On the equivalence of the frontier-to-root tree transducers I.

By Z. ZACHAR

It is known in finite automata theory that the equivalence problem can be traced back to the isomorphism of automata. Then, in a natural way, one can raise the question whether two frontier-to-root tree transducers (F-transducers) are isomorphic if they are equivalent.

In this paper we deal with this problem. We introduce the class of the connected F-transducers with adapted rules and that of the inferior F-transducers. It will be shown that for each F-transducer there are equivalent F-transducers from the above classes.

Moreover, in the second part we define a subclass of the class of deterministic F-transducers, namely the class of normalized F-transducers. It will be proved that two strongly normalized F-transducers are equivalent if and only if they are isomorphic.

The terminology is used in the sense of [1]. The algebraic notations developed by Gécseg and Steinby in [3, 4] will be used throughout this paper.

1. Notions and notations

By a *ranked alphabet* we mean a finite nonvoid union $F=\bigcup(F_k|k=0, 1, \dots)$ of pairwise disjoint sets F_k.

Take an arbitrary ranked alphabet F and a set R. Then the set of all *F-trees* over R (or trees, for short) is the smallest set $T_F(R)$ satisfying the following conditions.

(i) $F_0\cup R\subseteq T_F(R)$.

(ii) If $f\in F_k$ $(k>0)$ and $p_1, \dots, p_k\in T_F(R)$ then $f(p_1, \dots, p_k)\in T_F(R)$.

We can define the *height* $(h^S(p))$ and *frontier* $(fr^S(p))$ of a tree $p(\in T_F(R))$ with respect to $S(\subseteq R)$ in the following way:

(i) if $p\in T_F(R\setminus S)$ then $fr^S(p)=\varepsilon$, $h^S(p)$ is undefined,

(ii) if $p\in S$ then $fr^S(p)=p$, $h^S(p)=0$, and

(iii) if $p=f(p_1, \dots, p_k)(\in T_F(R)\setminus T_F(R\setminus S))$ then $fr(p)=fr(p_1)\dots fr(p_k)$ and $h^S(p)=\max(h^S(p_i)|i=1, \dots, k)+1$.

Here ε denotes the empty string. If $S=R$ then the symbol S can be omitted.

The set of *subtrees* $(\mathrm{sub}\,(p))$ and the set of proper subtrees $(\mathit{sub}\,(p))$ of a tree p are defined in the usual way.

In the rest of this paper the pairwise disjoint sets of variables $X=\{x_1, x_2, \ldots\}$, $Y=\{y_1, y_2, \ldots\}$ and $Z=\{z_1, z_2, \ldots\}$ are kept fixed. The symbols $z_1, z_2, \ldots$ are used as auxiliary variables. For arbitrary integer $n(\geqq 0)$, X_n, Y_n and Z_n denote the sets $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ and $\{z_1, \ldots, z_n\}$, respectively.

If $p\in T_F(X_n\cup Z_k)$ and $fr^Z(p)=z_{i_1}\ldots z_{i_l}$ then for p we also use the notations $p(z_1, \ldots, z_k)$ and $p\langle z_{i_1}, \ldots, z_{i_l}\rangle$. Substituting $t_i(\in T_F(X_n\cup Z))$ $(1\leqq i\leqq k)$ for the auxiliary variable z_i $(1\leqq i\leqq k)$ in a tree p we obtain another tree which is denoted by $p(t_1, \ldots, t_k)$. Let $p=q(z_{i_1}, \ldots, z_{i_l})$ where $q\in T_F(X_n\cup Z_l)$ and $fr^Z(q)=z_1\ldots z_l$. Then $p\langle t_1, \ldots, t_l\rangle$ will stand for $q(t_1, \ldots, t_l)$ $(t_i\in T_F(X_n\cup Z),\ i=1, \ldots, l)$, that is the tree $p\langle t_1, \ldots, t_l\rangle$ is obtained by replacing each variables of $z_{i_1}, \ldots, z_{i_l}$ by the tree $t_1, \ldots, t_l$ one after another.

The auxiliary variable z_1 of Z_1 will also be denoted by $\#$.

In the sequel we shall use the notations

$\hat{T}_F(X_n) = \{p \mid p\in T_F(X_n\cup Z_1),\ fr^{Z_1}(p) = \#\}$ and

$\tilde{T}_F(X_n) = T_F(X_n\cup Z_1)\setminus T_F(X_n)$.

If $\bar{p}\in\tilde{T}_F(X_n)$ and $p\in T_F(X_n)$ then we denote the tree $\bar{p}(p)$ by $p\cdot\bar{p}$.

Now we can define the set of the *supertrees* $(\sup\,(p))$ for a tree $p(\in T_F(X_n))$: $\bar{q}\in\tilde{T}_F(X_n)$ is in $\sup\,(p)$ if there exists a $q\in T_F(X_n)$ such that $p=q\cdot\bar{q}$.

We now turn to the definition of a frontier-to-root tree transducer (F-transducer). An *F-transducer* is a system $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', \Sigma)$, where F and G are ranked alphabets, A is a finite nonvoid set of *states*, $A'\subseteqq A$ is the set of *final states*, and Σ is a finite set of rewriting *rules* of the following two types:

(i) $x\to aq$ $(x\in X_n\cup F_0,\ a\in A,\ q\in T_G(Y_m))$ and

(ii) $f(a_1, \ldots, a_k)\to aq(z_1, \ldots, z_k)$ $(f\in F_k,\ k>0,\ a_1, \ldots, a_k,\ a\in A,\ q\in T_G(Y_m\cup Z_k))$.

The transformation induced by $\mathbf{A}$ will be denoted by $\tau_{\mathbf{A}}$. Moreover, let $\mathrm{dom}\,\tau_{\mathbf{A}}$ and $\mathrm{range}\,\tau_{\mathbf{A}}$ be, respectively, the domain and range of $\tau_{\mathbf{A}}$. For an arbitrary tree p we put $\tau_{\mathbf{A}}(p)=\{q \mid (p, q)\in\tau_{\mathbf{A}}\}$.

