to Y$ where σ is the name of the functional dependency and X and Y are sets of attributes. A functional dependency $\sigma\colon X\to Y$ holds in $R(\Omega)$ where X and Y are subsets of Ω, if for every tuple u and $v\in R$, $u[X]=v[X]$ implies $u[Y]=v[Y]$ ($u[X]$ denotes the projection of the tuple u on X).

Let F be a set of functional dependencies. A relation R defined over the attributes $\Omega=\{A_1, A_2, \ldots, A_n\}$ is said to be an *instance* of the *relational schema* $S=\langle\Omega, F\rangle$ iff each functional dependency $\sigma\in F$ holds in R.

There are inference rules — Armstrong's axioms — which can be applied to deriving further functional dependencies from F, and they are listed below. The system of Armstrong's rules is complete in the sense of Ullman [3].

Armstrong's axioms.[1]

For every $X, Y, Z\subseteq\Omega$,

[1] In fact we used here a system of axioms which is equivalent to that of Armstrong.

A1. (Reflexivity): if $Y \subseteq X$ then $X \to Y$.
A2. (Augmentation): if $X \to Y$ then $X \cup Z \to Y \cup Z$.
A3. (Transitivity): if $X \to Y$ and $Y \to Z$ then $X \to Z$.
From the Armstrong's axioms it is easy to prove the following inference rules:
Union rule: if $X \to Y$ and $X \to Z$ then $X \to Y \cup Z$.
Decomposition rule: if $X \to Y$ and $Z \subset Y$ then $X \to Z$.

Let F be a given set of FDs. The closure F^+ of F is the set of FDs which can be derived from F through Armstrong's inference rules.

It is shown in [3] that

$$(X \to Y) \in F^+ \Leftrightarrow Y \subseteq X^+$$

where

$$X^+ = \{A_i / (X \to A_i) \in F^+\}$$

is the closure of X w.r.t. F.

There is a linear-time algorithm, proposed by Beeri and Bernstein [4], for computing the closure X^+ of a given set X (w.r.t F):

1) Establish the sequence $X^{(0)}, X^{(1)}, \ldots,$ as follows:

$$X^0 \equiv X.$$

Suppose $X^{(i)}$ is computed then

$$X^{(i+1)} = X^{(i)} \cup Z^{(i)}$$

where

$$Z^{(i)} = \bigcup_{X_j \subseteq X^{(i)}} Y_j$$

$$(X_j \to Y_j) \in F$$

2) In view of the construction, it is obvious that

$$X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \ldots$$

Since Ω is a finite set, there exists a smallest non negative integer t such that

$$X^{(t)} = X^{(t+1)}$$

3) We have $X^+ = X^{(t)}$.

Keys of a relational schema.

Let $S = \langle \Omega, F \rangle$ be a relational schema and let X be a subset of Ω.
X is a key of S if it satisfies the following two conditions:

(i) $$(X \to \Omega) \in F^+,$$

(ii) $$\overline{\exists} X' \subset X \colon (X' \to \Omega) \in F^+.$$

The subset X which satisfies only (i) is called a *super key* of S.
In the following, instead of $(X \to Y) \in F^+$, we shall write $X \xrightarrow{*} Y$.

2. §.

Let $S=\langle \Omega, F\rangle$ be a relational schema, where

$$\Omega = \{A_1, A_2, \dots, A_n\},$$

$$F = \{L_i \to R_i / i = 1, 2, \dots, k; \quad L_i, R_i \subseteq \Omega\}.$$

Let us denote

$$L = \bigcup_{i=1}^{k} L_i \quad \text{and} \quad R = \bigcup_{i=1}^{k} R_i.$$

Without loss of generality, in this paper we assume that

$$L_i \cap R_i = \emptyset, \quad i = 1, 2, \dots, k.$$

(Results without this assumption can be found in [5], [6].) We have the following lemmas.

Lemma 1. Let $S=\langle \Omega, F\rangle$ be a relational schema.
If $A \bar{\in} L$ and $X \xrightarrow{*} Y$ then

$$X \setminus \{A\} \xrightarrow{*} Y \setminus \{A\}.$$

Proof. From the algorithm for computing the closure X^+ of X w.r.t. F (see 1. §.) it is easy to prove by induction that if $A \bar{\in} L$ then $(X \setminus A)^+$ is equal either to X^+ or to $(X^+ \setminus A)$.

On the other hand, $X \xrightarrow{*} Y$ implies $Y \subseteq X^+$.

