On the equivalence of the frontier-to-root tree transducers II.

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In this paper we continue our study started in [6] about the equivalent and isomorphic frontier-to-root transducers (F-transducers). First we introduce the superior F-transducer which can be seen the dual of the inferior F-transducer from part I. Then we deal with a subclass of the class of deterministic F-transducers, namely the class of normalized F-transducers. It will be proved that the strongly normalized forms of equivalent deterministic F-transducers are isomorphic.

Since this paper connects with [6] closely thus we use the notions, notations and results of part I.

1. Notions and notations

Take an arbitrary positive integer k. Let $p_1, p_2 \in T_F(X_n \cup Z_k)$ be arbitrary trees and $z_i \in Z_k$. Then the *z_i-product* $p_1 \cdot_i p_2$ of p_1 by p_2 is the tree

$$p_2(z_1, \ldots, z_{i-1}, p_1, z_{i+1}, \ldots, z_k).$$

For an F-transducer $\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$ and sets $A_i \subseteqq A$ $(i=0, \ldots, k)$ we denote by $\tau^{A_0}_{\mathbf{A}, A_1, \ldots, A_k}$ the transformation induced by

$$(T_F(X_n \cup Z_k), A, T_G(Y_m \cup Z_k), A_0, \Sigma \cup \{z_i \to a_i z_i \mid a_i \in A_i,\ i=1, \ldots, k\}).$$

Finally, when we will refer to a definition or a result from a part of our paper if the serial number of the part is I then it will be marked otherwise it will not be.

2. Superior F-transducers

Definition 1. Let $\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$ be an AF-transducer. The transformation induced by the state $a (\in A)$ can *be increased* by the tree $q^a \in \tilde{T}_G(Y_m) \setminus \{\#\}$ if for all $p \in dom\, \tau_{\mathbf{A},a}$ and $q \in \tau_{\mathbf{A},a}(p)$, there is a tree $\tilde{q} \in T_G(Y_m \cup \{\#\})$ satisfying $q = q^a \cdot \tilde{q}$, provided that range $\tau^a_{\mathbf{A}}$ is not a singleton. The tree q^a increases the transformation $\tau^a_{\mathbf{A}}$ maximally if the tree q^a is a proper subtree of a tree $\bar{q}^a$ then $\tau^{\mathbf{A}}_a$ cannot be increased by $\bar{q}^a$.

Definition 2. An AF-transducer $A=(T_F(X_n), A, T_G(Y_m), A', \Sigma)$ is called a *superior* AF-transducer if none of the transformations induced by its states can be increased by any trees.

Take an AF-transducer $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', \Sigma)$. Assume that for each state $a\in A$ the tree q^a increases τ^a maximally if τ^a can be increased and $q^a=\#$ otherwise. It means that for all $a\in A$, $p\in dom\, \tau_a$ and $q\in\tau_a(p)$ there is a tree $\tilde{q}$ such that $q=q^a\cdot\tilde{q}$. We suppose that the tree q^a is given for every state $a(\in A)$. Then the following lemma is valid under these notations.

Lemma 3. There is a superior AF-transducer $\bar{\mathbf{A}}=(T_F(X_n), A, T_G(Y_m), A', \bar{\Sigma})$ which is equivalent to $\mathbf{A}$.

Proof. We shall show that one can construct an AF-transducer

$$\bar{\mathbf{A}} = (T_F(X_n), A, T_G(Y_m), A', \bar{\Sigma})$$

such that for each state $a\in A$ the following conditions hold.

(1) $\text{dom}\,\tau^a_{\mathbf{A}}=\text{dom}\,\tau^a_{\bar{\mathbf{A}}}$.

(2) $\text{dom}\,\tau_{\mathbf{A},a}=\text{dom}\,\tau_{\bar{\mathbf{A}},a}$ and $dom\,\tau_{\mathbf{A},a}=dom\,\tau_{\bar{\mathbf{A}},a}$.

(3) $\{(p, q\cdot q^a)|q\in\tau^a_{\bar{\mathbf{A}}}(p),\ p\in\text{dom}\,\tau^a_{\mathbf{A}}\}=\tau^a_{\mathbf{A}}$.

(4) $\{(p, q^a\cdot q)|q\in\tau_{\bar{\mathbf{A}},a}(p),\ p\in\text{dom}\,\tau_{\mathbf{A},a}\}=\tau_{\mathbf{A},a}$.

In a way similar to that in the proof of Lemma I.7 we can see that $\bar{\mathbf{A}}$ is an equivalent superior AF-transducer for $\mathbf{A}$.

Next we define the rules of $\bar{\Sigma}$ in the following way:

(i) $x\to ar\in\Sigma$ $(x\in X_n\cup F_0)$ if and only if $x\to a\bar{r}\in\bar{\Sigma}$ where $\bar{r}=r\cdot q^a$,

(ii) $f(a_1, \ldots, a_k)\to ar\in\Sigma$ $(f\in F_k,\ k>0)$ if and only if $f(a_1, \ldots, a_k)\to a\bar{r}\in\bar{\Sigma}$ where the tree $\bar{r}(q^{a_1}(z_1), \ldots, q^{a_k}(z_k))$ equals the tree $r\cdot q^a$.

It is clear that this construction can be made for each rule of form (i). Assume that $f(a_1, \ldots, a_k)\to ar\in\Sigma$ $(f\in F_k,\ k>0)$.

Then let $p^j\in\text{dom}\,\tau^{a_j}_{\mathbf{A}}$ and $t_j\in\tau^{a_j}_{\mathbf{A}}(p^j)$ be arbitrary fixed trees $(j=1, \ldots, k)$. For each index j $(1\leqq j\leqq k)$ we use the following notations:

$$p_j = f(p^1, \ldots, p^{j-1}, \#, p^{j+1}, \ldots, p^k),$$

$$r_j = r(t_1, \ldots, t_{j-1}, \#, t_{j+1}, \ldots, t_k) \quad \text{and}$$

$$\bar{r}_j = r_j\cdot q^a.$$

It is sufficient to show that for each index j $(1\leqq j\leqq k)$ there is a tree $\bar{q}_j$ such that $\bar{r}_j=q^{a_j}\cdot\bar{q}_j$. From this we obtain easily that the tree $\bar{r}$ with

$$r\cdot q^a = \bar{r}(q^{a_1}(z_1), \ldots, q^{a_k}(z_k))$$

can be constructed.

Let j be an arbitrary index $(1\leqq j\leqq k)$. If $\bar{r}_j\in T_G(Y_m)$ or $q^{a_j}=\#$ then let $\bar{q}_j=\bar{r}_j$. In this case our statement holds obviously.

We may assume that $\bar{r}_j\in\tilde{T}_G(Y_m)$ and $\bar{q}^{a_j}\in\tilde{T}_G(Y_m)\setminus\{\#\}$. If range τ^a is a singleton then by the construction from Lemma I.2 the tree r is in $T_G(Y_m)$. It follows that $\bar{r}_j\in T_G(Y_m)$ which is a contradiction. It means that range τ^a is not a singleton. From this we obtain that there are trees $p \in dom\, \tau_a$ and $q \in \tau_a(p)$ such that $q \in \tilde{T}_G(Y_m)$. It implies that $r_j \cdot q \in \tau_{a_j}(p_j \cdot p)$ and $r_j \cdot q \in \tilde{T}_G(Y_m)$. By Definition 2 we know that $r_j \cdot q = q^{a_j} \cdot \tilde{q}$ under a suitable $\tilde{q}$. It means that $\tilde{q} \in \tilde{T}_G(Y_m)$. Moreover, one of the inclusions $q^{a_j} \in \mathrm{sub}\,(r_j)$ and $r_j \in sub\,(q^{a_j})$ holds.