For an F-transducer $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', \Sigma)$ and two sets $A_1, A_2(\subseteqq A)$ we denote by $\tau^{A_1}_{\mathbf{A}, A_2}$ the transformation induced by

$$(T_F(X_n\cup Z_1), A, T_G(Y_m\cup Z_1), A_1, \Sigma\cup\{\#\to a\# \mid a\in A_2\}).$$

Moreover, let

$\mathit{dom}\,\tau^{A_1}_{\mathbf{A}, A_2} = \mathrm{dom}\,\tau^{A_1}_{\mathbf{A}, A_2}\cap\hat{T}_F(X_n)$ and

$\mathit{range}\,\tau^{A_1}_{\mathbf{A}, A_2} = \{q \mid p\in\mathit{dom}\,\tau^{A_1}_{\mathbf{A}, A_2},\ q\in\tau^{A_1}_{\mathbf{A}, A_2}(p)\}$.

If $A_1=A'$ and $A_2=\emptyset$, then A_1 and A_2 will generally be omitted in $\tau^{A_1}_{\mathbf{A}, A_2}$. Furthermore, if there is no danger of confusion then we write τ instead of $\tau_{\mathbf{A}}$. Let us note that a singleton will also be denoted by its element.

Take an arbitrary F-transducer $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', \Sigma)$. If $\tau_{\mathbf{A}}$ is a partial mapping then $\mathbf{A}$ is called *functional*. Moreover, $\mathbf{A}$ is *deterministic*, if all its different rules have different left sides.

Let $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', \Sigma_A)$ and $\mathbf{B}=(T_F(X_n), B, T_G(Y_m), B', \Sigma_B)$ be two F-transducers and take a bijective mapping μ of A onto B. If the following three conditions are satisfied then μ is called an *isomorphism*.

(i) $x \to aq \in \Sigma_A$ $(x \in X_n \cup F_0, a \in A)$ if and only if $x \to \mu(a)q \in \Sigma_B$.
(ii) $f(a_1, \ldots, a_k) \to a_0 q \in \Sigma_A$ if and only if $f(\mu(a_1), \ldots, \mu(a_k)) \to \mu(a_0)q \in \Sigma_B$, where $f \in F_k$ $(k > 0)$ and $a_i \in A$ $(i = 0, 1, \ldots, k)$.
(iii) $\mu(A') = B'$.

We can say that $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic*.

Finally, two F-transducers are called *equivalent* if the transformations induced by them coincide.

2. Inferior F-transducers

Let $\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$ be an arbitrary F-transducer. It is called *connected*, if for each rule of the form $x \to aq$ $(x \in X_n \cup F_0)$ and $f(a_1, \ldots, a_k) \to aq$ $(f \in F_k, k > 0)$ in Σ, there are trees $p_1, \ldots, p_k$, $\bar{p}$ such that $\bar{p} \in \mathit{dom}\ \tau_a$ and $p_i \in \mathrm{dom}\ \tau^{a_i}$ $(i = 1, \ldots, k)$, moreover, the set A of states coincides with $\{a \mid p \to aq \in \Sigma\}$.

One can easily show that for every $\mathbf{A}$ there is a connected F-transducer $\mathbf{B}$ with $\tau_{\mathbf{A}} = \tau_{\mathbf{B}}$.

Definition 1. By a connected F-transducer with its *adapted rules* (AF-transducer), we mean a connected F-transducer $\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$ such that each state $a (\in A)$ satisfies the following conditions:
(i) if range τ^a is a singleton then for each tree $\bar{p} \in \mathrm{dom}\ \tau_a \setminus \{\#\}$, the inclusion $\tau_a(\bar{p}) \subseteq T_G(Y_m)$ holds,
(ii) if range $\tau_a \subseteq T_G(Y_m)$ then range $\tau^a = \{y_1\}$.

It is easy to prove that *range* $\tau_a \subseteqq T_G(Y_m)$ if and only if range $\tau_a \subseteqq T_G(Y_m)$. Thus the condition (ii) of the above definition can be replaced by the following:
(ii)′ if *range* $\tau_a \subseteqq T_G(Y_m)$ then range $\tau^a = \{y_1\}$.

Lemma 2. For any connected F-transducer an equivalent AF-transducer can be constructed.

Proof. Let $\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$ be an arbitrary connected F-transducer. We shall construct the F-transducer $\bar{\mathbf{A}} = (T_F(X_n), A, T_G(Y_m), A', \bar{\Sigma})$ by rewriting the rules of Σ.

Assume that range $\tau^a_{\mathbf{A}}$ is a singleton i.e., for each tree $p \in \mathrm{dom}\ \tau^a_{\mathbf{A}}$, $\tau^a_{\mathbf{A}}(p) = q$. Then we replace every rule $f(a_1, \ldots, a_k) \to a_0 r$ in Σ by the rule $f(a_1, \ldots, a_k) \to a_0 r(t_1, \ldots, t_k)$, where $t_i = q$ if $a_i = a$ and $t_i = z_i$ otherwise $(i = 1, \ldots, k)$.

If range $\tau_{\mathbf{A}, a} \subseteqq T_G(Y_m)$ then $a \notin A'$, thus every rule of the form $f(a_1, \ldots, a_k) \to ar$ and $x \to ar$ may be replaced by the rule of the form $f(a_1, \ldots, a_k) \to a y_1$ and $x \to a y_1$, resp.

It is clear that the set $\bar{\Sigma}$ of rules constructed in this way satisfies the conditions of Lemma 2.

Lemma 3. If the AF-transducer $\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$ is functional then for each state $a (\in A)$, τ^a and τ_a are mappings.

Proof. Assume that τ^a $(a \in A)$ is not a mapping. Then $a \notin A'$ and there are trees $p \in \mathrm{dom}\ \tau^a$ and $q_1, q_2 \in \tau^a(p)$ such that $q_1 \neq q_2$. Since range τ^a is not a singleton, thus by condition (ii) of Definition 1 there exist trees $\bar{p} \in \mathrm{dom}\ \tau_a$ and $\bar{q} \in \tau_a(\bar{p})$ such that the tree $\bar{q}$ contains the symbol $\#$ in its frontier. Then $p \cdot \bar{p} \in \mathrm{dom}\ \tau$, so

$q_i \cdot \bar{q} \in \tau(p \cdot \bar{p})$ $(i=1, 2)$. It means that $q_1 \cdot \bar{q} = q_2 \cdot \bar{q}$, therefore $q_1 = q_2$ which contradicts our assumption.