Hence
$$Y \setminus A \subseteq X^+ \setminus A.$$

(i) The case $(X \setminus A)^+ = X^+$. From $Y \setminus A \subseteq X^+ \setminus A$, it is clear that

$$Y \setminus A \subseteq X^+ = (X \setminus A)^+.$$

Hence

$$X \setminus A \xrightarrow{*} Y \setminus A.$$

(ii) The case $(X \setminus A)^+ = X^+ \setminus A$.
From $Y \setminus A \subseteq X^+ \setminus A$, it is obvious that

$$Y \setminus A \subseteq (X \setminus A)^+ = X^+ \setminus A.$$

Hence

$$X \setminus A \xrightarrow{*} Y \setminus A.$$

Lemma 2. Let $S=\langle \Omega, F\rangle$ be a relational schema, $X \subseteq \Omega$. If $A \in X$ and $X \setminus A \xrightarrow{*} A$ then X is not a key of S.

Proof. By the hypothesis of the lemma,

$$X \setminus A \xrightarrow{*} A.$$

On the other hand, it is obvious that

$$X \setminus A \xrightarrow{*} X \setminus A.$$

Applying the union rule, we obtain

$$X\setminus A \xrightarrow{*} X.$$

Since $A\in X$ then $X\setminus A\subset X$, showing that X is not a key.

We are now in a position to prove the following theorem.

Theorem 1. Let $S=\langle\Omega, F\rangle$ be a relational schema and X a key of S. Then

$$\Omega\setminus R \subseteqq X \subseteqq (\Omega\setminus R)\cup(L\cap R).$$

Proof. We shall begin with showing that

$$\Omega\setminus R \subseteqq X.$$

First we observe that $X^+\subseteqq X\cup R$. Since X is a key, obviously $X^+=\Omega$.

Hence $X\cup R=\Omega$. This implies that

$$\Omega\setminus R \subseteqq X.$$

It remained to show that:

$$X \subseteqq (\Omega\setminus R)\cup(L\cap R). \tag{1}$$

It is clear that

$$X \subseteqq \Omega = (\Omega\setminus R)\cup(L\cap R)\cup(R\setminus L). \tag{2}$$

To obtain (1), it is therefore sufficient to prove that

$$X\cap(R\setminus L) = \emptyset.$$

Assume the contrary. Then, there would exist an attribute $A\in X$, $A\in R$ and $A\bar{\in} L$. Since X is a key, we have $X\xrightarrow{*}\Omega$. Since $A\bar{\in} L$, we refer to Lemma 1 to deduce

$$X\setminus\{A\}\xrightarrow{*}\Omega\setminus\{A\}.$$

On the other hand, from $A\bar{\in} L$ and $L\subseteqq\Omega$, we have $L\subseteqq\Omega\setminus A$. Hence $\Omega\setminus A\xrightarrow{*}L$.

Applying the transitivity rule for the sequence $X\setminus A\xrightarrow{*}\Omega\setminus A\xrightarrow{*}L\xrightarrow{*}R\xrightarrow{*}A$ (since $A\in R$) we obtain

$$X\setminus A\xrightarrow{*} A \quad \text{with} \quad A\in X.$$

By virtue of Lemma 2, this contradicts the hypothesis that X is a key.

Thus we have proved that if X is a key, then $X\cap(R\setminus L)=\emptyset$.

From (2) we deduce that

$$X\subseteqq(\Omega\setminus R)\cup(L\cap R).$$

The proof is complete.

Remark 1. Theorem 1 can be deduced from Lemma 6 in Békéssy's and Demetrovics' paper [9]. Here another formal proof has been given. Theorem 1 is illustrated by Fig. 1, where X is an arbitrary key of the relational schema $S=\langle\Omega, F\rangle$.
In view of Theorem 1, it is seen that the keys of $S=\langle\Omega, F\rangle$ are different only on the attributes of $L\cap R$. In other words, if X_1 and X_2 are two different keys of S, then

$$X_1\setminus X_2\subset L\cap R \quad \text{and} \quad X_2\setminus X_1\subset L\cap R.$$

Let $\mathscr{K}(\Omega, F)$ denote the set of all keys of $S=\langle\Omega, F\rangle$, and $\mathscr{S}(Z)$ — the maximal cardinality Sperner system on a set Z [7].

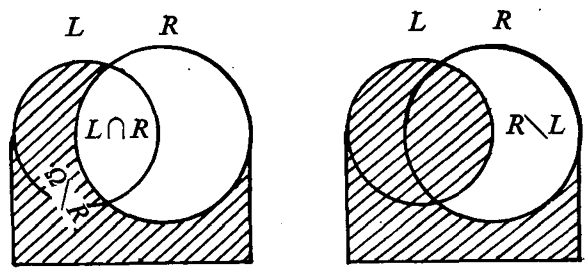


Fig. 1

As an immediate consequence of Theorem 1 and results in [8], [9], we have the following corollaries.