Firstly, assume that $q^{a_j} \in \mathrm{sub}\,(r_j)$. Then there exists a tree $\bar{q} \in \tilde{T}_G(Y_m)$ for which $r_j = q^{a_j} \cdot \bar{q}$. In this case let $\bar{q}_j = \bar{q} \cdot q^a$. It means that $\bar{r}_j = r_j \cdot q^a = q^{a_j} \cdot \bar{q} \cdot q^a = q^{a_j} \cdot \bar{q}_j$.

Secondly, assume that $r_j \in sub\,(q^{a_j})$. Then there is a tree $\bar{q} \in \tilde{T}_G(Y_m) \setminus \{\#\}$ for which $q^{a_j} = r_j \cdot \bar{q}$. We have that for each tree $p \in dom\, \tau_a$ and $q \in \tau_a(p)$, the inclusion $r_j \cdot q \in \tau_{a_j}(p_j \cdot p)$ holds. Moreover, there is a tree $\tilde{q}$ such that $r_j \cdot q = q^{a_j} \cdot \tilde{q}$. From this we obtain that $r_j \cdot q = r_j \cdot \bar{q} \cdot \tilde{q}$. Since $r_j \in \tilde{T}_G(Y_m)$ the equality $q = \bar{q} \cdot \tilde{q}$ holds, too. It means that τ^a can be increased by the tree $\bar{q}$.

On the other hand we have that $q = q^a \cdot \tilde{\tilde{q}}$ under a suitable tree $\tilde{\tilde{q}}$. Since the tree q^a increases τ^a maximally thus from the two equalities above we get $\bar{q} \in \mathrm{sub}\,(q^a)$ i.e., there exists a $\bar{\bar{q}}$ for which $\bar{q} \cdot \bar{\bar{q}} = q^a$. Let $\bar{q}_j = \bar{\bar{q}}$. It follows that $\bar{r}_j = r_j \cdot q^a = r_j \cdot \bar{q} \cdot \bar{\bar{q}} = q^{a_j} \cdot \bar{\bar{q}} = q^{a_j} \cdot \bar{q}_j$.

It means that our statement is valid, thus the rules of $\bar{\Sigma}$ can be constructed.

Finally, one can see easily that the F-transducer $\bar{\mathbf{A}} = (T_F(X_n), A, T_G(Y_m), A', \bar{\Sigma})$ constructed in this way satisfies conditions (1)—(4).

This ends the proof of Lemma 3.

Lemma 4. There is an algorithm to decide for each AF-transducer

$$\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$$

and arbitrary state $a (\in A)$ whether the transformation τ^a can be increased. Moreover, every tree q^a can be given effectively which increases τ^a.

Proof. We have that if the transformation τ^a can be increased by the tree q^a then $h(q^a) \leqq \min\,(\tau_a(p) | p \in dom\, \tau_a)$. It means that the number of trees which increases τ^a is finite. Moreover, by the proof of Lemma 3 it is easy to see that for each tree q^a the transformation τ^a is increased by q^a if and only if the rules of Σ can be rewritten according to the conditions (i)—(ii) from the proof of Lemma 3. From this the statement of our lemma is obtained obviously.

3. Normalized F-transducers

Definition 5. A deterministic AF-transducer $\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$ is called a *normalized* F-transducer (NF-transducer) if conditions (i) and (ii) below hold.

(i) For every state $a \in A$, range τ^a is either a singleton or infinite.

(ii) For all states a, $\bar{a}$ if both range τ^a and range $\tau^{\bar{a}}$ are infinite, $dom\, \tau_a = dom\, \tau_{\bar{a}}$ and there exist trees $q, \bar{q} \in T_G(Y_m)$ such that for each tree $\bar{p} \in dom\, \tau_a$, $q \cdot \tau_a(\bar{p}) = \bar{q} \cdot \tau_{\bar{a}}(\bar{p})$ then at least one of the following conditions are satisfied.

(ii_1) There are trees $r, \bar{r} \in \tilde{T}_G(Y_m)$ such that at least one of them is equal to the tree $\#$ and for each tree $\bar{p} \in dom\, \tau_a$ the equality $r \cdot \tau_a(\bar{p}) = \bar{r} \cdot \tau_{\bar{a}}(\bar{p})$ holds.

(ii_2) The sets range $\tau^a \cap q$ and range $\tau^{\bar{a}} \cap \bar{q}$ are empty.

The next lemma, in a different form, can be found in [2]. The proof can be performed easily thus it is omitted.

Lemma 6. Let $q_j, r_j \in T_G(Y_m \cup Z)$ be arbitrary trees ($j=1, \ldots, 5$). For each positive integer i the equalities (1)—(7) imply the equality (8).

(1) $r_1 \cdot_i r_5 = q_1 \cdot_i q_5$

(2) $r_1 \cdot_i r_2 \cdot_i r_5 = q_1 \cdot_i q_2 \cdot_i q_5$

(3) $r_1 \cdot_i r_3 \cdot_i r_5 = q_1 \cdot_i q_3 \cdot_i q_5$

(4) $r_1 \cdot_i r_4 \cdot_i r_5 = q_1 \cdot_i q_4 \cdot_i q_5$

(5) $r_1 \cdot_i r_2 \cdot_i r_3 \cdot_i r_5 = q_1 \cdot_i q_2 \cdot_i q_3 \cdot_i q_5$

(6) $r_1 \cdot_i r_2 \cdot_i r_4 \cdot_i r_5 = q_1 \cdot_i q_2 \cdot_i q_4 \cdot_i q_5$

(7) $r_1 \cdot_i r_3 \cdot_i r_4 \cdot_i r_5 = q_1 \cdot_i q_3 \cdot_i q_4 \cdot_i q_5$

(8) $r_1 \cdot_i r_2 \cdot_i r_3 \cdot_i r_4 \cdot_i r_5 = q_1 \cdot_i q_2 \cdot_i q_3 \cdot_i q_4 \cdot_i q_5$

Lemma 7. For any deterministic F-transducer $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', \Sigma)$ an equivalent NF-transducer can be constructed.

