Metric representations by ν_i-products

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The purpose of this paper is to compare the metric representation powers of the product and ν_i-products introduced in [1]. It is shown that a class of automata is metrically complete with respect to the product if and only if it is metrically complete regarding the ν_1-product. It is also proved that the ν_3-product is metrically equivalent to the product.

We start with some basic notions and notations.

An *alphabet* is a nonvoid finite set. The free monoid generated by an alphabet X will be denoted by X^*. An element $p=x_1 \dots x_n \in X^*$ $(x_i \in X,\ i=1, \dots, n)$ is a *word* over X, and n is the *length* of p, in notation, $|p|=n$. If $n=0$ then p is the *empty word*, which will be denoted by e. For arbitrary integer $n(\geqq 0)$, $X^{(n)}$ will stand for the subset of X^* consisting of all words with length less than or equal to n.

An *automaton* is a system $\mathfrak{A}=(X, A, \delta)$, where X is the *input alphabet*, A is a nonvoid finite set of *states* and the mapping $\delta\colon A \times X \to A$ is the *transition function* of $\mathfrak{A}$. We extend δ to a mapping $\delta\colon A \times X^* \to A$ in the following way: for arbitrary $a \in A$, $\delta(a, e)=a$ and $\delta(a, px)=\delta(\delta(a, p), x)$ $(p \in X^*, x \in X)$.

Take an automaton $\mathfrak{A}=(X, A, \delta)$, a state $a \in A$ and an integer $n(\geqq 0)$. We say that the system $(\mathfrak{A}, a)$ is *n-free* if $\delta(a, p) \neq \delta(a, q)$ for arbitrary $p, q \in X^{(n)}$ with $p \neq q$.

If we add an output to an automaton then we get the concept of a sequential machine. More precisely, a system $\mathfrak{A}=(X, A, Y, \delta, \lambda)$ is a *Mealy machine*, where (X, A, δ) is an automaton, Y is the *output alphabet* and the mapping $\lambda\colon A \times X \to Y$ is the *output function* of $\mathfrak{A}$. We can extend λ to a mapping $\lambda\colon A \times X^* \to Y^*$ in the following way: for every $a \in A$, $\lambda(a, e)=e$ and $\lambda(a, px)=\lambda(a, p)\lambda(\delta(a, p), x)$. A mapping $\mu\colon X^* \to Y^*$ is called an *automaton mapping* if there exist a Mealy machine $\mathfrak{A}=(X, A, Y, \delta, \lambda)$ and an $a \in A$ such that $\mu(p)=\lambda(a, p)$ $(p \in X^*)$. If this is the case then we say that μ can be *induced* by $\mathfrak{A}$ in the state a.

Take a Mealy machine $\mathfrak{A}=(X, A, Y, \delta, \lambda)$, an automaton mapping $\mu\colon X^* \to Y^*$ and an integer $n(\geqq 0)$. It is said that $\mathfrak{A}$ *induces μ in length n* if for some $a \in A$, $\mu(p)=\lambda(a, p)$ $(p \in X^{(n)})$.

Let $\mathfrak{A}_j=(X_j, A_j, \delta_j)$ $(j=1, \dots, t)$ be automata, X and Y alphabets, and

$$\varphi\colon A_1 \times \dots \times A_t \times X \to X_1 \times \dots \times X_t,$$
$$\psi\colon A_1 \times \dots \times A_t \times X \to Y$$

mappings. Then the Mealy machine $\mathfrak{A}=(X, A, Y, \delta, \lambda)$ is the *product (α_i-product, ν_i-product)* of $\mathfrak{A}_j$ $(j=1, \dots, t)$ with respect to X, Y and φ, ψ if the automaton (X, A, δ) is the product (α_i-product, ν_i-product) of $\mathfrak{A}_j$ $(j=1, \dots, t)$ with respect to X and φ, and for arbitrary $\mathbf{a}=(a_1, \dots, a_t)\in A$ and $x\in X$, $\lambda(\mathbf{a}, x)=\psi(a_1, \dots, a_t, x)$.

