The analysis of signal flow graph containing sampled-data elements

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Summary

In this paper the author reduces the input-output analysis of a signal flow graph containing sampled-data elements to the analysis of a signal flow graph consisting purely of basic (linear) elements. By this reduction, the algebraic formula of the output signals known from the literature can be substantially simplified. The new formula can advantageously be used to calculate the response signals by computer as well as by topological methods.

Introduction

The input-output analysis of a linear system excited by continuous and sampled-data signals is possible by calculating the output signals of a signal flow graph consisting of basic and sampled-data elements [1]. In practice, the methods for these calculations adopt algebraic [5] or topological apparatus [4]. The main advantage of a topological apparatus is in its graphic quality, but the application is recommendable in special cases only, for it is complicated in respect of computer technique. The algebraic method presented in [5] can be used in more general cases but the formula of the output signals is still complicated.

In this paper, a procedure is introduced which gives a simpler algebraic formula of the analysis. Hence, on the one hand, the earlier method [5] can be reduced from the point of view of computer implementation, on the other hand an effective topological procedure can be designed for more general cases. The present procedure is applicable to signal flow graphs containing sampled-data elements working synchronously, however there is no limitation either for the number of the elements or for the number of the input-output vertices.

The new algebraic formula of the analysis

Consider the signal flow graph G containing sampled-data elements of number r. Let the number of input vertices be m and that of the output vertices n. Associate with each edge of G a transfer function as a parameter, and excite the system at the input vertices by the vector $X=(x_1, ..., x_m)$ the components of which are Laplace transforms of the exciting signals. Due to this excitation, the response vector $Y=(y_1, ..., y_n)$ appears at the output of G, the components of which are Laplace transforms of the output (response) signals. For example Fig. 1 shows a signal flow graph with one input and one output and with two sampled-data elements. On Fig. 1 the sampled-data elements are marked by dotted lines and the transfer directions are also indicated by arrows. The task is to calculate the output vector Y.

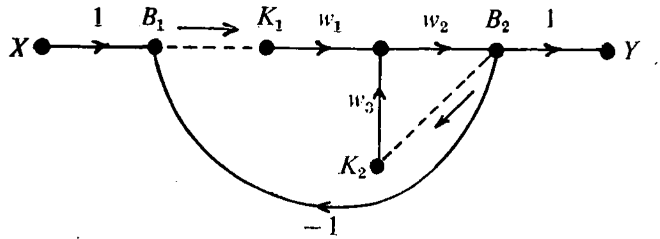

Fig. 1.

Let B be the vector of (the Laplace transform of) the signals appearing at the starting points of the sampled-data elements and denote by K the vector the components of which are (the Laplace transform of) the signals at the endpoints of the sampled-data elements.

Vector B and K are of size r.

For the calculation, we transform G as follows. Delete all sampled-data elements from G and consider $\begin{bmatrix} X \\ K \end{bmatrix}$ as the exciting vector of the remaining graph and let $\begin{bmatrix} B \\ Y \end{bmatrix}$ be the response vector. As the remaining graph is linear we can write:

$$\begin{bmatrix} B \\ Y \end{bmatrix} = \mathbf{W} \begin{bmatrix} X \\ K \end{bmatrix} \tag{1}$$

where $\mathbf{W}$ denotes the transfer matrix of the remaining graph. Moreover, the condition

$$K = B^* \tag{2}$$

must be fulfilled. The star in (2) refers to the sampling operation. During the transformation neither the topology nor the signals of the original graph change, particularly the vector Y remains the same.

Fig. 2 derived from Fig. 1 illustrates the transformation. Observe that now the input and output vectors of the remaining graph are of size 3. After partitioning $\mathbf{W}$,

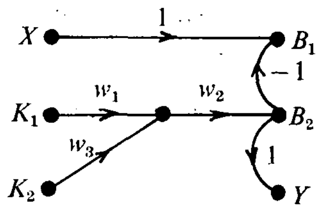

Fig. 2.