Corollary 1. Let $S=\langle\Omega, F\rangle$ be a relational schema. Then

$$\#\mathcal{K}(\Omega, F) \leqq \#\mathcal{S}(L\cap R) = C_h^{[h/2]}$$

where $h=\#(L\cap R)$.

Corollary 2. Let $S=\langle\Omega, F\rangle$ be a relational schema, and X a key of S. Then

$$\#(\Omega\setminus R) \leqq \#X \leqq \#(\Omega\setminus R)+\#(L\cap R).$$

Corollary 3. Let $S=\langle\Omega, F\rangle$ be a relational schema.
If $R\setminus L\neq\emptyset$ then there exists a key X such that $X\neq\Omega$ (non trivial key).

Corollary 4. Let $S=\langle\Omega, F\rangle$ be a relational schema.
If $L\cap R=\emptyset$ then $\#\mathcal{K}(\Omega, F)=1$ and $\Omega\setminus R$ is the unique key of S.

Theorem 2. Let $S=\langle\Omega, F\rangle$ be a relational schema, where

$$L\cap R = \{A_{t_1}, A_{t_2}, \dots, A_{t_h}\} \subseteqq \{A_1, A_2, \dots, A_n\} = \Omega.$$

Let us define

$$K(1) = (\Omega\setminus R)\cup(L\cap R),$$

$$K(i+1) = \begin{cases} K(i)\setminus A_{t_i} & \text{if } K(i)\setminus A_{t_i} \xrightarrow{*} A_{t_i}, \\ K(i) & \text{if } K(i)\setminus A_{t_i} \not\xrightarrow{*} A_{t_i} \end{cases}$$

$$\text{with } i = 1, 2, \dots, h.$$

Then $K(h+1)$ is a key of $S=\langle\Omega, F\rangle$.

Proof. We shall begin with showing that

$$K(i+1) \xrightarrow{*} K(i).$$

Two cases can occur:

a) If $K(i)\setminus A_{t_i} \not\xrightarrow{*} A_{t_i}$ then from the definition of $K(i+1)$ we have $K(i+1)= =K(i)$ and it is obvious that

$$K(i+1) \xrightarrow{*} K(i).$$

b) If $K(i)\setminus A_{t_i} \xrightarrow{*} A_{t_i}$, we have

$$K(i+1) = K(i)\setminus A_{t_i}.$$

On the other hand, it is obvious that

$$K(i)\setminus A_{t_i} \xrightarrow{*} K(i)\setminus A_{t_i}.$$

Applying the union rule, we get:

$$K(i)\setminus A_{t_i} \xrightarrow{*} K(i).$$

Therefore

$$K(i+1) \xrightarrow{*} K(i).$$

So we have:

$$K(h+1) \xrightarrow{*} K(h) \xrightarrow{*} \dots \xrightarrow{*} K(1).$$

From the above definition of $K(i+1)$ it is clear that

$$K(h+1) \subseteq K(h) \subseteq \dots \subseteq K(1).$$

We are now in a position to prove the theorem. As an immediate consequence of Theorem 1, $K(1)=(\Omega\setminus R)\cup(R\cap L)$ is a superkey of $\langle\Omega, F\rangle$. On the other hand $K(h+1) \xrightarrow{*} K(1)$ showing that $K(h+1)$ is a superkey. To complete the proof, it remains to show that $K(h+1)$ is a key.

Were it false, there would exist a key $\bar{X}$ such that $\bar{X} \subset K(h+1)$, and using the result of Theorem 1 we find

$$\Omega\setminus R \subseteq \bar{X} \subset K(h+1) \subseteq (\Omega\setminus R)\cup(L\cap R)$$

clearly, there exist

$$A_{t_i} \in K(h+1)\cap(L\cap R)\setminus\bar{X}$$

$$\text{with} \quad 1 \leqq i \leqq h.$$

From the definition of $K(i+1)$, we find $K(i)\setminus A_{t_i} \not\xrightarrow{*} A_{t_i}$. Since $K(h+1) \subseteq K(i)$ it follows that $K(h+1)\setminus A_{t_i} \not\xrightarrow{*} A_{t_i}$. On the other hand $\bar{X} \subseteq K(h+1)\setminus A_{t_i}$. Therefore $\bar{X} \not\xrightarrow{*} A$ which conflicts with the fact that $\bar{X}$ is a key of $\langle\Omega, F\rangle$.