Proof. Let $K=\max\left(h(\tau^a(p))\,|\,p\in \operatorname{dom}\tau^a,\ a\in A,\ h(p)\leqq \|A\|\right)$ and
$L_1 = \max\left(h(\tau_a(p))\,|\,p\in dom\ \tau_a,\ a\in A,\ h(p)\leqq 4\cdot\|A\|^2\right)$,
$L_2 = \max\left(h(\tau_a(p))\,|\,p\in dom\ \tau_a,\ a\in A,\ h(p)\leqq 2\cdot\|A\|,\ h^{\#}(p)\leqq\|A\|\right)$ and
$L=L_1+L_2$.
Moreover, set $Q=\{q\,|\,q\in T_G(Y_m), h(q)\leqq \max(K, L)\}$ and $C=Q\cup\{\#\}$.
Construct the deterministic F-transducer

$$\mathbf{A}^1 = \left(T_F(X_n), A\times C, T_G(Y_m), A'\times C, \Sigma^1\right)$$

such that $x\to(a,c)r\in\Sigma^1$ if and only if $x\to ar\in\Sigma$ and $c=r$, moreover,

$$f((a_1, c_1), \ldots, (a_k, c_k)) \to (a, c)r\in\Sigma^1$$

if and only if $f(a_1, \ldots, a_k)\to a\bar{r}\in\Sigma$ where c and r are defined in the following way. Let $q=\bar{r}(c_1(z_1), \ldots, c_k(z_k))$. If $q\in Q$ then $c=q$ otherwise $c=\#$. If $a\notin A'$ and $q\in Q$ then $r=y_1$ otherwise $r=q$. It is obvious that $\mathbf{A}$ and $\mathbf{A}^1$ are equivalent. Eliminating surplus states and rules in a standard way we get a connected deterministic F-transducer $\mathbf{B}=(T_F(X_n), B, T_G(Y_m), B', \Sigma_B)$ where $B\subseteqq A\times C$, $B'\subseteqq A'\times C$ and $\Sigma_B\subseteqq\Sigma^1$. It is clear that $\mathbf{B}$ and $\mathbf{A}^1$ are equivalent.

We will show that $\mathbf{B}$ is an NF-transducer. Take an arbitrary state $b=(a,c)\in B$. By our construction it is clear that $\operatorname{dom}\tau_{\mathbf{A},a}=\operatorname{dom}\tau_{\mathbf{B},b}$ and if $p\in\operatorname{dom}\tau^b_{\mathbf{B}}$ then $p\in\operatorname{dom}\tau^a_{\mathbf{A}}$, moreover, if $c=\#$ then the equality $\tau^a_{\mathbf{A}}(p)=\tau^b_{\mathbf{B}}(p)$ holds, too.

Assume that $c=\#$. Then for each tree $p\in\operatorname{dom}\tau^b_{\mathbf{B}}(\subseteqq\operatorname{dom}\tau^a_{\mathbf{A}})$ the inequality $h(\tau^b_{\mathbf{B}}(p))>\max(K, L)\geqq K$ holds. It follows that there are trees p_1, p_2, p_3 and a state $\tilde{a}$ such that $p=p_1\cdot p_2\cdot p_3$, $p_1\in\operatorname{dom}\tau^{\tilde{a}}_{\mathbf{A}}$, $p_2\in dom\ \tau^{\tilde{a}}_{\mathbf{A},\tilde{a}}$, $p_3\in dom\ \tau^a_{\mathbf{A},\tilde{a}}$ and $h^{\#}(\tau^{\tilde{a}}_{\mathbf{A},\tilde{a}}(p_2))>0$, $\tau^a_{\mathbf{A},\tilde{a}}(p_3)\in\tilde{T}_G(Y_m)$. From this we obtain that $p^k=p_1\cdot p_2^k\cdot p_3\in\operatorname{dom}\tau^a_{\mathbf{A}}$ $(k=1, 2, \ldots)$ and the trees $\tau^a_{\mathbf{A}}(p^k)$ are pairwise different. Since $\operatorname{range}\tau^b_{\mathbf{B}}=\operatorname{range}\tau^a_{\mathbf{A}}\setminus Q$ thus $\operatorname{range}\tau^b_{\mathbf{B}}$ is infinite. Moreover, we have that for all trees $\bar{p}\in\operatorname{dom}\tau_{\mathbf{B},b}$ and $p\in\operatorname{dom}\tau^b_{\mathbf{B}}$ the equality $\tau^a_{\mathbf{A}}(p)\cdot\tau_{\mathbf{A},a}(\bar{p})=\tau^b_{\mathbf{B}}(p)\cdot\tau_{\mathbf{B},b}(\bar{p})$ holds. From this we obtain that $\tau_{\mathbf{A},a}=$ $=\tau_{\mathbf{B},b}$.

Furthermore, we know that if $c \in Q$ then range $\tau_{\mathbf{B}}^{b}$ is a singleton and for each tree $\bar{p} \in \mathrm{dom}\, \tau_{\mathbf{B},b} \setminus \{\#\}$, $\tau_{\mathbf{B},b}(\bar{p}) \in T_G(Y_m)$. It follows that **B** is a deterministic AF-transducer and condition (i) of Definition 5 holds for **B**.

Then we have to prove that condition (ii) of Definition 5 can be satisfied. Take arbitrary states $b, \bar{b} \in B$ and trees $q, \bar{q} \in T_G(Y_m)$. Let $b=(a, c)$ and $\bar{b}=(\bar{a}, \bar{c})$. Assume that $dom\, \tau_b = dom\, \tau_{\bar{b}}$, both range τ^b and range $\tau^{\bar{b}}$ are infinite, moreover, for each tree $\bar{p} \in dom\, \tau_b$ the equality $q \cdot \tau_b(\bar{p}) = \bar{q} \cdot \tau_{\bar{b}}(\bar{p})$ holds. In this case we have that $c = \bar{c} = \#$ and $dom\, \tau_a = dom\, \tau_{\bar{a}}$.

It is sufficient to show that if at least one of two trees q and $\bar{q}$ is higher than L then the following condition $(\mathrm{ii}_1)'$ holds.

$(\mathrm{ii}_1)'$ There are trees $r, \bar{r} \in \tilde{T}_G(Y_m)$ such that at least one of them is equal to the tree $\#$ and for each tree $\bar{p} \in dom\, \tau_a \cap R$ the equality $r \cdot \tau_a(\bar{p}) = \bar{r} \cdot \tau_{\bar{a}}(\bar{p})$ holds where $R = \{\bar{p} \mid \bar{p} \in \tilde{T}_F(X_n), h(\bar{p}) \leqq 4 \cdot \|A\|^2\}$.

Now we prove this statement. First we show that $h(q), h(\bar{q}) > L_1$. Assume that $h(q) > L = L_1 + L_2$. It is clear that there is a tree $\bar{p} \in dom\, \tau_a$ for which $h(\bar{p}) \leqq 2 \cdot \|A\|$, $h^{\#}(\bar{p}) \leqq \|A\|$ and $\tau_a(\bar{p}) \in \tilde{T}_G(Y_m)$. Since $h(\tau_{\bar{a}}(\bar{p})) \leqq L_2$ and $q \cdot \tau_a(\bar{p}) = \bar{q} \cdot \tau_{\bar{a}}(\bar{p})$ thus $h(q \cdot \tau_a(\bar{p})) > L$ and $h(\bar{q} \cdot \tau_{\bar{a}}(\bar{p})) \leqq h(\bar{q}) + L_2$. It follows that $h(\bar{q}) > L_1$. Similarly, the inequality $h(\bar{q}) > L$ implies $h(q) > L_1$.

We have that there is a tree $\bar{p}$ for which $h(\bar{p}) \leqq 4 \cdot \|A\|^2$ and at least one of the trees $\tau_a(\bar{p})$ and $\tau_{\bar{a}}(\bar{p})$ is in $\tilde{T}_G(Y_m)$. In this case there exist an $s \in \tilde{T}_G(Y_m)$ and $q_1, \ldots, q_m, \bar{q}_1, \ldots, \bar{q}_m \in T_G(Y_m \cup \{\#\})$ $(m \geqq 1)$ such that the equalities $\tau_a(\bar{p}) = s\langle q_1, \ldots, q_m\rangle$ and $\tau_{\bar{a}}(\bar{p}) = s\langle \bar{q}_1, \ldots, \bar{q}_m\rangle$ hold, moreover, for each index j $(1 \leqq j \leqq m)$ at least one of the trees q_j and $\bar{q}_j$ equals $\#$. It means that $q \cdot q_j = \bar{q} \cdot \bar{q}_j$ $(j = 1, \ldots, m)$. Since $h(q_j), h(\bar{q}_j) \leqq L_1$ we get $q_j, \bar{q}_j \in \tilde{T}_G(Y_m)$ $(j = 1, \ldots, m)$.