Next let us consider the transformation τ_a. We have that $\mathrm{dom}\,\tau_a = \mathrm{dom}\,\tau \cup \cup \{p \mid p \in \tilde{T}_F(X_n) \cap \mathrm{dom}\,\tau_a\}$. Since τ is a mapping, it suffices to prove that if $\bar{p} \in \mathrm{dom}\,\tau_a \setminus T_F(X_n) \setminus \{\#\}$ and $\bar{q}_1, \bar{q}_2 \in \tau_a(\bar{p})$ then $\bar{q}_1 = \bar{q}_2$.

If range τ^a is a singleton then by condition (i) in the definition of an AF-transducer we know that $\bar{q}_1, \bar{q}_2 \in T_G(Y_m)$. It means that for an arbitrary tree $p \in \mathrm{dom}\,\tau^a$ the equalities $\tau(p \cdot \bar{p}) = q_1$ and $\tau(p \cdot \bar{p}) = q_2$ hold. Consequently $\bar{q}_1 = \bar{q}_2$.

If range τ^a is not a singleton then there are trees $p_1, p_2 \in \mathrm{dom}\,\tau^a$ for which $\tau^a(p_1) = q_1 \neq q_2 = \tau^a(p_2)$. We have that

$$\tau(p_1 \cdot \bar{p}) = q_1 \cdot \bar{q}_1 = q_1 \cdot \bar{q}_2$$

and

$$\tau(p_2 \cdot \bar{p}) = q_2 \cdot \bar{q}_1 = q_2 \cdot \bar{q}_2,$$

which imply that $\bar{q}_1 = \bar{q}_2$. This ends the proof of Lemma 3.

Definition 4. Let $\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$ be an AF-transducer. The transformation induced by the state $a (\in A)$ can *be cut* by the tree $q_a \in \tilde{T}_G(Y_m) \setminus \{\#\}$, if for all $\bar{a} \in A'$, $p \in \mathrm{dom}\,\tau^a_{\bar{a}}$ and $q \in \tau^a_{\bar{a}}(p)$ there is a tree $\tilde{q}$ such that $q = \tilde{q} \cdot q_a$. The tree q_a cuts the transformation τ^a maximally, if τ^a can not be cut by any tree $\bar{q} \cdot q_a$, where $\bar{q} \in \tilde{T}_G(Y_m) \setminus \{\#\}$.

By the above definition the transformation τ^a can be cut by the tree q_a if and only if q_a is a supertree of each tree from the set $\{q \mid q \in \mathrm{range}\,\tau^a_{\bar{a}},\ \bar{a} \in A'\}$.

Theorem 5. There is an algorithm to decide for each AF-transducer $\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$ and arbitrary state $a (\in A)$ whether the transformation τ^a can be cut. Moreover, every tree q_a cutting τ^a can be given effectively.

Proof. Let $K = \max(q \mid q \in \tau^a_{\bar{a}}(p),\ p \in \mathrm{dom}\,\tau^a_{\bar{a}},\ h(p) \leqq \|A\|,\ a \in A,\ \bar{a} \in A')$ and $L = (K+6) \cdot \|A\|$. We denote by Q the set $\{p \mid p \in \tilde{T}_F(X_n),\ h(p) \leqq L\}$. Let $a \in A$ and $q_a \in \tilde{T}_G(Y_m) \setminus \{\#\}$ be arbitrary. It is sufficient to show that the following statement is valid:

if for all $\bar{a} \in A'$, $p \in \mathrm{dom}\,\tau^a_{\bar{a}} \cap Q$ and $q \in \tau^a_{\bar{a}}(p)$ there exists a tree $\tilde{q}$ such that $q = \tilde{q} \cdot q_a$, then the transformation τ^a can be cut by q_a i.e., for all $\bar{a} \in A'$, $p \in \mathrm{dom}\,\tau^a_{\bar{a}}$ and $q \in \tau^a_{\bar{a}}(p)$ the tree q_a is a supertree of q. Obviously, every such q_a can be given effectively.

The proof of this statement can be performed by induction. If $h(p) \leqq L$ then by our assumption the tree q_a is a supertree of each tree from the sets range $\tau^a_{\bar{a}}$ $(\bar{a} \in A')$. Now let $h(p) > L$ and assume that our statement holds for all trees which have less number of occurrences of symbols from F than p has. Then there are two sequences $p_0, \ldots, p_{K+6}$ and $q_0, \ldots, q_{K+6}$ of trees and a state $\tilde{a} (\in A)$ such that $q_0 \in \tau^a_{\tilde{a}}(p_0)$, $q_i \in \tau^{\tilde{a}}_{\tilde{a}}(p_i)$ $(i=1, \ldots, K+5)$, $q_{K+6} \in \tau^{\tilde{a}}_{\bar{a}}(p_{K+6})$, $p_0 \cdot \ldots \cdot p_{K+6} = p$ and $q_0 \cdot \ldots \cdot q_{K+6} = q$.

Now there are three cases.

Firstly, we assume that there is an index j $(2 \leqq j \leqq K+6)$ for which $q_j \in T_G(Y_m)$. Then $q = q_j \cdot \ldots \cdot q_{K+6} = q_0 \cdot q_j \cdot \ldots \cdot q_{K+6} \in \tau^a_{\bar{a}}(p_0 \cdot p_j \cdot \ldots \cdot p_{K+6})$. By the induction hypothesis concerning the tree $p_0 \cdot p_j \cdot \ldots \cdot p_{K+6}$ we have that q_a is a supertree of q.

Secondly, we suppose that there is an index j $(2 \leqq j \leqq K+5)$ for which $q_j = \#$.

It means that $q=q_0\cdot\ldots\cdot q_{j-1}\cdot q_{j+1}\cdot\ldots\cdot q_{K+6}\in\tau_{\bar a}^{a}(p_0\cdot\ldots\cdot p_{j-1}\cdot p_{j+1}\cdot\ldots\cdot p_{K+6})$. Again by the induction hypothesis, we get that there exists a tree $\tilde q$ for which $q=\tilde q\cdot q_a$.