A class K of automata is *metrically complete* with respect to the product (α_i-product, ν_i-product) if for arbitrary automaton mapping $\mu\colon X^*\to Y^*$ and integer $n(\geqq 0)$ there exists a product (α_i-product, ν_i-product) $\mathfrak{A}=(X, A, Y, \delta, \lambda)$ of automata from K inducing μ in length n. Moreover, the ν_i-product is *metrically equivalent* to the product provided that for every class K of automata and non-negative integer n an automaton mapping $\mu\colon X^*\to Y^*$ can be induced in length n by a ν_i-product $\mathfrak{A}=(X, A, Y, \delta, \lambda)$ of automata from K if and only if it can be induced in length n by a product $\mathfrak{B}=(X, B, Y, \delta', \lambda')$ of automata from K.

Let $\mathfrak{A}_i=(X_i, A_i, \delta_i)$ $(i=1, \dots, t)$ be automata, and take a product

$$\mathfrak{A} = (X, A, \delta) = \prod_{i=1}^{t} A_i[X, \varphi].$$

Then for arbitrary $\mathbf{a}=(a_1, \dots, a_t)\in A$, $p\in X^*$ and i $(1\leqq i\leqq t)$ define $\varphi_i(\mathbf{a}, p)$ in the following way: $\varphi_i(\mathbf{a}, e)=e$ and $\varphi_i(\mathbf{a}, qx)=\varphi_i(\mathbf{a}, q)\varphi_i(\delta(\mathbf{a}, q), x)$ $(q\in X^*, x\in X)$.

For notions and notations not defined here, see [3] and [4].

Now we are ready to state and prove

Theorem 1. A class K of automata is metrically complete with respect to the product if and only if K is metrically complete with respect to the ν_1-product.

Proof. The condition is obviously sufficient.

To show the necessity assume that K is metrically complete with respect to the product. We prove that for every alphabet Y and integer $k(\geqq 0)$ there exist a ν_1-product $\mathfrak{D}=(Y, D, \delta'')$ of automata from K and a state $d\in D$ such that the system $(\mathfrak{D}, d)$ is k-free. This obviously implies that K is metrically complete with respect to the ν_1-product.

It is shown in [2] that K is metrically complete with respect to the product if and only if for arbitrary integer $k(\geqq 0)$ there exist an $\mathfrak{A}=(X, A, \delta)$ in K a state $a_0\in A$ and a word $p\in X^*$ with $|p|=k$ such that $\delta(a_0, p)$ is ambiguous, that is $\delta(a_0, px)\neq\delta(a_0, px')$ for some $x, x'\in X$. Let us distinguish the following two cases.

Case 1. K contains an $\mathfrak{A}=(X, A, \delta)$ such that for certain pairwise distinct states $a_0, a_1, \dots, a_{n-1}, a_1'$ and inputs $x_0, x_1, \dots, x_{n-1}, x_1'$ we have

$$\delta(a_0, x_1) = a_1,\ \delta(a_1, x_2) = a_2, \dots, \delta(a_{n-2}, x_{n-1}) = a_{n-1},\ \delta(a_{n-1}, x_0) = a_0$$

$$\text{and}\quad \delta(a_0, x_1') = a_1'.$$

Let $k(>0)$ be an integer, and take two words $p=y_1\dots y_r y_{r+1}\dots y_s$, $q=y_1\dots y_r z_{r+1}\dots$ $\dots z_t\in Y^{(k)}$ $(y_1, \dots, y_s, z_{r+1}, \dots, z_t\in Y)$ with $t\leqq s$, and $y_{r+1}\neq z_{r+1}$ if $t\neq r$, where Y is an arbitrarily fixed alphabet. Consider the ν_1-product

$$\mathfrak{B} = (Y, B, \delta') = \prod_{i=1}^{s+1} \mathfrak{B}_i[Y, \varphi, \nu]$$

given as follows.