The proof is complete.

It is natural to ask whether the results formulated in Theorem 1 can be improved. The answer is in the affirmative as the following lemmas and theorems show.

Lemma 3. Let $S=\langle\Omega, F\rangle$ be a relational schema and X a key of S. Then

$$X\cap R\cap(L\setminus R)^{+} = \emptyset.$$

Proof. Suppose it were not so then there would exist on attribute A such that $A \in X\cap R\cap(L\setminus R)^{+}$, thus $A \in X$, $A \in R$ and $L\setminus R \xrightarrow{*} A$. Since $A \in R$, it follows that $A \bar{\in} (L\setminus R)$.

On the other hand, it is clear that

$$L\setminus R \subseteq \Omega\setminus R.$$

Taking Theorem 1 into account we get

$$L\setminus R \subseteq \Omega\setminus R \subseteq X.$$

Thus

$$L\setminus R \subseteq X\setminus A \quad (\text{since } A \bar{\in} L\setminus R).$$

Evidently $X\setminus A \xrightarrow{*} L\setminus R \xrightarrow{*} A$, where $A\in X$.
By Lemma 2, this contradicts the hypothesis that X is a key of S. The proof is complete.
We define $a(L, R)=(L\setminus R)^+\cap(L\cap R)$.
It is clear that $a(L, R)\subseteqq(L\setminus R)^+\cap R$.
From this: $X\cap a(L, R)=\emptyset$.
Combining with Theorem 1, the following theorem is obvious.

Theorem 3. Let $S=\langle\Omega, F\rangle$ be a relational schema, and X a key of S. Then:

$$(\Omega\setminus R)\subseteqq X\subseteqq(\Omega\setminus R)\cup((L\cap R)\setminus a(L, R)).$$

Here is an example where $a(L, R)\neq\emptyset$.

Example 1. $\Omega=\{A, B, H, G, Q, M, N, V, W\}$

$$F=\{A\to B,\ B\to H,\quad G\to Q,\quad V\to W,\quad W\to V\}$$

From this we have

$L=ABGVW;\quad R=BHQVW,\quad L\cap R=BVW;$

$L\setminus R=AG;\quad (L\setminus R)^+=AGBHQ$

$a(L, R)=(L\setminus R)^+\cap(L\cap R)=\{B\}\neq\emptyset.$

Remark 2. It is worth noticing that $(\Omega\setminus R)^+=(\Omega\setminus(L\cup R))\cup(L\setminus R)^+$. Therefore, if X is a key of S then obviously:

$$X\cap R\cap(\Omega\setminus R)^+=X\cap R\cap(L\setminus R)^+=\emptyset$$

and

$$(\Omega\setminus R)\cup\{(L\cap R)\setminus(\Omega\setminus R)^+\}=(\Omega\setminus R)\cup\{(L\cap R)\setminus a(L, R)\}.$$

Remark 3. Using Theorem 3, Corollaries 1, 2, 3 deduced from Theorem 1 above, can be improved, as well.

3. §.

Based on Theorems 1 and 2, we now propose some algorithms for the key finding and key recognition problem.

It is worth recalling that:
(i) X is a superkey of $S=\langle\Omega, F\rangle$ iff $X^+=\Omega$;
(ii) $X\xrightarrow{*}Y$ iff $Y\subseteqq X^+$.

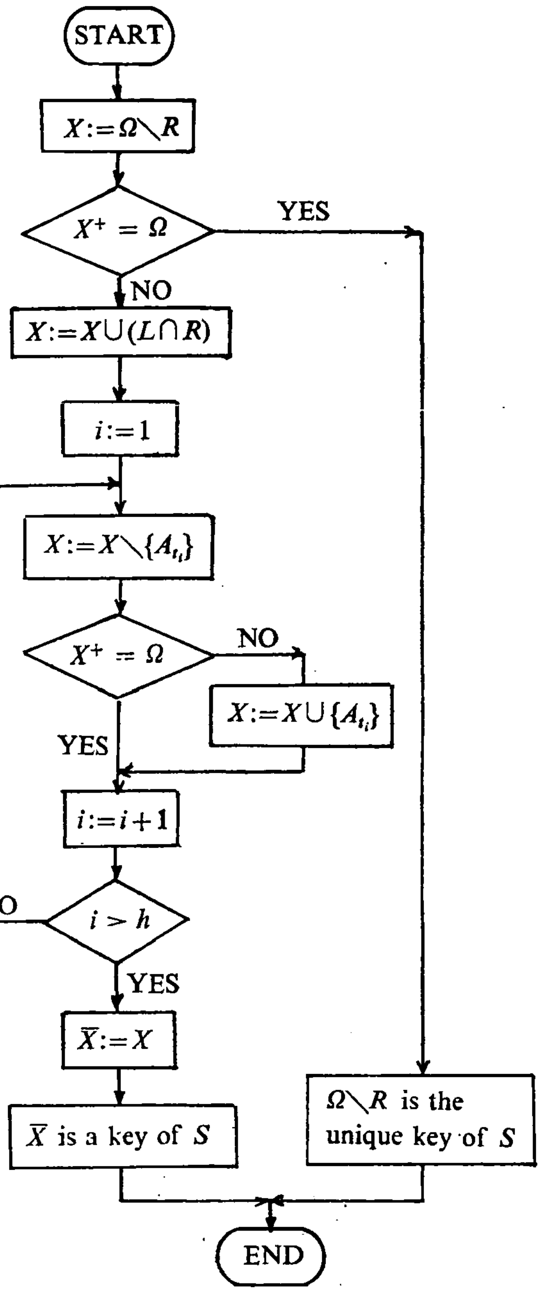
Algorithm 1.
Algorithm for finding one key of the relational schema $S=\langle\Omega, F\rangle$ where