Finally, we may assume that $h^{\#}(q_j)>0$ $(j=2,\ldots,K+5)$ and $q_{K+6}\in\tilde T_G(Y_m)$. Let $\bar q=q_5\cdot\ldots\cdot q_{K+6}$. Furthermore, we have that $r=q_0\cdot q_1\cdot q_2\neq q_0\cdot q_1\cdot q_2\cdot q_3=s$. By the induction hypothesis there are trees $\tilde r$ and $\tilde s$ such that $r\cdot\bar q=\tilde r\cdot q_a$ and $s\cdot\bar q=\tilde s\cdot q_a$. We know that $h(q_a)\leqq K$ and $h(\bar q)>K$. From this we obtain that the tree $\bar q$ can be given in the form $\hat q\cdot q_a$, i.e. q_a is a supertree of $\bar q$. Since $q=q_0\cdot q_1\cdot q_2\cdot q_3\cdot q_4\cdot\bar q=q_0\cdot q_1\cdot q_2\cdot q_3\cdot q_4\cdot\hat q\cdot q_a$, thus there is a tree $\tilde q$ for which $q=\tilde q\cdot q_a$. This ends the proof of our lemma.

Definition 6. An AF-transducer $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', \Sigma)$ is called *inferior* if none of the transformations induced by its states can be cut by any trees.

Take an AF-transducer $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', \Sigma)$. Assume that the transformations induced by the states $a_1,\ldots,a_l$ can be cut and the tree q_{a_i} cuts τ^{a_i} maximally $(i=1,\ldots,l)$. For a state a, if τ^a can not be cut $(a\notin\{a_1,\ldots,a_l\})$ then let $q_a=\#$. It means that for all $a\in A$, $\bar a\in A'$, $p\in\operatorname{dom}\tau_{\bar a}^{a}$ and $q\in\tau_{\bar a}^{a}(p)$, the equality $q=\tilde q\cdot q_a$ holds under a suitable $\tilde q$.

The following lemma is valid under these notations.

Lemma 7. There is an inferior F-transducer $\bar{\mathbf{A}}$ which is equivalent to $\mathbf{A}$.

Proof. We shall show that one can construct an F-transducer

$$\bar{\mathbf{A}}=(T_F(X_n), A, T_G(Y_m), A', \bar\Sigma)$$

such that for all states $a\in A$ and $\bar a\in A'$ the following conditions are satisfied.

(1) $\operatorname{dom}\tau_{\mathbf{A}}^{a}=\operatorname{dom}\tau_{\bar{\mathbf{A}}}^{a}$ and $\operatorname{dom}\tau_{\mathbf{A},\bar a}^{a}=\operatorname{dom}\tau_{\bar{\mathbf{A}},\bar a}^{a}$.

(2) $\operatorname{dom}\tau_{\mathbf{A},a}=\operatorname{dom}\tau_{\bar{\mathbf{A}},a}$.

(3) $\{(p, q\cdot q_a)\mid q\in\tau_{\bar{\mathbf{A}},\bar a}^{a}(p), p\in\operatorname{dom}\tau_{\mathbf{A},\bar a}^{a}\}=\tau_{\mathbf{A},\bar a}^{a}$.

(4) $\{(p, q_a\cdot q)\mid q\in\tau_{\mathbf{A},a}(p), p\in\operatorname{dom}\tau_{\mathbf{A},a}\}=\tau_{\bar{\mathbf{A}},a}$.

From this Lemma 7 will follow. Indeed, from (3) we get that $\bar{\mathbf{A}}$ is equivalent to $\mathbf{A}$. If range $\tau_{\bar{\mathbf{A}}}^{a}$ is a singleton then range $\tau_{\mathbf{A}}^{a}$ is a singleton by (3), too. Using condition (i) of Definition 1 we have that for each $\bar p\in\operatorname{dom}\tau_{\mathbf{A},a}\setminus\{\#\}$, $\tau_{\mathbf{A},a}(\bar p)\subseteq T_G(Y_m)$. Therefore, by (4), $\tau_{\bar{\mathbf{A}},a}(\bar p)\subseteq T_G(Y_m)$. It means that (i) of Definition 1 holds for $\bar{\mathbf{A}}$. Similarly, we obtain that $\bar{\mathbf{A}}$ satisfies condition (ii). Consequently, $\bar{\mathbf{A}}$ is an AF-transducer. It is also clear that $\bar{\mathbf{A}}$ is an inferior F-transducer, too. In the opposite case we would arrive at a contradiction by assuming the maximality of the trees q_a $(a\in A)$.

Next we define the rules of $\bar{\mathbf{A}}$ in the following way.

(i) $x\to ar\in\Sigma$ $(x\in X_n\cup F_0)$ if and only if
$x\to a\bar r\in\bar\Sigma$, where $r=\bar r\cdot q_a$.

(ii) $f(a_1,\ldots,a_k)\to ar\in\Sigma$ $(f\in F_k,\ k>0)$ if and only if
$f(a_1,\ldots,a_k)\to a\bar r$, where the tree $\bar r=r(q_{a_1}(z_1),\ldots,q_{a_k}(z_k))$ is equal to $\bar r\cdot q_a$.

First, we show that the rules of $\bar\Sigma$ can be constructed. It is obvious, that this construction can be performed if the rule satisfies the assumption (i) or (ii) provided the equality $q_a=\#$.

Then let $f(a_1, ..., a_k) \to ar \in \Sigma$ ($f \in F_k$, $k > 0$) be an arbitrary rule such that $q_a \in \tilde{T}_G(Y_m) \setminus \{\#\}$. We have that for every final states $\bar{a}$ and all trees $p_i \in \operatorname{dom} \tau_{\bar{a}}^{a_i}$, $q_i \in \tau_{\bar{a}}^{a_i}(p_i)$ ($i = 1, ..., k$) the following conditions hold.