(1) can be written in the following form:

$$\begin{bmatrix} B \\ Y \end{bmatrix} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{bmatrix} \begin{bmatrix} X \\ K \end{bmatrix},$$

and hence the system of equations

$$\begin{cases} B = \mathbf{W}_{11}X + \mathbf{W}_{12}K \\ Y = \mathbf{W}_{21}X + \mathbf{W}_{22}K \end{cases} \tag{3}$$

follows.

Let us consider the sampled form of the second equation of (3). Taking into account (2), we can write:

$$B^* = (\mathbf{W}_{11}X)^* + \mathbf{W}_{12}^* B^*. \tag{4}$$

Hence we obtain:

$$(\mathbf{1} - \mathbf{W}_{12}^*)B^* = (\mathbf{W}_{11} \cdot X)^*, \tag{5}$$

where $\mathbf{1}$ is the unit matrix of size $r \times r$. If $\det(\mathbf{1} - \mathbf{W}_{12}^*) \neq 0$, from (5)

$$B^* = (\mathbf{1} - \mathbf{W}_{12}^*)^{-1} \cdot (\mathbf{W}_{11}X)^* \tag{6}$$

follows, where the upper index -1 refers to the inverse matrix. Finally, substituting the right-hand side of formula (6) into the second equation of (3), and taking (2) into consideration, we have:

$$Y = \mathbf{W}_{21}X + \mathbf{W}_{22}(1 - \mathbf{W}_{12}^*)^{-1} \cdot (\mathbf{W}_{11} \cdot X)^*. \tag{7}$$

Notice that the practical application of the formula (7) requires the calculation of the transfer matrix of a linear signal flow graph, which is possible by the method elaborated in the reference [5].

Computer implementation

Comparing the response vector (7) with the transfer matrix formula of a signal flow graph consisting of basic elements only given in [5] it can be observed that both of these calculations require the same matrix operations (i.e. partitioning, substraction from the unit matrix, calculation of the inverse, multiplication). Since the application of (7) also requires the calculation of the transfer matrix of a signal flow graph with basic elements, there is a possibility to construct a common program for the analysis of signal flow graphs without as well as with sampled-data elements. Such a program gives either the transfer matrix (first case) or the output vector (second case). Fig. 3 shows the scheme of the program mentioned above.

The input data of the program are as follows: m and n denote the numbers of the input and output vertices, r is the number of the sampled-data elements while v stands for the number of the internal vertices. The basic elements are given by