$$\Omega=\{A_1, A_2, \ldots, A_n\},$$

$$F=\{L_i\to R_i/i=1, 2, \ldots, k;\ L_i, R_i\subseteqq\Omega\},$$

$$L=\bigcup_{i=1}^{k}L_i,\quad R=\bigcup_{i=1}^{k}R_i,$$

$$L\cap R=\{A_{t_1}, A_{t_2}, \ldots, A_{t_h}\}.$$


START
X:=Ω\R
X⁺ = Ω
YES
NO
X:=X∪(L∩R)
i:=1
X:=X\{A_{t_i}}
X⁺ = Ω
NO
X:=X∪{A_{t_i}}
YES
i:=i+1
NO
i > h
YES
X̄:=X
X̄ is a key of S
Ω\R is the unique key of S
END

Example 2.

The following example illustrates the performance of the Algorithm 1. Let $S=\langle\Omega, F\rangle$ be a relation scheme with

$$\Omega = \{A, B, C, D, E, G\}$$

and

$$F = \{B \to C, \quad C \to B, \quad A \to GD\}.$$

From this we have

$$L = BCA, \quad R = BCGD,$$

$$\Omega \setminus R = EA, \quad L \cap R = BC.$$

Since $(\Omega\setminus R)^+ = (EA)^+ = EAGD \neq \Omega$, $(\Omega\setminus R)$ is not a key of $S=\langle\Omega, F\rangle$. From the block ③, the algorithm begins with the superkey $X=EABC$. With $A_{t_1}=B$ and $A_{t_2}=C$, we have the sequence

$$X := X\setminus\{B\} = EAC; \quad (EAC)^+ = EACBGD = \Omega,$$

$$X := X\setminus\{C\} = EA; \quad (EA)^+ = EAGD \neq \Omega,$$

$$X := X\cup\{C\} = EAC; \quad \overline{X} := EAC.$$

We obtained a key of S, being $\overline{X}=EAC$.
Similarly, if we start with the same superkey

$$X = EABC$$

but with $A_{t_1}=C$ and $A_{t_2}=B$, then after the termination of Algorithm 1, we obtain another key of the relational schema $S=\langle\Omega, F\rangle$, being EAB.

Algorithm 2.
Algorithm for finding one key of the relational schema $S=\langle\Omega, F\rangle$ included in a given superkey X.
Suppose that $\overline{X}$ is a key included in X.
Then $\overline{X}\subseteq X$.
On the other hand, from Theorem 1:

$$\Omega\setminus R \subseteq \overline{X} \subseteq (\Omega\setminus R)\cup(L\cap R).$$