(a) $r(q_1, ..., q_k) \in \tau_{\bar{a}}^{a}(f(p_1, ..., p_k))$ and

(b) $r(q_1, ..., q_k) = r(\tilde{q}_1 \cdot q_{a_1}, ..., \tilde{q}_k \cdot q_{a_k}) =$

$$= r(q_{a_1}(z_1), ..., q_{a_k}(z_k))(\tilde{q}_1, ..., \tilde{q}_k) = \bar{r}(\tilde{q}_1, ..., \tilde{q}_k).$$

Let $f(p_1, ..., p_k) = p$ and $r(q_1, ..., q_k) = q$. By Definition 4, $q = \tilde{q} \cdot q_a$. Therefore, $\bar{r}(\tilde{q}_1, ..., \tilde{q}_k) = \tilde{q} \cdot q_a$.

Let s be a tree for which there exist trees $r_1, ..., r_m \in T_G(Y_m \cup Z_k)$ and $t_1, ..., t_m \in T_G(Y_m \cup \{\#\})$ ($m > 0$) such that $s\langle r_1, ..., r_m\rangle = \bar{r}$ and $s\langle t_1, ..., t_m\rangle = q_a$, moreover, for each index j ($1 \leqq j \leqq m$) at least one of the conditions $r_j \in Z_k$ and $t_j = \#$ holds. It means that for an arbitrary index j ($1 \leqq j \leqq m$), $r_j(\tilde{q}_1, ..., \tilde{q}_k) = \tilde{q} \cdot t_j$.

Assume that $r_j \in Z_k$, i.e. there is an index l ($1 \leqq l \leqq k$) satisfying $r_j = z_l$. Thus for each tree $p_l \in \operatorname{dom} \tau_{\bar{a}}^{a_l}$ and $q_l \in \tau_{\bar{a}}^{a_l}(p_l)$ the equalities $\tilde{q}_l \cdot q_{a_l} = q_l$ and $\tilde{q}_l = \tilde{q} \cdot t_j$ hold, that is t_j is a supertree of $\tilde{q}_l$.

If $t_j \in T_G(Y_m)$ then $q_l = \tilde{q} \cdot t_j \cdot q_{a_l} = t_j \cdot q_{a_l}$ implies that range τ^{a_l} is a singleton. On the other hand the symbol z_l is contained in the frontier of the tree $\bar{r}$. Therefore, it should occur in the frontier of r, too. This means that range $\tau_{a_l} \nsubseteq T_G(Y_m)$, thus by the condition (i) of Definition 1 range τ^{a_l} is not a singleton which is a contradiction. Then we have that $t_j \in \tilde{T}_G(Y_m)$.

If $t_j \neq \#$ then, by $q_l = \tilde{q} \cdot t_j \cdot q_{a_l}$, the transformation τ^{a_l} can be cut by the tree $t_j \cdot q_{a_l}$ which contradicts the maximality of q_{a_l}.

Now we have that for each index j ($1 \leqq j \leqq m$), $t_j = \#$. It implies that $s = q_a$. Therefore, $\bar{r} = q_a\langle r_1, ..., r_m\rangle$. Using (b) we obtain that

$$\bar{r}(\tilde{q}_1, ..., \tilde{q}_k) = q_a\langle r_1(\tilde{q}_1, ..., \tilde{q}_k), ..., r_m(\tilde{q}_1, ..., \tilde{q}_k)\rangle = \tilde{q} \cdot q_a,$$

consequently, $\tilde{q} = r_j(\tilde{q}_1, ..., \tilde{q}_k)$ ($j = 1, ..., m$).

We shall prove that the trees $r_1, ..., r_m$ are equal to each other. Let $s_1, s_2 \in \{r_1, ..., r_m\}$ be arbitrary. Then the equality $s_1(\tilde{q}_1, ..., \tilde{q}_k) = s_2(\tilde{q}_1, ..., \tilde{q}_k)$ holds for each $p_i \in \operatorname{dom} \tau^{a_i}$ and $q_i \in \tau^{a_i}(p_i)$ ($i = 1, ..., k$). Let j ($1 \leqq j \leqq k$) be an arbitrary index. Let $p_i \in \operatorname{dom} \tau^{a_i}$ and $t_i \in \tau^{a_i}(p_i)$ ($i = 1, ..., k$; $i \neq j$) be arbitrary fixed trees, moreover $\tilde{t}_j = \#$. Denote the trees $s_1(\tilde{t}_1, ..., \tilde{t}_k)$ and $s_2(\tilde{t}_1, ..., \tilde{t}_k)$ by u_j and v_j, respectively. We have that for each $p_j \in \operatorname{dom} \tau^{a_j}$ and $q_j \in \tau^{a_j}(p_j)$ the equality $\tilde{q}_j \cdot u_j = \tilde{q}_j \cdot v_j$ holds. It is obvious that $u_j \in T_G(Y_m)$ if and only if $v_j \in T_G(Y_m)$, moreover, if $u_j \in \tilde{T}_G(Y_m)$ then range τ^{a_j} is not a singleton. From this we obtain that $u_j = v_j$. It means that for all indices j ($1 \leqq j \leqq k$) the equality $u_j = v_j$ holds, which implies that $s_1 = s_2$.

We now have that $r_1 = r_2 = ... = r_m$, and this tree is denoted by $\bar{r}$. It follows that $\bar{r} = \bar{r} \cdot q_a$, thus the rules of $\bar{\Sigma}$ can be constructed.

Consider the F-transducer $\overline{\mathbf{A}} = (T_F(X_n), A, T_G(Y_m), A', \bar{\Sigma})$ constructed in this way. We will show that $\overline{\mathbf{A}}$ has the properties (1)—(4). By the construction, it is easy to see that (1) and (2) hold. The property (3) shall be proved by induction.

Let $a\in A$ and $\bar{a}\in A'$ be arbitrary states and $p\in\operatorname{dom}\tau^a_{\mathbf{A},\bar{a}}$. Assume that $h(p)=0$. If $p\in X_n\cup F_0$ then $p\to a\tilde{q}\in\bar{\Sigma}$ if and only if $p\to a\tilde{q}\cdot q_a\in\Sigma$. Therefore, $(p,\tilde{q})\in\tau^{\bar{a}}_{\bar{\mathbf{A}},\bar{a}}$ if and only if $(p,\tilde{q}\cdot q_a)\in\tau^a_{\mathbf{A},\bar{a}}$.