$$\mathfrak{B}_i = \mathfrak{A} \quad (i = 1, \ldots, s+1).$$

$$\nu(1) = \emptyset \quad \text{and} \quad \nu(i) = i-1 \quad (i = 2, \ldots, s+1).$$

$$\varphi_i(a_j, y) = x_j \quad (i=2, \ldots, s+1;\ j=0, \ldots, n-1;\ y \in Y).$$

$$\varphi_i(a_1', y) = \begin{cases} x_1 & \text{if } i = r+2 \text{ and } y = z_{r+1}, \\ x_1' & \text{otherwise} \end{cases} \quad (i = 2, \ldots, s+1;\ y \in Y).$$

In all other cases φ is given arbitrarily such that the resulting product is a ν_1-product.

Take the state $\mathbf{b}=(b_1, b_2, \ldots, b_{s+1}) \in B$ with $b_1 = a_1'$, $b_i = a_{n-(i-2)}$ $(i=2, \ldots, s+1)$, where the indices of a's are taken modulo n. One can easily show by induction on j that for every $j(=1, \ldots, s)$

$$\delta'(\mathbf{b}, y_1 \ldots y_j) = (c_1, \ldots, c_{j+1}, c_{j+2}, \ldots, c_{s+1})$$

where $c_{j+1} = a_1'$ and $c_i = a_{n-(i-2)+j}$ $(i=j+2, \ldots, s+1)$. Moreover, for every $j(=r+1, \ldots, t)$

$$\delta'(\mathbf{b}, y_1 \ldots y_r z_{r+1} \ldots z_j) = (c_1, \ldots, c_{j+1}, \ldots, c_{s+1})$$

where $c_i = a_{n-(i-2)+j}$ $(i=j+1, \ldots, s+1)$. (The indices of a's are considered modulo n in the latter two cases, too.)

Therefore, the last component of $\delta'(\mathbf{b}, p)$ is a_1', and the last component of $\delta'(\mathbf{b}, q)$ is in the set $\{a_0, a_1, \ldots, a_{n-1}\}$. Thus $\delta'(\mathbf{b}, p) \neq \delta'(\mathbf{b}, q)$.

Case 2. K does not satisfy the conditions of Case 1. Then for every integer $k(\geqq 0)$ there is an $\mathfrak{A}=(X, A, \delta)$ in K with pairwise distinct states $a_0, a_1, \ldots$ $\ldots, a_k, a_{k+1}, a_{k+1}'$ and inputs $x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+1}'$ such that $\delta(a_i, x_{i+1}) = a_{i+1}$ $(i=0, \ldots, k)$ and $\delta(a_k, x_{k+1}') = a_{k+1}'$. Again take the alphabet Y and the words p, q of Case 1. Consider the ν_1-product

$$\mathfrak{B} = (Y, B, \delta') = \prod_{i=1}^{s} \mathfrak{B}_i[Y, \varphi, \nu]$$

given in the following way.

$$\mathfrak{B}_i = \mathfrak{A} \quad (i = 1, \ldots, s).$$

$$\nu(1) = \emptyset \quad \text{and} \quad \nu_i = i-1 \quad (i = 2, \ldots, s).$$

$$\varphi_1(y_1) = x_{k+1} \quad (\text{and} \quad \varphi_1(z_1) = x_{k+1}' \quad \text{if} \quad r = 0 \quad \text{and} \quad t \neq 0).$$

$$\varphi_i(a_j, y) = \begin{cases} x_{k+1}' & \text{if} \quad i = r+1, \quad j = k+1 \quad \text{and} \quad y = z_{r+1}, \\ x_j & \text{otherwise} \end{cases}$$

$$(i = 2, \ldots, s; \quad j = 1, \ldots, k+1).$$

$$\varphi_i(a_{k+1}', y) = x_{k+1}' \quad (i=2, \ldots, s).$$

In all other cases φ is given arbitrarily in accordance with the definition of the ν_1-product.