Therefore

$$\overline{X} \subseteq (\Omega\setminus R)\cup(X\cap(L\cap R)).$$

Thus we can start with the superkey

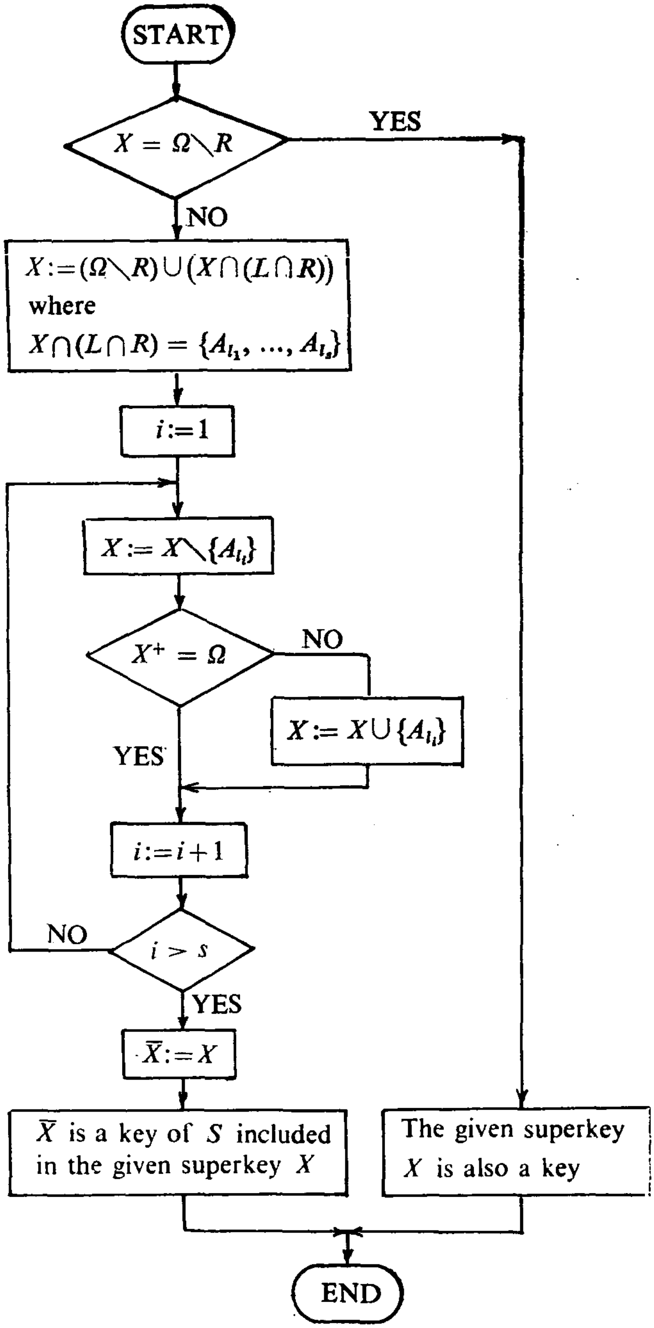
$$(\Omega\setminus R)\cup(X\cap(L\cap R))$$

for finding a key included in a given superkey X.

It is easily seen that Algorithm 2 is similar to Algorithm 1 but block ③ is replaced by the assignment

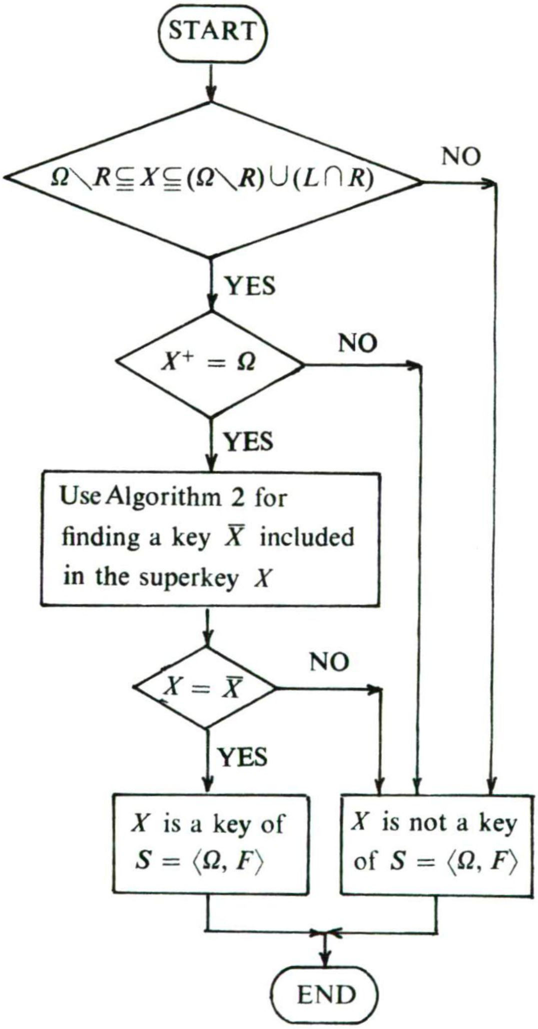
$$X := (\Omega \setminus R) \cup (X \cap (L \cap R))$$

with $X \cap (L \cap R) = \{A_{l_1}, A_{l_2}, \ldots, A_{l_s}\}$ and there are, in addition, some non significant modifications.