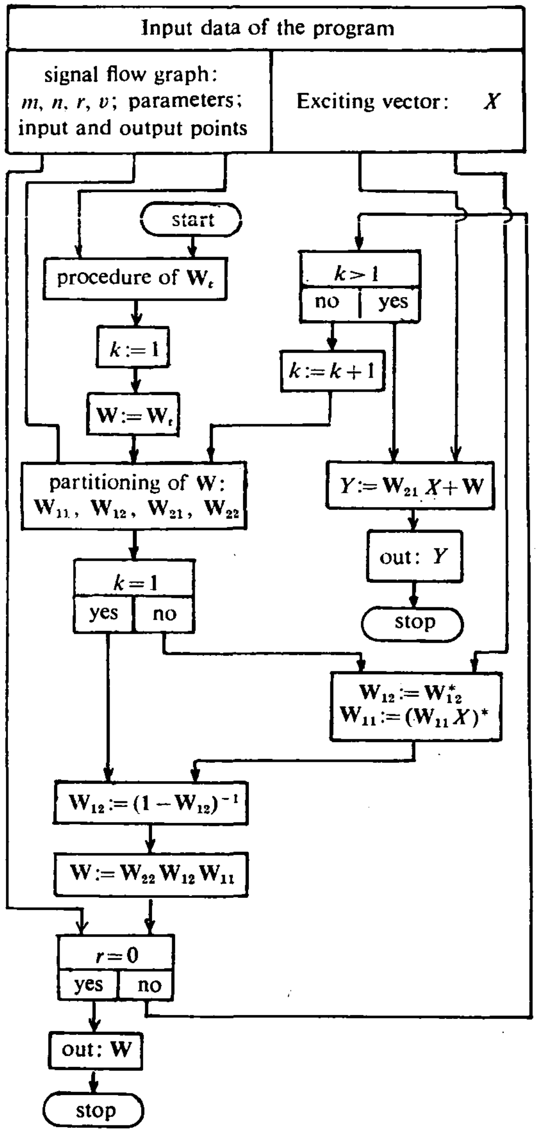

Fig. 3.

their endpoints. For the calculation of the transfer matrix, the components of vector X can be chosen arbitrarily. The program is executed in one cycle if the task is to determine the transfer matrix, or in two cycles if we wish to calculate the output vector Y. The program parameter k counts the necessary cycles and $\mathbf{W}_t$ denotes the node matrix of the original signal flow graph.

For the sake of comprehensibility we summarize the calculation of the transfer matrix on the left side of Fig. 3 (case $r=0$). If sampled-data elements are also present ($r \neq 0$), the calculation proceeds on the right-hand side of the scheme and the second cycle starts. For performing the iterative steps, the program returns to the appropriate blocks on the left-hand side of Fig. 3. The location of the necessary sampling operations can also be easily seen in the scheme.

A topological procedure

Notice that matrices $\mathbf{W}_{11}, \ldots, \mathbf{W}_{22}$ occuring in (3) can be obtained by topological formulas from the signal flow graph without sampled-data elements, namely each of them can also be regarded as a transfer matrix belonging to a special excitation. (For example to determine $\mathbf{W}_{11}$ let the input vector be $\begin{bmatrix} X \\ O \end{bmatrix}$, the output one B'; in case of $\mathbf{W}_{12}$ the input vector $\begin{bmatrix} O \\ K \end{bmatrix}$, the output one B''; because of the linearity $B=B'+B''$, and so on). For determining Y by topological method it is enough to show that K can also be produced by a topological formula.

For this purpose, let us introduce the notations $S=(S_1, \ldots, S_r)=(\mathbf{W}_{11}X)^*$. Taking (2) into account, we have:

$$K = S+\mathbf{W}_{12}^* K \tag{8}$$

Now, let us consider the signal flow graph G_M associated with the linear system of equations (8) in the usual manner (MASON graph, [2]). In the general case,

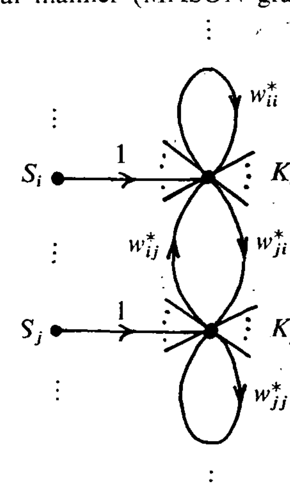

Fig. 4.

G_M arises from the directed full graph with r vertices, the edges of which are parametrized with the elements of $\mathbf{W}_{12}^*$, namely using the notation $\mathbf{W}_{12}=(w_{ij})_{r\times r}$ the parameter of the loop coinciding to the i-th vertex is w_{ii}^*, while the parameter of the edge directed from the i-th vertex to the j-th vertex is w_{ji}^* $(i,j=1, ..., r)$. This full graph has to be supplemented by r edges, each of which has a parameter 1 as transmission. The starting points of the supplementary edges are the inputs of G_M, the endpoints are the vertices of the full graph which are at the same time the outputs of G_M. Exciting G_M by S at the inputs, the vector K appears as the response vector in the outputs.

Fig. 4 indicates a part of G_M in a general case. From G_M the vector K can be obtained by a topological formula, and finally, the first equation of (3) gives the vector Y.

Application

Let us consider the signal flow graph with sampled-data elements given in Fig. 1 and let our first task be the calculation of the response vector Y by formula (7).