Take the state $\mathbf{b}=(a_k, a_{k-1}, \ldots, a_{k-s+1}) \in B$. Again it is easy to show that for every $j(=1, \ldots, s)$

$$\delta'(\mathbf{b}, y_1 \ldots y_j) = (c_1, \ldots, c_j, \ldots, c_s),$$

where $c_i=a_{k-(i-1)+j}$ $(i=j, \ldots, s)$. Moreover, for every $j(=r+1, \ldots, s)$

$$\delta'(\mathbf{b}, y_1 \ldots y_r z_{r+1} \ldots z_j) = (c_1, \ldots, c_j, \ldots, c_s)$$

with $c_j=a'_{k+1}$ and $c_i=a_{k-(i-1)+j}$ $(i=j+1, \ldots, s)$.

Therefore, the last component of $\delta'(\mathbf{b}, p)$ is a_{k+1}. If $s=t$ then the last component of $\delta'(\mathbf{b}, q)$ is a'_{k+1}. Moreover, if $t<s$ then the last component of $\delta'(\mathbf{b}, q)$ is $a_{k-(s-1)+t}$. In both cases we have $\delta'(\mathbf{b}, p)\neq\delta'(\mathbf{b}, q)$.

To end the proof of Theorem 1 take an integer $k(\geqq 1)$ and an alphabet Y. Moreover, set $I=\{(p, q)|p, q\in Y^{(k)}, p\neq q\}$. As it has been shown for every pair $(p, q)\in I$ there exist a ν_1-product $\mathfrak{D}_{(p,q)}=(Y, D_{(p,q)}, \delta_{(p,q)})$ of automata from K and a state $d_{(p,q)}\in D_{(p,q)}$ such that $\delta_{(p,q)}(d_{(p,q)}, p)\neq\delta_{(p,q)}(d_{(p,q)}, q)$. Form the direct product $\mathfrak{D}=\Pi(\mathfrak{D}_{(p,q)}|(p, q)\in I)$, and take the state $\mathbf{d}\in D$ with $pr_{(p,q)}(\mathbf{d})=d_{(p,q)}$, where $pr_{(p,q)}$ denotes the $(p, q)^{\text{th}}$ projection. Obviously, $(\mathfrak{D}, \mathbf{d})$ is a k-free system. Since the direct product of ν_1-products of automata is isomorphic to a ν_1-product of the same automata this completes the proof of Theorem 1.

Let us note that the ν_1-product used in the proof of Theorem 1 is also an α_0-product.

Next we prove

Theorem 2. The product is metrically equivalent to the ν_3-product.

Proof. Let K be a class of automata. If K is metrically complete with respect to the product then, by Theorem 1, for arbitrary integer $k(\geqq 0)$ every automaton mapping μ: $X^* \to Y^*$ can be induced in length $k+1$ by a ν_1-product $\mathfrak{A}=(X, A, Y, \delta, \lambda)$ of automata from K. Thus we assume that K is not metrically complete with respect to the product. Therefore, none of Case 1 and Case 2 holds for K. This implies that either there is no ambiguous state in any of the automata from K or there is a maximal positive integer k such that for some $\mathfrak{A}=(X, A, \delta)\in K$, $a\in A$ and $p\in X^*$ with $|p|=k-1$, $\delta(a, p)$ is ambiguous. In the first case every product of automata from K can be given as a quasi-direct product of the same automata. Thus we suppose the existence of the above k.

Let

$$\mathfrak{A} = (X, A, \delta) = \prod_{i=1}^{s} \mathfrak{A}_i[X, \varphi] \quad (\mathfrak{A}_i = (X_i, A_i, \delta_i)\in K, \ i = 1, \ldots, s)$$

be a product and $\mathbf{a}=(a_1, \ldots, a_s)\in A$ a state. We shall prove the existence of a ν_3-product

$$\mathfrak{B} = (X, B, \delta') = \prod_{i=1}^{t} \mathfrak{B}_i[X, \varphi', \nu] \quad (\mathfrak{B}_i = (X'_i, B_i, \delta'_i), \ i = 1, \ldots, t)$$

with a state $\mathbf{b}=(b_1, \ldots, b_t)\in B$ such that the following conditions are satisfied.