Algorithm 3.

Algorithm for recognition whether a given subset X ($X \subseteq \Omega$) is a key of $S = \langle \Omega, F \rangle$.



Remark 4. The Algorithms 1, 2, and 3 can easily be improved using Theorem 3.

Lemma 4. Let $S=\langle \Omega, F\rangle$ be a relational schema.
Then

$$G \cap R = \emptyset,$$

where $G = \bigcap_{X_i \in \mathcal{X}\langle \Omega, F\rangle} X_i$ is the intersection of all keys of S.

Proof. It is sufficient to prove that for each $A \in R$ there exists a key X of S such that $A \bar{\in} X$.

In fact, from $A \in R$ we deduce that A belongs to some R_i. Consider the functional dependency

$$L_i \to R_i, \quad (L_i \cap R_i = \emptyset)$$

therefore

$$A \bar{\in} L_i.$$

It is easily seen that:

$$L_i \cup \{\Omega \setminus (L_i \cup R_i)\} \xrightarrow{*} \Omega$$

and

$$A \bar{\in} L_i \cup \{\Omega \setminus (L_i \cup R_i)\},$$

showing that $L_i \cup \{\Omega \setminus (L_i \cup R_i)\}$ is a superkey of S. This superkey includes a key X such that $A \bar{\in} X$.

From this $G \cap R = \emptyset$.

Theorem 4. Let $S=\langle \Omega, F\rangle$ be a relational schema. Then

$$G = \Omega \setminus R.$$

Proof. As an immediate consequence of Lemma 4 we have

$$G \subseteqq \Omega \setminus R.$$

On the other hand it is easily seen by Theorem 1 that

$$\Omega \setminus R \subseteqq G.$$

Hence

$$G = \Omega \setminus R.$$

The proof is complete.

In most cases it is easier to compute $\Omega \setminus R$ than to compute first all keys directly and take their intersection.

Theorem 5. Let $S=\langle \Omega, F\rangle$ be a relational schema satisfying the following condition

$$\forall_i (R_i \cap L \neq \emptyset \Rightarrow L_i \cap R = \emptyset).$$

Then S has exactly one key and $\Omega \setminus R$ is this unique key.

Proof. Let $C = \Omega \setminus (L \cup R)$. Since $L \xrightarrow{*} R$, consequently

$$L \cup C \xrightarrow{*} L \cup C \cup R = \Omega.$$

Let $I = \{i, R_i \cap L \neq \emptyset\}$.

Evidently

$$\bigcup_{i\in I} L_i \cap R = \emptyset \tag{3}$$

and

$$L\cap R \subseteq \bigcup_{i\in I} R_i. \tag{4}$$

It is obvious that

$$\bigcup_{i\in I} R_i \xrightarrow{*} L\cap R.$$

On the other hand we have

$$\bigcup_{i\in I} L_i \xrightarrow{*} \bigcup_{i\in I} R_i.$$

From (4), clearly

$$\bigcup_{i\in I} L_i \xrightarrow{*} L\cap R.$$

From (3) we have

$$\bigcup_{i\in I} L_i \subseteq L\setminus R.$$

Hence

$$L\setminus R \xrightarrow{*} \bigcup_{i\in I} L_i \xrightarrow{*} L\cap R.$$

From this we get

$$L\setminus R \xrightarrow{*} (L\setminus R)\cup(L\cap R).$$

That is $L\setminus R \xrightarrow{*} L$.

Using $L\cup C \xrightarrow{*} \Omega$, we have

$$(L\setminus R)\cup C \xrightarrow{*} \Omega.$$

Evidently $(L\setminus R)\cup C=\Omega\setminus R$ is a superkey of S. By Theorem 1, $S=\langle\Omega, F\rangle$ has $(\Omega\setminus R)$ as the unique key.

Theorem 6. Let $S=\langle\Omega, F\rangle$ be a relational schema, X a superkey of S. If $X\cap R=\emptyset$ then X is the unique key of S.

Proof. From $X\cap R=\emptyset$, it is obvious that $X\subseteq \Omega\setminus R$. Since X is a superkey of S, there exist a key $\bar{X}\subseteq X$. Using Theorem 1, clearly

$$(\Omega\setminus R)\subseteq \bar{X}\subseteq X\subseteq(\Omega\setminus R)$$

showing that $\Omega\setminus R$ is the unique key of S.

Theorem 7. Let $S=\langle\Omega, F\rangle$ be a relational schema and X a superkey. Then X is a unique key of S iff $X\cap R=\emptyset$.