Let j be an arbitrarily fixed index $(1 \leqq j \leqq m)$ and let $r = q_j$ and $\bar{r} = \bar{q}_j$. It follows that $q \cdot r = \bar{q} \cdot \bar{r}$. We show that for each tree $\bar{p} \in dom\, \tau_a \cap R$ the equality $r \cdot \tau_a(p) = \bar{r} \cdot \tau_{\bar{a}}(\bar{p})$ holds.

Take an arbitrary tree $\bar{p} \in dom\, \tau_a \cap R$. If both $\tau_a(\bar{p})$ and $\tau_{\bar{a}}(\bar{p})$ are in $T_G(Y_m)$ then $r \cdot \tau_a(\bar{p}) = \bar{r} \cdot \tau_{\bar{a}}(\bar{p})$ because of the equality $q \cdot \tau_a(\bar{p}) = \bar{q} \cdot \tau_{\bar{a}}(\bar{p})$.

In the opposite case the equalities $\tau_a(\bar{p}) = s\langle q_1, \ldots, q_m\rangle$ and $\tau_{\bar{a}}(\bar{p}) = s\langle \bar{q}_1, \ldots, \bar{q}_m\rangle$ hold where the trees s, $q_1, \ldots, q_m, \bar{q}_1, \ldots, \bar{q}_m$ satisfy the above conditions. Similarly, we have that $q \cdot q_j = \bar{q} \cdot \bar{q}_j$ and $q_j, \bar{q}_j \in \tilde{T}_G(Y_m)$ $(j = 1, \ldots, m)$.

Assume that $r = \#$, consequently, $q = \bar{q} \cdot \bar{r}$. It follows that $\bar{q} \cdot \bar{r} \cdot q_j = \bar{q} \cdot \bar{q}_j$ $(j = 1, \ldots, m)$. If $\bar{r} = \#$ then $\bar{q} \cdot q_j = \bar{q} \cdot \bar{q}_j$. Since $h(\bar{q}) > L_1$ and $h(q_j), h(\bar{q}_j) \leqq L_1$ thus $q_j = \bar{q}_j = \#$ $(j = 1, \ldots, m)$. From this we obtain $\tau_a(\bar{p}) = \tau_{\bar{a}}(\bar{p})$ i.e. $r \cdot \tau_a(\bar{p}) = \bar{r} \cdot \tau_{\bar{a}}(\bar{p})$. We may suppose that $\bar{r} \neq \#$. Then $\bar{q}_j \neq \#$, because in the opposite case $\bar{q} \cdot \bar{r} \cdot q_j = \bar{q}$ which is a contradiction $(j = 1, \ldots, m)$. It means that for each index j $(1 \leqq j \leqq m)$, $q_j = \#$ and $\bar{q} \cdot \bar{r} = \bar{q} \cdot \bar{q}_j$. From this we obtain that $\bar{r} = \bar{q}_j$ $(j = 1, \ldots, m)$. It implies that $r \cdot \tau_a(\bar{p}) = \bar{r} \cdot \tau_{\bar{a}}(\bar{p})$.

From the assumption $\bar{r} = \#$ we arrive at the equality $r \cdot \tau_a(\bar{p}) = \bar{r} \cdot \tau_{\bar{a}}(\bar{p})$ in a similar way.

Now we can prove that conditions (ii) of Definition 5 are satisfied. If $h(q), h(\bar{q}) \leqq L$ then (ii_2) holds because each tree of both range τ^b and range $\tau^{\bar{b}}$ is higher than L. In the opposite case condition $(\mathrm{ii}_1)'$ holds. We will show by induction that $r \cdot \tau_a(\bar{p}) = \bar{r} \cdot \tau_{\bar{a}}(\bar{p})$ for each tree $\bar{p} \in dom\, \tau_a$.

If $h(\bar{p}) \leqq 4 \cdot \|A\|^2$ then $r \cdot \tau_a(\bar{p}) = \bar{r} \cdot \tau_{\bar{a}}(\bar{p})$ by $(\mathrm{ii}_1)'$. Now let $h(\bar{p}) > 4 \cdot \|A\|^2$. We have that there are trees p_1, p_2, p_3, p_4, p_5 and states $\tilde{a}, \tilde{\tilde{a}} \in A$ such that

$\bar{p}=p_1 \cdot_2 p_2 \cdot_2 p_3 \cdot_2 p_4 \cdot_2 p_5$ and $p_1 \in \mathrm{dom}\, \tau_a^{\bar{a}} \cap \mathrm{dom}\, \tau_{\bar{a}}^{\bar{\bar{a}}}$, $p_i \in \mathrm{dom}\, \tau_{a,\bar{a}}^{\bar{a}} \cap \mathrm{dom}\, \tau_{\bar{a},\bar{\bar{a}}}^{\bar{\bar{a}}}$ $(i=2, 3, 4)$, $p_5 \in \mathrm{dom}\, \tau_{a,\bar{a}} \cap \mathrm{dom}\, \tau_{\bar{a},\bar{\bar{a}}}$, where $p_i \in T_F(X_n \cup Z_2)$ $(i=1, \ldots, 5)$ and the symbol z_2 occurs exactly once in the frontier of the tree p_i $(i=2, \ldots, 5)$.
Let

$$q_1 = r \cdot \tau_a^{\bar{a}}(p_1), \quad r_1 = \bar{r} \cdot \tau_{\bar{a}}^{\bar{\bar{a}}}(p_1),$$

$$q_2 = \tau_{a,\bar{a}}^{\bar{a}}(p_2), \quad r_2 = \tau_{\bar{a},\bar{\bar{a}}}^{\bar{\bar{a}}}(p_2),$$

$$q_3 = \tau_{a,\bar{a}}^{\bar{a}}(p_3), \quad r_3 = \tau_{\bar{a},\bar{\bar{a}}}^{\bar{\bar{a}}}(p_3),$$

$$q_4 = \tau_{a,\bar{a}}^{\bar{a}}(p_4), \quad r_4 = \tau_{\bar{a},\bar{\bar{a}}}^{\bar{\bar{a}}}(p_4),$$

$$q_5 = \tau_{a,\bar{a}}(p_5), \quad r_5 = \tau_{\bar{a},\bar{\bar{a}}}(p_5).$$

By the induction hypothesis it is clear that the trees r_i, q_i $(i=1, \ldots, 5)$ satisfy the conditions of Lemma 6. It means that $q_1 \cdot_2 q_2 \cdot_2 q_3 \cdot_2 q_4 \cdot_2 q_5 = r_1 \cdot_2 r_2 \cdot_2 r_3 \cdot_2 r_4 \cdot_2 r_5$ i.e., $r \cdot \tau_a(\bar{p}) = \bar{r} \cdot \tau_{\bar{a}}(\bar{p})$.