(i) $(\mathfrak{B}_1, b_1)$ is k-free, $X'_1=X$, φ'_1 is the identity mapping on X and $\mathfrak{B}_1$ is a ν_1-product of automata from K.

(ii) $\mathfrak{B}_2$ is a ν_1-product of automata from K, $X'_2=X$ and for any two words $p, q\in X^*$ with $|p|<k$ and $|q|\geqq k$, $\delta'_2(b_2, \varphi'_2(\mathbf{b}, p))\neq\delta'_2(b_2, \varphi'_2(\mathbf{b}, q))$.

(iii) $\mathfrak{B}_i\in K$ $(i=3, \ldots, t)$.

(iv) For arbitrary two words $p, q\in X^*$ with $|p|=|q|=k$ and integer i $(1\leqq i\leqq s)$ there is a j $(1\leqq j\leqq t)$ with $\mathfrak{B}_j=\mathfrak{A}_i$, $b_j=a_i$, $\delta'_j(b_j, \varphi'_j(\mathbf{b}, p))=\delta_i(a_i, \varphi_i(\mathbf{a}, p))$ and $\delta'_j(b_j, \varphi'_j(\mathbf{b}, q))=\delta_i(a_i, \varphi_i(\mathbf{a}, q))$.

This will imply that the subautomaton of $\mathfrak{A}$ generated by $\mathbf{a}$ is a homomorphic image of the subautomaton of $\mathfrak{B}$ generated by $\mathbf{b}$. Indeed, take two words $p, q \in X^*$ with $\delta(\mathbf{a}, p) \neq \delta(\mathbf{a}, q)$. It is enough to show that $\delta'(\mathbf{b}, p) \neq \delta'(\mathbf{b}, q)$. Let us distinguish the following cases.

(I) $|p|, |q| \leqq k$. Then $\delta'(\mathbf{b}, p) \neq \delta'(\mathbf{b}, q)$ since they differ at least in their first components.

(II) $|p| < k$ and $|q| > k$. Then $\delta'(\mathbf{b}, p)$ and $\delta'(\mathbf{b}, q)$ are different at least in their 2^{nd} components.

(III) $|p|, |q| \geqq k$. First of all observe that, by the maximality of k, for arbitrary automaton $\mathfrak{C} = (Y, C, \delta'') \in K$, state $c \in C$ and words $r, r_1, r_2 \in Y^*$ with $|r| = k$ and $|r_1| = |r_2|$, $\delta''(c, rr_1) = \delta''(c, rr_2)$. Let $p = p_1 p_2$ and $q = q_1 q_2$ ($|p_1| = |q_1| = k$). Moreover, let i $(1 \leqq i \leqq s)$ be an index for which $\delta_i(a_i, \varphi_i(\mathbf{a}, p)) \neq \delta_i(a_i, \varphi_i(\mathbf{a}, q))$. Take the index j given by (iv) to this i and p_1, q_1. Then by our remark above $\delta'_j(b_j, \varphi'_j(\mathbf{b}, p_1 p_2)) = \delta'_j(b_j, \varphi'_j(\mathbf{b}, p_1) p'_2) = \delta_i(a_i, \varphi_i(\mathbf{a}, p_1) p'_2) = \delta_i(a_i, \varphi_i(\mathbf{a}, p_1 p_2))$ where $p'_2 \in X_i^*$ is a word with $|p'_2| = |p_2|$. Similarly, $\delta'_j(b_j, \varphi'_j(\mathbf{b}, q_1 q_2)) = \delta_i(a_i, \varphi_i(\mathbf{a}, q_1 q_2))$. Therefore, $\delta'(\mathbf{b}, p) \neq \delta'(\mathbf{b}, q)$ since they differ at least in their j^{th} components.