Proof. The sufficiency of this theorem is essentially Theorem 6. We have only to prove the necessity. Let X be the unique key of $S=\langle\Omega, F\rangle$. Assume the contrary that $X\cap R\neq\emptyset$. Then we should have $A\in X\cap R$.

Evidently $A\in R$ and $A\in X$. In view of Lemma 4, there exists a key $\bar{X}$ such that $A\notin\bar{X}$. Thus $X, \bar{X}$ are different keys of S, which contradicts the condition that X is the unique key of S.

Theorem 8. Let $S=\langle\Omega, F\rangle$ be a relational schema and let $A\in\Omega$ satisfies the following conditions: for all L_i,

(i) $$A \in L_i \Rightarrow L_i \setminus A \xrightarrow{*} A, \quad \text{and}$$

(ii) $$A \bar{\in} L_i \Rightarrow A \in L_i^+.$$

Then A is a non prime attribute, that is $A \bar{\in} H$, where

$$H = \bigcup_{X_i \in \mathscr{X}(\Omega, F)} X_i.$$

Proof. The proof is by contradiction. Assume the contrary that $A \in H$. Then there would exist a key X of S such that $A \in X$, and an L_i such that $L_i \subseteq X$.

(i) If $A \in L_i$ then by the hypothesis of the theorem, we have

$$L_i \setminus A \xrightarrow{*} A.$$

Consequently

$$X \setminus A \xrightarrow{*} L_i \setminus A \xrightarrow{*} A$$

which, by Lemma 2, contradicts the fact that X is a key of S.

(ii) If $A \bar{\in} L_i$ then by the hypothesis of the theorem, we have $A \in L_i^+$.
Since $A \bar{\in} L_i$, consequently

$$L_i \subseteq X \setminus A.$$

Hence

$$X \setminus A \xrightarrow{*} L_i \xrightarrow{*} A$$

which contradicts the fact that X is a key of S.

Thus $A \bar{\in} H$. The proof is complete.

Example 3.

$$\Omega = \{A_1, A_2, A_3, A_4, A_5, A_6\} \quad \text{and}$$

$$F = \{A_1 \to A_3A_5;\ A_3A_4 \to A_1A_6;\ A_1A_5A_6 \to A_2A_4\}.$$

It is easy to verify that A_5 satisfies all conditions of Theorem 8. Therefore $A_5 \bar{\in} H$.

Acknowledgment

The authors would like to take this opportunity to express their sincere thanks to Prof. Dr. J. Demetrovics and Dr. A. Békéssy for their valuable comments and suggestions.

Abstract

In this paper we investigate some characteristic properties of a given relational schema $S = \langle \Omega, F \rangle$, in particular the necessary conditions under which a subset X of Ω is a key.

Basing on these results, some effective algorithms are proposed for the key finding problem and key recognition problem. Moreover, a simple explicit formula is given for computing the intersection of all keys of S, as well as sufficient conditions for which a relational schema has exactly one key, and a criterion for which an attribute is a non prime one.

AUTHOR'S ADDRESS:
HO THUAN, LE VAN BAO,
· COMPUTER AND AUTOMATION INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES, BUDAPEST VICTOR HUGO STR. 18—22.
· INSTITUTE OF COMPUTER SCIENCES AND CYBERNETICS, BA DINH—LIEU GIAI—HANOI—VIET-NAM

References

[1] DELOBEL, C., Theoretical aspects of modeling in relational data base. Rapport de recherche № 177, July 1979, Université scientifique et médicale et Institut national polytechnique de Grenoble.
[2] ARMSTRONG, W. W., Dependency structures of database relationships, Information processing 74. North Holland Publ. Co. Amsterdam 1974.
[3] ULLMAN, J., Principles of data base systems. Computer Science Press, 1980.
[4] BEERI, C. and BERNSTEIN, P. A., Computational problems related to the design of normal forms for relational schemas, ACM Transactions on Data base systems, vol 4, № 1 March 1979.
[5] LE VAN BAO and HO THUAN, On some properties of keys of relational schemas, Preprint series, № 9, 1983, Hanoi. Institute of Mathematics and Institute of Computer science and Cybernetics.
[6] LE VAN BAO and HO THUAN, Sufficient conditions for which a relational schema has precisely one key, Preprint series, № 7, 1983. Hanoi.
[7] SPERNER, E., Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928), 544—548.
[8] DEMETROVICS, J., On the number of candidate keys, Information processing letters, vol.7. number 6, October 1978.
[9] BÉKÉSSY A. and J. DEMETROVICS, Contributions to the theory of data bases relations, Discrete Math., 27 (1979) 1—10.

(Received Dec. 12, 1983)