We have that $\tau_a = \tau_b$ and $\tau_{\bar{a}} = \tau_{\bar{b}}$. It follows that for each tree $\bar{p} \in dom\, \tau_b$ the equality $r \cdot \tau_b(\bar{p}) = \bar{r} \cdot \tau_{\bar{b}}(\bar{p})$ holds. It means that **B** is an NF-transducer. Consequently the statement of this lemma is valid.

Lemma 8. Let $\bar{\mathbf{A}} = (T_F(X_n), A, T_G(Y_m), A', \bar{\Sigma}_A)$ be a deterministic F-transducer. Then there is an superior NF-transducer **B** which is equivalent to $\bar{\mathbf{A}}$.

Proof. By Lemma 3 we can construct a superior F-transducer

$$\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma_A)$$

with $\tau_{\mathbf{A}} = \tau_{\bar{\mathbf{A}}}$. From the proof of Lemma 3 one can see that **A** is deterministic, too. Next we consider the NF-transducer $\mathbf{B} = (T_F(X_n), B, T_G(Y_m), B', \Sigma_B)$ constructed for **A** by Lemma 7. From the proof we have that for each state $b \in B$ if range $\tau_{\mathbf{B}}^b$ is not a singleton then there exists a state $a \in A$ such that $\tau_{\mathbf{A},a} = \tau_{\mathbf{B},b}$. It follows that $\tau_{\mathbf{B}}^b$ can not be increased by any tree q^b, because in the opposite case the transformation $\tau_{\mathbf{A}}^a$ is increased by q^b which is a contradiction. It means that **B** is a superior NF-transducer equivalent to $\bar{\mathbf{A}}$.

Definition 9. Let $\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma)$ be a superior NF-transducer. We say that the state $\bar{a} (\in A)$ can *be substituted* by the state $a (\in A)$ if the condition (i) holds or there is a tree $\bar{q} \in T_G(Y_m)$ such that the conditions (ii_1)—(ii_6) are satisfied.

(i) $\tau_a = \tau_{\bar{a}}$, and if range τ^a is a singleton then range τ^a = range $\tau^{\bar{a}}$.

(ii_1) $dom\, \tau_a = dom\, \tau_{\bar{a}}$.

(ii_2) range τ^a is infinite.

(ii_3) range $\tau^{\bar{a}}$ is a singleton.

(ii_4) For each tree $\bar{p} \in dom\, \tau_a \setminus \{\#\}$ the equality $\bar{q} \cdot \tau_a(\bar{p}) = \tau_{\bar{a}}(\bar{p})$ holds.

(ii_5) If $\bar{a} \in A'$ then range $\tau^{\bar{a}} = \bar{q}$.

(ii_6) If there is a state $\bar{\bar{a}} (\in A \setminus \{a, \bar{a}\})$ for which $dom\, \tau_a = dom\, \tau_{\bar{\bar{a}}}$ and range $\tau^{\bar{\bar{a}}}$ is infinite, moreover, there exist trees $q, \bar{q} \in T_G(Y_m)$ such that for each tree $\bar{p} \in dom\, \tau_a$ the equality $q \cdot \tau_a(\bar{p}) = \bar{q} \cdot \tau_{\bar{\bar{a}}}(\bar{p})$ holds then either $q \neq \bar{q}$ or $\tau_a(\bar{p}) = \tau_{\bar{\bar{a}}}(\bar{p})$ for each tree $\bar{p} \in dom\, \tau_a$.

We note that $\tau_a=\tau_{\bar{a}}$ if and only if $dom\ \tau_a=dom\ \tau_{\bar{a}}$ and for each tree $\bar{p}\in dom\ \tau_a$ the equality $\tau_a(\bar{p})=\tau_{\bar{a}}(\bar{p})$ holds.

Definition 10. A superior NF-transducer $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', \Sigma)$ is called a *strongly normalized* F-transducer (SNF-transducer) if none of the states can be substituted by another state.

Theorem 11. For each deterministic F-transducer

$$\bar{\mathbf{A}} = (T_F(X_n), \bar{A}, T_G(Y_m), \bar{A}', \bar{\Sigma}_A)$$

an equivalent SNF-transducer can be constructed.

Proof. By Lemma 8 we construct a superior NF-transducer

$$\mathbf{A} = (T_F(X_n), A, T_G(Y_m), A', \Sigma_A)$$

which is equivalent to $\bar{\mathbf{A}}$. Next we will show that by rewriting rules and eliminating states an equivalent SNF-transducer is obtained.

Assume that the states $a, \bar{a}\in A$ satisfy the condition (i) of Definition 9. Then we construct the F-transducer $\mathbf{A}^1=(T_F(X_n), A\setminus\{\bar{a}\}, T_G(Y_m), A'\setminus\{\bar{a}\}, \Sigma_A^1)$ in the following way. Let us eliminate the rules of Σ_A wherein the state $\bar{a}$ is in the left side. Then we replace the state $\bar{a}$ by a in each rule. It is clear that $\mathbf{A}^1$ is deterministic. Moreover, for each state $\tilde{a}\in A\setminus\{a, \bar{a}\}$, $\tau_{\mathbf{A},\tilde{a}}=\tau_{\mathbf{A}^1,\tilde{a}}$ and $\tau_{\mathbf{A}}^{\tilde{a}}=\tau_{\mathbf{A}^1}^{\tilde{a}}$ hold. We have that dom $\tau_{\mathbf{A}^1}^{a}=$dom $\tau_{\mathbf{A}}^{a}\cup$dom $\tau_{\mathbf{A}}^{\bar{a}}$ and range $\tau_{\mathbf{A}^1}^{a}=$range $\tau_{\mathbf{A}}^{a}\cup$range $\tau_{\mathbf{A}}^{\bar{a}}$. From this one can easily show that $\mathbf{A}^1$ is a superior NF-transducer equivalent to $\mathbf{A}$. It means that for $\mathbf{A}$ we construct an equivalent superior NF-transducer $\mathbf{B}=(T_F(X_n), B, T_G(Y_m), B', \Sigma_B)$ such that there are no states $b, \bar{b}\in B$ satisfying condition (i) of Definition 9.

Next we assume that there are states $a, \bar{a}\in B(\subseteq A)$ and a tree $\bar{q}\in T_G(Y_m)$ for which the conditions (ii$_1$)—(ii$_6$) hold. In this case we eliminate the rules containing the state $\bar{a}$ in their left side. Then we replace by $a\bar{q}$ the right side of rules wherein the state $\bar{a}$ occurs. Let $\mathbf{B}^1=(T_F(X_n), B\setminus\{\bar{a}\}, T_G(Y_m), B'\setminus\{\bar{a}\}, \Sigma_B^1)$ be the F-transducer obtained this way. By the construction it is obvious that $\mathbf{B}^1$ is deterministic. We have that for each state $\tilde{a}\in B\setminus\{a, \bar{a}\}$, $\tau_{\mathbf{B}^1,\tilde{a}}=\tau_{\mathbf{B},\tilde{a}}$ and $\tau_{\mathbf{B}^1}^{\tilde{a}}=\tau_{\mathbf{B}}^{\tilde{a}}$ hold. Moreover, dom $\tau_{\mathbf{B}^1}^{a}=$dom $\tau_{\mathbf{B}}^{a}\cup$dom $\tau_{\mathbf{B}}^{a}$ and range $\tau_{\mathbf{B}^1}^{a}=$range $\tau_{\mathbf{B}}^{a}\cup\bar{q}$. It is clear that $\mathbf{B}^1$ is a superior AF-transducer equivalent to $\mathbf{B}$.