The k-free automaton in (i) can be constructed by using the same method as in the proof of Theorem 1 (according to Case 2).

To give $\mathfrak{B}_2$ take an automaton $\mathfrak{C} = (Y, C, \delta'') \in K$ with pairwise distinct states $c_0, c_1, \ldots, c_{k-1}, c_k, c'_k$ and inputs $y_1, \ldots, y_{k-1}, y_k, y'_k$ such that $\delta''(c_0, y_1) = c_1, \ldots, \delta''(c_{k-2}, y_{k-1}) = c_{k-1}$, $\delta''(c_{k-1}, y_k) = c_k$ and $\delta''(c_{k-1}, y'_k) = c'_k$. Form the single factor ν_1-product

$$\mathfrak{B}_2 = \mathfrak{C}[X, \varphi'', \nu']$$

where $\nu'(1) = 1$ and $\varphi''(c_i, x) = y_{i+1}$ $(i = 0, \ldots, k-1;\ x \in X)$. Moreover, in all other cases φ'' is given arbitrarily. Since K is not metrically complete $\mathfrak{B}_2$ satisfies (ii).

Next we show that for arbitrary words $p, q \in X^*$ with $|p| = |q| = k$ and integer i $(1 \leqq i \leqq s)$ there are a ν_3-product

$$\mathfrak{D} = (X, D, \delta'') = \prod_{i=1}^{r} \mathfrak{C}_i[X, \varphi'', \nu']$$

$(\mathfrak{C}_i = (Y_i, C_i, \delta''_i) \in K,\ i = 1, \ldots, r)$ and a state $\mathbf{d} = (d_1, \ldots, d_r) \in D$ such that $\mathfrak{C}_r = \mathfrak{A}_i$, $d_r = a_i$, $\delta''_r(d_r, \varphi''_r(\mathbf{d}, p)) = \delta_i(a_i, \varphi_i(\mathbf{a}, p))$ and $\delta''_r(d_r, \varphi''_r(\mathbf{d}, q)) = \delta_i(a_i, \varphi_i(\mathbf{a}, q))$. Then taking the direct product of $\mathfrak{B}_1$, $\mathfrak{B}_2$ and these automata $\mathfrak{D}$ the resulting automaton $\mathfrak{B}$ with a suitable $\mathbf{b} \in B$ will obviously satisfy (i)—(iv).

Since the case

$$(*) \qquad \delta_i(a_i, \varphi_i(\mathbf{a}, p)) = \delta_i(a_i, \varphi_i(\mathbf{a}, q))$$

is trivial we may assume that $(*)$ does not hold. Then $p \neq q$. Let $p = x_1 \ldots x_m x_{m+1} \ldots x_k$, $q = x_1 \ldots x_m y_{m+1} \ldots y_k$, $x_{m+1} \neq y_{m+1}$, $\varphi_i(\mathbf{a}, p) = \bar{p} = u_1 \ldots u_m u_{m+1} \ldots u_k$ and $\varphi_i(\mathbf{a}, q) = \bar{q} = u_1 \ldots u_m v_{m+1} \ldots v_k$. Moreover, set $p_j = x_1 \ldots x_j$; $\bar{p}_j = u_1 \ldots u_j$ $(j = 0, 1, \ldots, k)$ and

$$q_j = \begin{cases} x_1 \ldots x_j & \text{if } 0 \leqq j \leqq m, \\ x_1 \ldots x_m y_{m+1} \ldots y_j & \text{if } m < j \leqq k, \end{cases}$$

$$\bar{q}_j = \begin{cases} u_1 \ldots u_j & \text{if } 0 \leqq j \leqq m, \\ u_1 \ldots u_m v_{m+1} \ldots v_j & \text{if } m < j \leqq k. \end{cases}$$