If $p=\#$ then $a=\bar{a}$ and $q_a=\#$, thus $\tau^a_{\mathbf{A},\bar{a}}(p)=\tau^{\bar{a}}_{\bar{\mathbf{A}},\bar{a}}(p)=\#$.

Assume that $p=f(p_1,\dots,p_k)$ and $q\in\tau^a_{\mathbf{A},\bar{a}}(p)$. There is a rule $f(a_1,\dots,a_k)\to ar\in\Sigma$ and there exist trees $q_i\in\tau^{a_i}_{\mathbf{A},\bar{a}}(p_i)$ $(i=1,\dots,k)$ such that $q=r(q_1,\dots,q_k)$. By the induction hypothesis we have that there are trees $\tilde{q}_i\in\tau^{\bar{a}_i}_{\bar{\mathbf{A}},\bar{a}}(p_i)$ for which $\tilde{q}_i\cdot q_{a_i}=q_i$ $(i=1,\dots,k)$. Therefore, $q=r(q_{a_1}(z_1),\dots,q_{a_k}(z_k))(\tilde{q}_1,\dots,\tilde{q}_k)$.

By our construction there is a rule $f(a_1,\dots,a_k)\to a\bar{r}\in\bar{\Sigma}$, where

$$r(q_{a_1}(z_1),\dots,q_{a_k}(z_k))=\bar{r}\cdot q_a.$$

Then $\tilde{q}=\bar{r}(\tilde{q}_1,\dots,\tilde{q}_k)\in\tau^{\bar{a}}_{\bar{\mathbf{A}},\bar{a}}(p)$ and $q=\tilde{q}\cdot q_a$. Similarly, we get that if $\tilde{q}\in\tau^{\bar{a}}_{\bar{\mathbf{A}},\bar{a}}(p)$ then $\tilde{q}\cdot q_a\in\tau^a_{\mathbf{A},\bar{a}}(p)$. It means that $\bar{\mathbf{A}}$ has property (3).

Let $\bar{p}\in\operatorname{dom}\tau_{\mathbf{A},a}$ and $r\in\tau_{\mathbf{A},a}(\bar{p})$ be arbitrary trees. By the proof of (3), there is a tree $\tilde{r}\in\tau_{\bar{\mathbf{A}},a}(\bar{p})$ such that for each $p\in\operatorname{dom}\tau^a_{\mathbf{A}}$ and $q\in\tau^a_{\mathbf{A}}(p)$, $q\cdot r=\tilde{q}\cdot\tilde{r}$ and $\tilde{q}\cdot q_a=q$ under a suitable tree $\tilde{q}$. It is easy to show that $r\in T_G(Y_m)$ if and only if $\tilde{r}\in T_G(Y_m)$. It follows that if $r\in T_G(Y_m)$ then $\tilde{r}=r=q_a\cdot r$. If $r\in\tilde{T}_G(Y_m)$, then none of range $\tau^a_{\mathbf{A}}$ and range $\tau^{\bar{a}}_{\bar{\mathbf{A}}}$ is a singleton. Using this we obtain that $\tilde{r}=q_a\cdot r$. It means that if $(\bar{p},r)\in\tau_{\mathbf{A},a}$ then $(\bar{p},q_a\cdot r)\in\tau_{\bar{\mathbf{A}},a}$. The inverse claim can be shown in a similar way.

This ends the proof of Lemma 7.

Let $\mathbf{A}=(T_F(X_n),A,T_G(Y_m),A',\Sigma_A)$ and $\mathbf{B}=(T_F(X_n),B,T_G(Y_m),B',\Sigma_B)$ be AF-transducers for which $\operatorname{dom}\tau_{\mathbf{A}}=\operatorname{dom}\tau_{\mathbf{B}}$. We construct the F-transducers $\mathbf{A}^1=(T_F(X_n),A\times C,T_G(Y_m),A'\times C',\Sigma^1_A)$ and $B^1=(T_F(X_n),B\times C,T_G(Y_m),B'\times C',\Sigma^1_B)$, where $C=A\times B$, $C'=A'\times B'$ and the sets of rules satisfy the following conditions.

(a) For each $c=(a,b)\in C$ and $x\in X_n\cup F_0$,
$x\to(a,c)q\in\Sigma^1_A$ and $x\to(b,c)r\in\Sigma^1_B$ if and only if
$x\to aq\in\Sigma_A$ and $x\to br\in\Sigma_B$.

(b) For each $f\in F_k$ $(k>0)$ and $c_i=(a_i,b_i)$ $(i=0,1,\dots,k)$,
$f((a_1,c_1),\dots,(a_k,c_k))\to(a_0,c_0)q\in\Sigma^1_A$ and
$f((b_1,c_1),\dots,(b_k,c_k))\to(b_0,c_0)r\in\Sigma^1_B$ if and only if
$f(a_1,\dots,a_k)\to a_0q\in\Sigma_A$ and $f(b_1,\dots,b_k)\to b_0r\in\Sigma_B$.

Using a standard construction we get two connected F-transducers $\mathbf{A}^2=(T_F(X_n),\overline{A\times C},T_G(Y_m),\overline{A'\times C'},\Sigma^2_A)$ and $\mathbf{B}^2=(T_F(X_n),\overline{B\times C},T_G(Y_m),\overline{B'\times C'},\Sigma^2_B)$ such that $\mathbf{A}^2$ is equivalent to $\mathbf{A}^1$ and $\mathbf{B}^2$ is equivalent to $\mathbf{B}^1$. Moreover, using the constructions of the proofs of Lemmas 2 and 7 we obtain two inferior F-transducers $\bar{\mathbf{A}}=(T_F(X_n),\overline{A\times C},T_G(Y_m),\overline{A'\times C'},\bar{\Sigma}_A)$ and $\bar{\mathbf{B}}=(T_F(X_n),\overline{B\times C},T_G(Y_m),\overline{B'\times C'},\bar{\Sigma}_B)$ which are equivalent to $\mathbf{A}^2$ and $\mathbf{B}^2$, resp. Let us denote the inferior F-transducers $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ by $\mathbf{A}(\mathbf{B})$ and $\mathbf{B}(\mathbf{A})$, respectively. Since $\tau_{\mathbf{A}}=\tau_{\mathbf{A}(\mathbf{B})}$ both $\tau_{\mathbf{A}}$ and $\tau_{\mathbf{A}(\mathbf{B})}$ will be denoted by φ. Similarly, ψ will denote $\tau_{\mathbf{B}}$ and $\tau_{\mathbf{B}(\mathbf{A})}$.