We will show that $\mathbf{B}^1$ is normalized. By the construction of $\mathbf{B}^1$ condition (i) of Definition 5 holds. Let $b, \bar{\bar{b}}\in B\setminus\{\bar{a}\}$ be arbitrary states of $\mathbf{B}^1$. Assume that $dom\,\tau_{\mathbf{B}^1,b}=$ $=dom\ \tau_{\mathbf{B}^1,\bar{\bar{b}}}$, both range $\tau_{\mathbf{B}^1}^{b}$ and range $\tau_{\mathbf{B}^1}^{\bar{\bar{b}}}$ are infinite, moreover, there are trees $q, \bar{\bar{q}}\in T_G(Y_m)$ such that for each tree $\bar{p}\in dom\ \tau_{\mathbf{B}^1,b}$, $q\cdot\tau_{\mathbf{B}^1,b}(\bar{p})=\bar{\bar{q}}\cdot\tau_{\mathbf{B}^1,\bar{\bar{b}}}(\bar{p})$. If none of the states $b, \bar{\bar{b}}$ is a then by the above connections it is obvious that condition (ii) holds.

We may assume that $b=a$. In this case we know that for $\mathbf{B}$ condition (ii$_6$) are satisfied by the states $a, \bar{a}, \bar{\bar{b}}$ and the trees $q, \bar{q}, \bar{\bar{q}}$. Furthermore, we have that the equality $\tau_{\mathbf{B},a}(\bar{p})=\tau_{\mathbf{B},\bar{\bar{b}}}(\bar{p})$ does not hold for each tree $\bar{p}\in dom\ \tau_{\mathbf{B},a}$. Indeed, in the opposite case $\tau_{\mathbf{B},a}=\tau_{\mathbf{B},\bar{\bar{b}}}$ which is a contradiction. By condition (ii$_6$) it means that $q\neq\bar{q}$. From this we obtain that range $\tau_{\mathbf{B}^1,a}\cap q=\emptyset$ and range $\tau_{\mathbf{B}^1,\bar{\bar{b}}}\cap\bar{\bar{q}}=\emptyset$. Consequently, condition (ii) of Definition 5 holds for $\mathbf{B}^1$ thus $\mathbf{B}^1$ is a superior normalized F-transducer.

Applying this construction we can get an SNF-transducer which is equivalent to $\overline{\mathbf{A}}$.

Let $\mathbf{A}=(T_F(X_n), A, T_G(Y_m), A', \Sigma_A)$ and $\mathbf{B}=(T_F(X_n), B, T_G(Y_m), B', \Sigma_B)$ be SNF-transducers such that $\mathbf{A}$ and $\mathbf{B}$ are equivalent. First we construct the inferior AF-transducers $\mathbf{A}(\mathbf{B})$ and $\mathbf{B}(\mathbf{A})$ as in part I. Next we consider the superior AF-transducers $\overline{\mathbf{A}}(\mathbf{B})=(T_F(X_n), \overline{A\times C}, T_G(Y_m), \overline{A'\times C'}, \bar{\Sigma}_A)$ and

$$\overline{\mathbf{B}}(\mathbf{A}) = (T_F(X_n), \overline{B\times C}, T_G(Y_m), \overline{B'\times C'}, \bar{\Sigma}_B)$$

which are constructed from $\mathbf{A}(\mathbf{B})$ and $\mathbf{B}(\mathbf{A})$ by Lemma 3, respectively. We have that $\overline{A\times C}\subseteq A\times C$, $\overline{A'\times C'}\subseteq A'\times C'$ and $\overline{B\times C}\subseteq B\times C$, $\overline{B'\times C'}\subseteq B'\times C'$ where $C=A\times B$ and $C'=A'\times B'$. From Theorem I.11 we know that $\mathbf{A}(\mathbf{B})$ and $\mathbf{B}(\mathbf{A})$ are isomorphic. By conditions (i)—(iv) from the proof of Lemma 3, it follows that $\overline{\mathbf{A}}(\mathbf{B})$ and $\overline{\mathbf{B}}(\mathbf{A})$ are isomorphic, too. It means that $\tau_{\mathbf{A}}=\tau_{\overline{\mathbf{A}}(\mathbf{B})}$ thus both of these transformations shall be denoted by φ. Similarly, ψ can be used instead of $\tau_{\mathbf{B}}$ and $\tau_{\overline{\mathbf{B}}(\mathbf{A})}$.

The next lemmas are valid under the above notations.

Lemma 12. For each state $(a, c)\in\overline{A\times C}$ $(c=(a, b))$ the following conditions hold.

(i) $dom\ \varphi_a=dom\ \varphi_{a,c}=dom\ \psi_{b,c}=dom\ \psi_b$.

(ii) If range $\varphi^{a,c}$ is infinite then for each tree $\bar{p}\in dom\ \varphi_a$ the equalities $\varphi_a(\bar{p})=\varphi_{a,c}(\bar{p})=\psi_{b,c}(\bar{p})=\psi_b(\bar{p})$ hold.

(iii) If range $\varphi^{a,c}$ is a singleton then there are trees $q, \bar{q}\in T_G(Y_m)$ such that for each tree $\bar{p}\in dom\ \varphi_a\setminus\{\#\}$ the equalities $q\cdot\varphi_a(\bar{p})=\varphi_{a,c}(\bar{p})=\psi_{b,c}(\bar{p})=\bar{q}\cdot\psi_{b,c}(\bar{p})$ hold, moreover $q=\varphi^a(p)$ and $\bar{q}=\psi^b(p)$ where $p\in\mathrm{dom}\ \varphi^{a,c}$.

Proof. Let $(a, c)\in\overline{A\times C}$ be an arbitrary state where $c=(a, b)$. Let $p\in\mathrm{dom}\ \varphi^{a,c}$ be a fixed tree. Then $p\in\mathrm{dom}\ \varphi^a\cap\mathrm{dom}\ \psi^{b,c}\cap\mathrm{dom}\ \psi^b$. Let $\bar{p}\in dom\ \varphi_a$. Since $p\cdot\bar{p}\in\mathrm{dom}\ \varphi$ the tree $\bar{p}$ is in $dom\ \varphi_{a,c}$. Consequently, $dom\ \varphi_a\subseteq dom\ \varphi_{a,c}$. In the same way one can prove the inclusions $dom\ \varphi_a\subseteq dom\ \varphi_{a,c}\subseteq dom\ \psi_{b,c}\subseteq dom\ \psi_b\subseteq\subseteq dom\ \varphi_a$. From this we obtain that condition (i) holds.

Assume that range $\varphi^{a,c}$ is infinite. It follows that range $\psi^{b,c}$ is infinite, too. Then there are trees $p_1, p_2\in\mathrm{dom}\ \varphi^{a,c}$ such that $\varphi^{a,c}(p_1)\neq\varphi^{a,c}(p_2)$. For each tree $\bar{p}\in dom\ \varphi_a$ the equality $\varphi^{a,c}(p_i)\cdot\varphi_{a,c}(\bar{p})=\psi^{b,c}(p_i)\cdot\psi_{b,c}(\bar{p})$ holds $(i=1, 2)$. In a similar way as in the proof of Lemma 3, we can obtain that there exist trees r, $\bar{r}$ such that at least one of them equals the tree $\#$ and for each tree $\bar{p}\in dom\ \varphi_a$, $r\cdot\varphi_a(\bar{p})=\bar{r}\cdot\varphi_{a,c}(\bar{p})$.