Denote a_i by c_0. Let l_1 be the smallest integer u for which there is a v with $u < v \leqq k$ such that $\delta_i(c_0, \bar{p}_u) = \delta_i(c_0, \bar{p}_v)$. If there are no such u and v then let $l_1 = k$. Similarly, let l_2 be the least integer u such that for some v $(u<v\leqq k)$, $\delta_i(c_0,\bar{q}_u)=\delta_i(c_0,\bar{q}_v)$. Again if there are no such u and v then let $l_2=k$. Assume that $l_1\geqq l_2$. Finally, denote by w the maximal number with $\delta_i(c_0,\bar{p}_w)=\delta_i(c_0,\bar{q}_w)$ $(0\leqq w\leqq k)$. Since $\delta_i(c_0,\bar{p})\neq\delta_i(c_0,\bar{q})$ the inequality $l_2>w$ holds. Moreover, $w\geqq m$. Let us introduce the notations $\delta_i(c_0,\bar{p}_j)=c_j$ $(j=0,\ldots,l_1)$ and $\delta_i(c_0,\bar{q}_j)=c'_j$ $(j=0,\ldots,l_2)$. Then the elements $c_0,\ldots,c_w,c_{w+1},c'_{w+1}$ are pairwise distinct, and so are the elements of the sets $\{c_0,\ldots,c_{l_1}\}$ and $\{c'_0,\ldots,c'_{l_2}\}$. We continue the proof by distinguishing the following two cases.

Case 1. $w=m$. Then let $r=2$ and $\mathfrak{C}_1=\mathfrak{C}_2=\mathfrak{A}_i$. Moreover, $v'(1)=1$, $v'(2)==\{1,2\}$ and

$$\varphi''_1(c_j,x)=u_{j+1}\quad(j=0,\ldots,l_1-1;\ x\in X),$$
$$\varphi''_2(c_j,c_j,x_{j+1})=u_{j+1}\quad(j=0,\ldots,l_1-1),$$
$$\varphi''_2(c_j,c'_j,y_{j+1})=v_{j+1}\ (j=m,\ldots,l_2-1).$$

In all other cases φ'' is given arbitrarily. φ'' is well defined. It is obvious that φ''_1 is a function. Assume that $(c_j,c_j,x_{j+1})=(c_j,c'_j,y_{j+1})$ holds for some j $(m<j<l_2)$. But this would imply $w>m$.

It is seen immediately that by taking $\mathbf{d}=(c_0,c_0)$ the equalities

$$\delta''(\mathbf{d},p_j)=(c_j,c_j)\ (j=0,\ldots,l_1)$$

and

$$\delta''(\mathbf{d},q_j)=(c_j,c'_j)\quad(j=0,\ldots,l_2)$$

hold. Since K is not metrically complete with respect to the product, by the choice of l_1 and l_2, this implies

$$\delta''(\mathbf{d},p)=\big(c,\delta_i(c_0,\bar{p})\big)\quad(c\in A_i)$$

and

$$\delta''(\mathbf{d},q)=\big(c',\delta_i(c_0,\bar{q})\big)\quad(c'\in A_i).$$

Case 2. $w>m$. Let $r=w-m+2$ and $\mathfrak{C}_1=\ldots=\mathfrak{C}_r=\mathfrak{A}_i$. Moreover, $v'(1)=1$, $v'(j)=j-1$ $(j=2,\ldots,r-2)$, $v'(r-1)=r-1$ and $v'(r)=\{r-2,r-1,r\}$. Furthermore,