Investigating Fig. 2 one can write the vector equation (1) in the following form:

$$\begin{bmatrix} B_1 \\ B_2 \\ Y \end{bmatrix} = \mathbf{W} \begin{bmatrix} X \\ K_1 \\ K_2 \end{bmatrix} \tag{9}$$

Using the method described in [5] the transfer matrix is:

$$\mathbf{W} = \left[\begin{array}{c:cc} 1 & -w_1 w_2 & -w_3 w_2 \\ 0 & w_1 w_2 & w_3 w_2 \\ \hdashline 0 & w_1 w_2 & w_3 w_2 \end{array}\right] \tag{10}$$

In (10) the dotted lines indicate the partitioning of $\mathbf{W}$. After some calculation

$$\mathbf{1}-\mathbf{W}_{12}^* = \begin{bmatrix} 1+(w_1 w_2)^* & (w_3 w_2)^* \\ -(w_1 w_2)^* & 1-(w_3 w_2)^* \end{bmatrix}$$

arises. Finally we get:

$$(\mathbf{1}-\mathbf{W}_{12}^*)^{-1} = \frac{1}{1-(w_3 w_2)^* + (w_1 w_2)^*} \begin{bmatrix} 1-(w_3 w_2)^* & -(w_3 w_2)^* \\ (w_1 w_2)^* & 1+(w_1 w_2)^* \end{bmatrix} \tag{11}$$

Taking into account (10) and (11), from (7) we can write for the output vector:

$$Y = [w_1 w_2 \quad w_3 w_2] \cdot \frac{1}{1-(w_3 w_2)^* + (w_1 w_2)^*} \begin{bmatrix} 1-(w_3 w_2)^* & -(w_3 w_2)^* \\ (w_1 w_2)^* & 1+(w_1 w_2)^* \end{bmatrix} \begin{bmatrix} X^* \\ 0 \end{bmatrix} =$$

$$= \frac{w_1 w_2 - w_1 w_2 (w_3 w_2)^* + w_3 w_2 (w_1 w_2)^*}{1-(w_3 w_2)^* + (w_1 w_2)^*} \cdot X^* \tag{12}$$

As a second task let us determine Y in the previous example by topological procedure. Using the MASON formula, from Fig. 2 the matrices $\mathbf{W}_{11}, ..., \mathbf{W}_{22}$ can immediately be read. Then it is sufficient to determine K by topological formula.

Taking into account the elements of $\mathbf{W}_{12}^*$, the signal flow graph G_M associated with the system of equations (8) can be given. G_M is drawn on Fig. 5. Applying the MASON formula directly to Fig. 5

$$K_1 = \frac{X^*(1-(w_3w_2)^*)}{1+(w_1w_2)^*-(w_3w_2)^*}, \quad \text{and}$$

$$K_2 = \frac{X^*(w_1w_2)^*}{1+(w_1w_2)^*-(w_3w_2)^*} \tag{13}$$

are fulfilled.

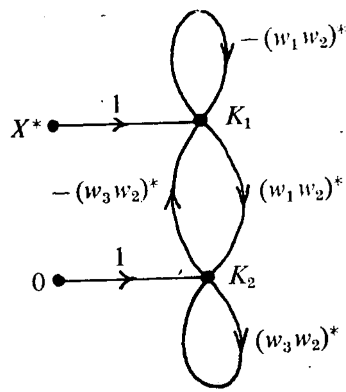

Fig. 5.

Finally, by (10) and (13) we get from the first equation of (3)

$$Y = [w_1w_2 \quad w_3w_2] \cdot \begin{bmatrix} \dfrac{X^*(1-(w_3w_2)^*)}{1+(w_1w_2)^*-(w_3w_2)^*} \\ \dfrac{X^*(w_1w_2)^*}{1+(w_1w_2)^*-(w_3w_2)^*} \end{bmatrix} \tag{14}$$

After calculating the prescribed operations in (14) the result is identical with (12).

References

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