On the other hand we have that both $\overline{\mathbf{A}}(\mathbf{B})$ and $\overline{\mathbf{B}}(\mathbf{A})$ are superior F-transducers. It means that $r=\bar{r}=\#$ i.e., for each tree $\bar{p}\in dom\ \varphi_a$ the equality $\varphi_a(\bar{p})=\varphi_{a,c}(\bar{p})$ holds.

Furthermore we know that range $\psi^{b,c}$ is infinite. In the same way we get that for each tree $\bar{p}\in dom\ \psi_b$, $\psi_{b,c}(\bar{p})=\psi_b(\bar{p})$. Since $\overline{\mathbf{A}}(\mathbf{B})$ and $\overline{\mathbf{B}}(\mathbf{A})$ are isomorphic it follows that condition (ii) of our lemma holds.

Next we assume that range $\varphi^{a,c}$ is a singleton. Let $p\in\mathrm{dom}\ \varphi^{a,c}$ be an arbitrarily fixed tree. We have that $p\in\mathrm{dom}\ \varphi^a$. Let $q=\varphi^a(p)$. It means that for each tree $\bar{p}\in dom\ \varphi_a\setminus\{\#\}$ the equalities $q\cdot\varphi_a(\bar{p})=\varphi(p\cdot\bar{p})=\varphi^{a,c}(p)\cdot\varphi_{a,c}(\bar{p})=\varphi_{a,c}(\bar{p})$ hold. In this case range $\psi^{b,c}$ is a singleton, too. It follows that if $\psi^{b,c}(p)=\bar{q}$ then for each tree $\bar{p}\in dom\,\psi_a\setminus\{\#\}$, $\bar{q}\cdot\psi_b(\bar{p})=\psi_{b,c}(\bar{p})$. It is clear that if $a\in A'$ then b, (a, c), (b, c) are final states. From this we obtain $q=\bar{q}$. It means that condition (iii) holds, too.

Lemma 13. For each state $a\in A$ there is exactly one state $b(\in B)$ satisfying the inclusion $(a, (a, b))\in\overline{A\times C}$, and conversely, for each state $b\in B$ there is exactly one state $a(\in A)$ with $(b, (a, b))\in\overline{B\times C}$. Moreover, if $(a, c)\in\overline{A\times C}$ $(c=(a, b))$ then for each tree $\bar{p}\in dom\,\varphi_a$ the equalities $\varphi_a(\bar{p})=\varphi_{a,c}(\bar{p})=\psi_{b,c}(\bar{p})=\psi_b(\bar{p})$ hold.

Proof. Let $a\in A$ be an arbitrary state. Denote by B_a the set

$$\{b|c = (a, b),\ (a, c)\in\overline{A\times C}\}.$$

It is clear that B_a is a nonvoid set.

Firstly, we assume that range φ^a is infinite. Then there are trees $p_i\in\text{dom}\,\varphi^a$ $(i=1, 2, \ldots)$ such that the trees $\varphi^a(p_i)$ are pairwise different. Moreover, we know that there exists a state $b_i(\in B_a)$ such that $p_i\in\text{dom}\,\psi^{b_i}$ $(i=1, 2, \ldots)$. Since B_a is a finite set of states there are indices k, l $(k<l)$ satisfying $b_k=b_l$. Denote by b this state. Let $c=(a, b)$. It is clear that neither range $\varphi^{a,c}$ nor range $\psi^{b,c}$ are a singleton. By Lemma 12 we get that for each tree $\bar{p}\in dom\,\varphi_a$ the equalities $\varphi_a(\bar{p})=\varphi_{a,c}(\bar{p})=\psi_{b,c}(\bar{p})=\psi_b(\bar{p})$ hold.

Next we show that the set B_a is a singleton. Assume that there is a state $\bar{b}\in B_a$ differing from b. Let $\bar{c}=(a, \bar{b})$. Now there are three cases.

First, suppose that range $\varphi^{a,\bar{c}}$ is infinite. By Lemma 12 we have that $\varphi_a(\bar{p})=\varphi_{a,\bar{c}}(\bar{p})=\psi_{\bar{b},\bar{c}}(\bar{p})=\psi_{\bar{b}}(\bar{p})$ hold for each tree $\bar{p}\in dom\,\varphi_a$. It means that the state $\bar{b}$ can be substituted by b which is a contradiction because **B** is an *SNF*-transducer.

In the second case assume that both range $\varphi^{a,\bar{c}}$ and range $\psi^{\bar{b}}$ are singleton. Then we know that for each tree $\bar{p}\in dom\,\varphi_a\setminus\{\#\}$ the equalities $q\cdot\varphi_a(\bar{p})=\varphi_{a,\bar{c}}(\bar{p})=\psi_{\bar{b},\bar{c}}(\bar{p})=\psi_{\bar{b}}(\bar{p})$ hold, where $q=\varphi^a(p)$ and $p\in\text{dom}\,\varphi^{a,\bar{c}}$. It is clear that $q=\text{range}\,\psi^{\bar{b}}$ if $\bar{b}$ is a final state. From this we obtain $q\cdot\psi_b(\bar{p})=\psi_{\bar{b}}(\bar{p})$ for each tree $\bar{p}\in dom\,\psi^b\setminus\{\#\}$. Since the state $\bar{b}$ cannot be substituted by the state b and conditions (ii_1)—(ii_5) of Definition 9 hold for the states b, $\bar{b}$ and the tree q condition (ii_6) can not be satisfied. It means that there is a state $\bar{\bar{b}}\in B\setminus\{b, \bar{b}\}$ and a tree $\bar{q}\in T_G(Y_m)$ for which $dom\,\psi_b=dom\,\psi_{\bar{\bar{b}}}$ and range $\psi^{\bar{\bar{b}}}$ is infinite, moreover, for each tree $\bar{p}\in dom\,\psi_b$ the equality $q\cdot\psi_b(\bar{p})=\bar{q}\cdot\psi_{\bar{\bar{b}}}(\bar{p})$ holds. One can see easily that there is a state $\bar{\bar{a}}\in A\setminus\{a, \bar{a}\}$ such that $dom\,\varphi_{\bar{\bar{a}}}=dom\,\psi_{\bar{\bar{b}}}$ and for each tree $\bar{p}\in dom\,\psi_{\bar{\bar{b}}}$, $\varphi_{\bar{\bar{a}}}(\bar{p})=\psi_{\bar{\bar{b}}}(\bar{p})$. It implies that for each tree $\bar{p}\in dom\,\varphi_a$, $q\cdot\varphi_a(\bar{p})=\bar{q}\cdot\varphi_{\bar{\bar{a}}}(\bar{p})$. Since **A** is an *NF*-transducer condition (ii) of Definition 5 has to hold. We have that $q\cap\text{range}\,\varphi^a\neq\emptyset$ thus there are trees $r, \bar{r}\in\tilde{T}_G(Y_m)$ such that $r\cdot\varphi_a(\bar{p})=\bar{r}\cdot\varphi_{\bar{\bar{a}}}(\bar{p})$ for each tree $\bar{p}\in dom\,\varphi_a$, where at least one of the trees r, $\bar{r}$ equals $\#$. It is clear that $r=\bar{r}=\#$ because **A** is a superior *NF*-transducer. It implies that for each tree $\bar{p}\in dom\,\varphi_a$ the equality $\varphi_a(\bar{p})=\varphi_{\bar{\bar{a}}}(\bar{p})$ holds which is a contradiction.