$$\varphi''_1(c_{w-m+l},x_{l+1})=u_{w-m+l+1}\quad(l=0,\ldots,m),$$
$$\varphi''_1(c_w,y_{m+1})=v_{w+1},$$
$$\varphi''_j(c_{w-m-j+2+l},x_{l+1})=u_{w-m-j+2+l}\quad(j=2,\ldots,r-2;\quad l=0,\ldots,m+j-1),$$
$$\varphi''_j(c_{w-m-j+2+l},y_{l+1})=u_{w-m-j+2+l}\quad(j=2,\ldots,r-2;\quad l=m,\ldots,m+j-2),$$
$$\varphi''_j(c'_{w+1},y_{m+j})=v_{w+1}\ (j=2,\ldots,r-2),$$
$$\varphi''_{r-1}(c_l,x_{l+1})=u_{l+1}\quad(l=0,\ldots,l_1-1),$$
$$\varphi''_{r-1}(c_l,y_{l+1})=u_{l+1}\quad(l=m,\ldots,l_2-1),$$
$$\varphi''_r(c_{l+1},c_l,c_l,x_{l+1})=u_{l+1}\quad(l=0,\ldots,w),$$
$$\varphi''_r(c_{l+1},c_l,c_l,y_{l+1})=u_{l+1}\quad(l=m,\ldots,w-1),$$
$$\varphi''_r(c'_{w+1},c_w,c_w,y_{w+1})=v_{w+1},$$
$$\varphi''_r(c,c_{w+l},c_{w+l},x_{w+l+1})=u_{w+l+1}\quad\big(c\in A_i,\ l=1,\ldots,l_1-(w+1)\big),$$
$$\varphi''_r(c,c_{w+l},c'_{w+l},y_{w+l+1})=v_{w+l+1}\quad\big(c\in A_i,\ l=1,\ldots,l_2-(w+1)\big).$$

In all other cases φ'' is given arbitrarily in accordance with the definition of the ν_3-product. φ'' is well defined. This is clear in all cases except when

$$(c, c_{w+l}, c_{w+l}, x_{w+l+1}) = (c', c_{w+l}, c'_{w+l}, y_{w+l+1})$$

for an l $(1 \leqq l \leqq l_2-(w+1))$. But this would contradict the choice of w.

One can easily show by induction on l that for $\mathbf{d}=(c_{w-m}, c_{w-m-1}, \ldots, c_1, c_0, c_0)$ the following equalities hold.

$$\delta''(\mathbf{d}, p_l) = (c_{w-m+l}, c_{w-m-1+l}, \ldots, c_{1+l}, c_l, c_l) \quad (l = 0, \ldots, m),$$

$$\delta''(\mathbf{d}, p_{m+l}) = (c''_1, \ldots, c''_{l-1}, c_{w+1}, c_w, \ldots, c_{m+l+1}, c_{m+l}, c_{m+l})$$
$$(c''_1, \ldots, c''_{l-1} \in A_i;\ l = 1, \ldots, w-m),$$

$$\delta''(\mathbf{d}, q_{m+l}) = (c''_1, \ldots, c''_{l-1}, c'_{w+1}, c_w, \ldots, c_{m+l+1}, c_{m+l}, c_{m+l})$$
$$(c''_1, \ldots, c''_{l-1} \in A_i;\ l = 1, \ldots, w-m),$$

$$\delta''(\mathbf{d}, p_l) = (c''_1, \ldots, c''_{r-2}, c_l, c_l) \quad (c''_1, \ldots, c''_{r-2} \in A_i;\ l = w+1, \ldots, l_1),$$

$$\delta''(\mathbf{d}, q_l) = (c''_1, \ldots, c''_{r-2}, c_l, c'_l) \quad (c''_1, \ldots, c''_{r-2} \in A_i;\ l = w+1, \ldots, l_2).$$

Since K is not metrically complete with respect to the product, by the choice of l_1 and l_2, the last two equalities imply

$$\delta''(\mathbf{d}, p) = (c''_1, \ldots, c''_{r-1}, \delta_i(c_0, \bar{p})) \quad \text{and} \quad \delta''(\mathbf{d}, q) = (\bar{c}_1, \ldots, \bar{c}_{r-1}, \delta_i(c_0, \bar{q}))$$
$$(c''_1, \ldots, c''_{r-1}, \bar{c}_1, \ldots, \bar{c}_{r-1} \in A_i)$$

which ends the proof of Theorem 2.

Let us note that the ν_3-product $\mathfrak{B}$ in the proof of Theorem 2 is also an α_1-product.

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