In the next lemmas and Theorem 11 we shall use the above notations.

Lemma 8. Let $(a,b)=c$, $(\bar{a},\bar{b})=\bar{c}\in C$. Then the following conditions are satisfied:

(i) $(a, c)\in\overline{A\times C}$ if and only if $(b, c)\in\overline{B\times C}$,
(ii) $(a, c)\in\overline{A'\times C'}$ if and only if $(b, c)\in\overline{B'\times C'}$,
(iii) $\mathrm{dom}\,\varphi^{a,c}=\mathrm{dom}\,\psi^{b,c}$,
(iv) $dom\,\varphi_{a,c}=dom\,\psi_{b,c}$,
(v) $\mathrm{dom}\,\varphi^{a,c}_{\bar{a},\bar{c}}=\mathrm{dom}\,\psi^{b,c}_{\bar{b},\bar{c}}$.

Proof. By the definitions of Σ^1_A and Σ^1_B there is a natural bijective mapping of Σ^1_A onto Σ^1_B. It is easy to see that the restriction of the above mapping to $\bar{\Sigma}_A$ is a bijective mapping, too. Using this the statement this lemma is obvious.

In Lemmas 9 and 10 and in Theorem 11 we assume that the AF-transducers **A** and **B** are equivalent i.e., $\varphi=\psi$. Then $\mathrm{dom}\,\tau_{\mathbf{A}}=\mathrm{dom}\,\tau_{\mathbf{B}}$, thus we may use the above notations and Lemma 8.

Lemma 9. Let $c=(a, b)\in C$ and $\bar{p}\in dom\,\varphi_{a,c}$ be arbitrary. Assume that the AF-transducer **A** is functional. Then $\varphi_{a,c}(\bar{p})\in T_G(Y_m)$ if and only if $\psi_{b,c}(\bar{p})\subseteq T_G(Y_m)$.

Proof. First of all we note that, by Lemma 3, the transformations ψ, $\psi^{b,c}$ and $\psi_{b,c}$ are mappings. Assume that there is a tree $\bar{p}$ for which the conclusion of this lemma does not hold. Let $\varphi_{a,c}(\bar{p})=q$ and $\psi_{b,c}(\bar{p})=r$. Then exactly one of q and r is in $T_G(Y_m)$, say $r\in T_G(Y_m)$ and $q\in\tilde{T}_G(Y_m)$. We have that $\bar{p}\neq\#$. Thus by condition (i) of Definition 1, $\mathrm{range}\,\varphi^{a,c}$ is not a singleton. It means that there are trees $p_1, p_2(\in\mathrm{dom}\,\varphi^{a,c})$ for which $q_1=\varphi^{a,c}(p_1)\neq\varphi^{a,c}(p_2)=q_2$. Then $q_i\cdot q=\varphi(p_i\cdot\bar{p})=\psi(p_i\cdot\bar{p})=r$ $(i=1, 2)$, consequently, $q_1\cdot q=q_2\cdot q$, which contradicts the assumption $q_1\neq q_2$. Similarly, we arrive at a contradiction by assuming $r\in\tilde{T}_G(Y_m)$ and $q\in T_G(Y_m)$.

Lemma 10. If **A** is functional, then $\varphi^{a,c}=\psi^{b,c}$ for all $(a, b)=c(\in C)$.

Proof. First we note that if (a, c) and (b, c) are final states then the equality $\varphi=\psi$ implies $\varphi^{a,c}=\psi^{b,c}$. We may assume that (a, c) and (b, c) are not final states. By Lemma 9, $\mathrm{range}\,\varphi^{a,c}$ is a singleton if and only if $\mathrm{range}\,\psi^{b,c}$ is a singleton, too. If both $\mathrm{range}\,\varphi^{a,c}$ and $\mathrm{range}\,\psi^{b,c}$ are singletons then the equality $\varphi^{a,c}(p)=y_1=\psi^{b,c}(p)$ holds for each tree $p\in\mathrm{dom}\,\varphi^{a,c}$. Therefore, in this case $\varphi^{a,c}=\psi^{b,c}$.

Suppose that $\mathrm{range}\,\varphi^{a,c}$ is not a singleton. By the note following Definition 1. we have that $range\,\varphi_{a,c}\subsetneqq T_G(Y_m)$ i.e., there is a tree $\bar{p}\in dom\,\varphi_{a,c}$ satisfying the inclusion $\varphi_{a,c}(\bar{p})\in\tilde{T}_G(Y_m)$. Let $\varphi_{a,c}(\bar{p})=\bar{q}$ and $\psi_{b,c}(\bar{p})=\bar{r}$. In the same way as in the proof of Lemma 7, one can see that there exist trees $s\in\tilde{T}_G(Y_m)$, $r_1, \ldots, r_m$ and $q_1, \ldots, q_m$ $(m>0)$ such that the equalities $\bar{r}=s\langle r_1, \ldots, r_m\rangle$ and $\bar{q}=s\langle q_1, \ldots, q_m\rangle$ hold, moreover, at least one of q_i and r_i is $\#$ for each index i $(1\leqq i\leqq m)$. It is easy to show that $q_i, r_i\in\tilde{T}_G(Y_m)$ $(i=1, \ldots, m)$.