In the third case suppose that range $\varphi^{a,\bar{c}}$ is a singleton and range $\psi^{\bar{b}}$ is infinite. We have that for each tree $\bar{p}\in dom\,\varphi_a\setminus\{\#\}$ the equalities $q\cdot\varphi_a(\bar{p})=\varphi_{a,\bar{c}}(\bar{p})=\psi_{\bar{b},\bar{c}}(\bar{p})=\bar{q}\cdot\psi_{\bar{b}}(\bar{p})$ hold where $q\in\text{range}\,\varphi^a$ and $\bar{q}\in\text{range}\,\psi^{\bar{b}}$. We have that if a is a final state then $\bar{b}$ is also a final state and $q=\bar{q}$. It implies that for each tree $\bar{p}\in dom\,\psi_b$, $q\cdot\psi_b(\bar{p})=\bar{q}\cdot\psi_{\bar{b}}(\bar{p})$. From Definition 5 we obtain that either for each tree $\bar{p}\in dom\,\psi_b$ the equality $\psi_b(\bar{p})=\psi_{\bar{b}}(\bar{p})$ holds or range $\psi^{\bar{b}}\cap\bar{q}=\emptyset$. It contradicts the above statements. It means that if range φ^a is infinite then B_a is a singleton.

Similarly, we can show that for each state $b \in B$ if range ψ^b is infinite then there is exactly one state $a \in A$ satisfying the inclusion $(b, c) \in \overline{B \times C}$ where $c=(a, b)$. Moreover, for each tree $\bar{p} \in dom\, \psi_b$ the equalities $\varphi_a(\bar{p})=\varphi_{a,c}(\bar{p})=\psi_{b,c}(\bar{p})=\psi_b(\bar{p})$ hold.

Secondly, we may assume that range φ^a is a singleton. It is clear that $B_a \neq \emptyset$ and for each state $b \in B_a$ range ψ_b is a singleton, too. Let $b \in B_a$ be arbitrary. Then for each tree $\bar{p} \in dom\, \varphi_a$ the equalities $\varphi_a(\bar{p})=\varphi_{a,c}(\bar{p})=\psi_{b,c}(\bar{p})=\psi_b(\bar{p})$ hold where $c=(a, b)$. We have that b is a final state if and only if a is a final state. It implies that range $\varphi^a=$ range ψ^b. From this and Definition 9 we get that B_a is a singleton.

In a similar way we can see that for each $b \in B$ if range ψ^b is a singleton then there is exactly one state $a \in A$ such that $(b, c) \in \overline{B \times C}$ $(c=(a, b))$ and for each tree $\bar{p} \in dom\, \varphi_a$ the equalities $\varphi_a(\bar{p})=\varphi_{a,c}(\bar{p})=\psi_{b,c}(\bar{p})=\psi_b(\bar{p})$ hold. This ends the proof of Lemma 13.

Lemma 14. The *SNF*-transducers **A** and **B** are isomorphic.

Proof. Let us define a mapping $\mu: A \to B$ such that $\mu(a)=b$ if and only if $(a, (a, b)) \in \overline{A \times C}$. By Lemma 13 it is clear that μ is a bijective mapping of A onto B, moreover, $\mu(A')=B'$.

Next suppose that $x \to aq \in \Sigma_A$ $(x \in X_n \cup F_0)$ and $b=\mu(a)$. We have that $x \to br \in \Sigma_B$ and for each tree $\bar{p} \in dom\, \varphi_a = dom\, \psi_b$ the equality $\varphi_a(\bar{p})=\psi_b(\bar{p})$ holds. It implies that $q \cdot \varphi_a(\bar{p})=\varphi(x \cdot \bar{p})=\psi(x \cdot \bar{p})=r \cdot \psi_b(\bar{p})$. From this we can obtain that $q=r$. It means that $x \to bq \in \Sigma_B$. Similarly, if $x \to br \in \Sigma_B$ and $a=\mu^{-1}(b)$ then $x \to ar \in \Sigma_A$.

Let $f(a_1, \ldots, a_k) \to a_0 q \in \Sigma_A$ where $f \in F_k$ $(k>0)$ and $a_i \in A$ $(i=0, 1, \ldots, k)$. We have that there is a rule of the form $f(b_1, \ldots, b_k) \to b_0 r$ in Σ_B where $b_i=\mu(a_i)$ $(i=0, 1, \ldots, k)$. Moreover, it is clear that dom $\varphi^{a_i}=$ dom ψ^{b_i}, and for each tree $p_i \in dom\, \varphi^{a_i}$, $\varphi^{a_i}(p_i)=\psi^{b_i}(p_i)$ $(i=0, 1, \ldots, k)$. From the proof of Lemma 13 we know that if range φ^{a_0} is a singleton then $q=$ range $\varphi^{a_0}=$ range $\psi^{b_0}=r$.

Next we may assume that range φ^{a_0} is infinite. In this case we have that there is a tree $\bar{p} \in dom\, \varphi_{a_0}$ for which $\varphi_{a_0}(\bar{p}) \in \tilde{T}_G(Y_m)$. Let $p_i \in$ dom φ^{a_i} $(i=1, \ldots, k)$ be arbitrary trees and let j be an arbitrary index $(1 \leqq j \leqq k)$. We define the trees s_i, t_i $(i=1, \ldots, k)$ in the following way. If $i=j$ then $s_i=t_i=\sharp$, otherwise $s_i=p_i$ and $t_i=\varphi^{a_i}(p_i)=\psi^{b_i}(p_i)$ $(i=1, \ldots, k)$. Denote by $\bar{s}_j$, $\bar{q}_j$ and $\bar{r}_j$ the trees $f(s_1, \ldots, s_k)$, $q(t_1, \ldots, t_k)$ and $r(t_1, \ldots, t_k)$, respectively. By Lemma 13 we have that the equalities $\bar{q}_j \cdot \varphi_{a_0}(\bar{p})=\varphi_{a_j}(\bar{s}_j \cdot \bar{p})=\psi_{b_j}(\bar{s}_j \cdot \bar{p})=\bar{r}_j \cdot \psi_{b_0}(\bar{p})$ and $\varphi_{a_0}(\bar{p})=\psi_{b_0}(\bar{p})$ hold. It follows that $\bar{q}_j=\bar{r}_j$. Since j is arbitrary we get $r=q$. It means that $f(b_1, \ldots, b_k) \to b_0 q \in \Sigma_B$.

Similarly, one can see that if $f(b_1, \ldots, b_k) \to b_0 r \in \Sigma_B$ then $f(a_1, \ldots, a_k) \to a_0 r \in \Sigma_A$ where $a_i=\mu^{-1}(b_i)$ $(i=0, 1, \ldots, k)$. Therefore, the *SNF*-transducers **A** and **B** are isomorphic.

By this lemma we get the following theorem.

Theorem 15. The *SNF*-transducers **A** and **B** are equivalent if and only if they are isomorphic.

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