Next we prove that all the r_i and q_i are equal to $\#$ $(i=1, \ldots, m)$. Let i be an arbitrary index $(1\leqq i\leqq m)$. Assume that $q_i=\#$ and $r_i\in\tilde{T}_G(Y_m)\setminus\{\#\}$. For each final state $(\bar{a}, \bar{c})$ $((\bar{a}, \bar{b})=\bar{c})$ and for all trees $p\in\mathrm{dom}\,\varphi^{a,c}_{\bar{a},\bar{c}}(p)=\mathrm{dom}\,\psi^{b,c}_{\bar{b},\bar{c}}$, if $q\in\varphi^{a,c}_{\bar{a},\bar{c}}(p)$ and $r\in\psi^{b,c}_{\bar{b},\bar{c}}(p)$ then

$$\varphi_{\bar{a},\bar{c}}(p\cdot\bar{p}) = q\cdot\bar{q} = s\langle q\cdot q_1, \ldots, q\cdot q_m\rangle \quad\text{and}$$
$$\psi_{\bar{b},\bar{c}}(p\cdot\bar{p}) = r\cdot\bar{r} = s\langle r\cdot r_1, \ldots, r\cdot r_m\rangle.$$

Since $(\bar{a}, \bar{c})$ and $(\bar{b}, \bar{c})$ are final states $\varphi^{\bar{a},\bar{c}}=\psi^{\bar{b},\bar{c}}$, which implies $\varphi_{\bar{a},\bar{c}}(p\cdot\bar{p})=\psi_{\bar{b},\bar{c}}(p\cdot\bar{p})$. Therefore, $r\cdot r_i=q\cdot q_i$, i.e. $r\cdot r_i=q$. It means that the transformation $\varphi^{a,c}$ can be cut by the tree r_i, which is a contradiction. Similarly, the assumptions $r_i=\#$ and $q_i\in\tilde{T}_G(Y_m)\setminus\{\#\}$ imply the equality $q_i=\#$.

Now we have that $\bar{r}=\bar{q}=s$ and $r=q$. It means that for each

$$p\in\operatorname{dom}\varphi^{a,c}(\subseteq\operatorname{dom}\varphi^{a,c}_{\bar{a},\bar{c}}),\ \varphi^{a,c}(p)=\psi^{b,c}(p)$$

holds. This ends the proof of Lemma 10.

Theorem 11. If the AF-transducer **A** is functional then the inferior F-transducers **A**(**B**) and **B**(**A**) are isomorphic.

Proof. Let us define a mapping $\mu\colon \overline{A\times C}\to\overline{B\times C}$ such that for an arbitrary state $(a,c)(\in\overline{A\times C})$ the equality $\mu(a,c)=(b,c)$ holds if $c=(a,b)$. It is clear that μ is a bijective mapping of $\overline{A\times C}$ onto $\overline{B\times C}$, moreover, $\mu(\overline{A'\times C'})=\overline{B'\times C'}$.

Next suppose that $x\to(a,c)q\in\bar{\Sigma}_A$ $(x\in X_n\cup F_0)$, where $c=(a,b)$. We have $x\in\operatorname{dom}\varphi^{a,c}$ thus $x\in\operatorname{dom}\psi^{b,c}$. By Lemma 10, $q=\varphi^{a,c}(x)=\psi^{b,c}(x)$ implies $x\to(b,c)q\in\bar{\Sigma}_B$. Similarly, if $x\to(b,c)r\in\bar{\Sigma}_B$ then we get $x\to(a,c)r\in\bar{\Sigma}_A$.

Let $f((a_1,c_1),\ldots,(a_k,c_k))\to(a_0,c_0)q\in\bar{\Sigma}_A$ where $c_i=(a_i,b_i)$ $(i=0,1,\ldots,k)$. By the construction of **A**(**B**) and **B**(**A**) we know that there is a rule of the form $f((b_1,c_1),\ldots,(b_k,c_k))\to(b_0,c_0)r$ in $\bar{\Sigma}_B$. Let $p_i(\in\operatorname{dom}\varphi^{a_i,c_i}=\operatorname{dom}\psi^{b_i,c_i})$ be arbitrary trees $(i=1,\ldots,k)$ and let j be an arbitrary index $(1\leq j\leq k)$. We define the trees s_i $(i=1,\ldots,k)$ in the following way. If $i=j$ then $s_i=\#$, otherwise $s_i=\varphi^{a_i,c_i}(p_i)(=\psi^{a_i,c_i}(p_i))$ $(i=1,\ldots,k)$.

Denote by $\bar{q}_j$ and $\bar{r}_j$ the tree $q(s_1,\ldots,s_k)$ and $r(s_1,\ldots,s_k)$, respectively. We have that $\varphi^{a_j,c_j}(p_j)=\psi^{b_j,c_j}(p_j)$ for each $p_j\in\operatorname{dom}\varphi^{a_j,c_j}$. From this it follows easily that the equality $\bar{r}_j=\bar{q}_j$ holds. Since j is arbitrary we get $r=q$. It means that $f((b_1,c_1),\ldots,(b_k,c_k))\to(b_0,c_0)q\in\bar{\Sigma}_B$.

Similarly, one can see that if $f((b_1,c_1),\ldots,(b_k,c_k))\to(b_0,c_0)r\in\bar{\Sigma}_B$ then the rule $f((a_1,c_1),\ldots,(a_k,c_k))\to(a_0,c_0)r$ is in $\bar{\Sigma}_A$.

Therefore, the inferior F-transducers **A**(**B**) and **B**(**A**) are isomorphic.

The next corollary is known from [2], where the result has been achieved in a different way.

Corollary 12. There exists an algorithm to decide for an arbitrary F-transducer $\bar{\mathbf{B}}$ and a functional F-transducer $\bar{\mathbf{A}}$ whether they are equivalent, i.e. $\tau_{\bar{\mathbf{A}}}=\tau_{\bar{\mathbf{B}}}$.

Proof. Let **A** and **B** be AF-transducers equivalent to $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$, respectively. Clearly, $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are equivalent if and only if so are **A** and **B**. By Theorem 11, $\tau_{\mathbf{A}}=\tau_{\mathbf{B}}$ if and only if $\operatorname{dom}\tau_{\mathbf{A}}=\operatorname{dom}\tau_{\mathbf{B}}$ and the inferior transducers **A**(**B**) and **B**(**A**) are isomorphic. It is known that the equality $\operatorname{dom}\tau_{\mathbf{A}}=\operatorname{dom}\tau_{\mathbf{B}}$ is decidable (c.f. [3, 4]). Obviously, **A**(**B**) and **B**(**A**) can be constructed. Moreover the isomorphism of these inferior transducers can be verified. Thus the statement of Corollary 12 is valid.

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TATABÁNYA COAL MINES
VÉRTANÚK TERE 1
2800 TATABÁNYA, HUNGARY