Varieties and general products of top-down algebras

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Unrestricted, i.e. both finite and infinite general products of unoids were treated in [2]. It has been shown that the unoid varieties arising with general products are exactly those classes of unoids which have equational presentation in terms of so-called p-identities. In addition, these type independent varieties coincide with the varieties obtainable with the more special α_0-products. In other words this means that the unrestricted general product is homomorphically as general as the α_0-product. Although unoids do have certain specialities as shown in [1] and [2], using a new method, the above mentioned results have been extended to arbitrary algebras in [1]. Due to the specific nature of unoids, all type-independent varieties of unoids have been described in [2]. No similar description is attainable for the general case of algebras at present.

The aim of this paper is to give similar results for top-down algebras, a less well-known type of algebraic structures originating from tree automata theory. Top-down algebras are elsewhere called root-to-frontier algebras or ascending algebras as eg. in [3] and [4], due to a converse visualization of trees. The whole treatment will be done parallel with [1].

1. Top-down algebras and general products

Let R be a nonvoid subset of the natural numbers $N=\{1, 2, ...\}$. R is called a rank type and will be fixed throughout the paper. A type of rank type R is a collection $F=\bigcup(F_n| n\in N)$ so that $F_n\neq\emptyset$ if and only if $n\in R$. In the sequel every type F is supposed to belong to the fixed rank type R. A top-down F-algebra is an ordered pair $\mathfrak{A}=(A, F)$ with A a nonvoid set and a realization $f: A\to A^n$ for each operational symbol $f\in F_n$. The class of all top-down F-algebras is denoted $\mathbf{K}_F$. $\mathbf{K}_R=\bigcup(\mathbf{K}_F|F$ has rank type $R)$.

Suppose we are given two top-down F-algebras $\mathfrak{A}=(A, F)$ and $\mathfrak{B}=(B, F)$. A mapping $\varphi: A\to B$ is called a homomorphism $\mathfrak{A}\to\mathfrak{B}$ if $\varphi f pr_i=f pr_i\varphi$ for every $\in F_n$ and $i\in[n]=\{1, ..., n\}$, where pr_i denotes the i-th projection and functional composition is written juxtaposition from left to right. If φ is onto, $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$. Further, $\mathfrak{A}$ is called a subalgebra of $\mathfrak{B}$ if $A\subseteq B$ and the natural inclusion is a homomorphism $\mathfrak{A}\to\mathfrak{B}$.

Let $\mathbf{K}$ be an arbitrary class of top-down algebras (of rank type R). Then $\mathscr{H}(\mathbf{K})$

and $\mathcal{S}(\mathbf{K})$ will respectively denote the class of all homomorphic images and the class of all subalgebras of top-down algebras from $\mathbf{K}$.

Now we are going to introduce general products of top-down algebras. To this, take types F and F^i $(i\in I)$ as well as top-down F^i-algebras $\mathfrak{A}_i=(A_i, F^i)$ $(i\in I)$. Let φ be a family of feedback functions $\varphi_i: \Pi(A_i|i\in I)\times F\to F^i$. The φ_i's must preserve the rank, i.e.,

$$(a, f)\varphi_i\in F_n^i$$

if $f\in F_n$, $a\in \Pi(A_i|i\in I)$. The general product of the $\mathfrak{A}_i$'s w.r.t. φ and F is that top-down F-algebra

$$\mathfrak{A} = (A, F) = \Pi(\mathfrak{A}_i|i\in I, F, \varphi)$$

satisfying

$$A = \Pi(A_i|i\in I),$$

$$afpr_j\, pr_i = apr_i\, f^i\, pr_j$$

where $a\in A, f\in F_n, i\in I, j\in[n]$ and $f^i=(a,f)\varphi_i$. A general product of a finite number of top-down algebras is denoted $\Pi(\mathfrak{A}_1, ..., \mathfrak{A}_n|\varphi, F)$.

Two restricted forms of the general product will be of particular interest. These are the α_0-product and the direct product. The general product defined above is an α_0-product if the index set I is linearly ordered, and for every $i\in I$, the feedback function φ_i assigning value in F^i to $((a_i|i\in I), f)\in\Pi(A_i|i\in I)\times F$ is independent of the a_j's with $j\geqq i$. In case of an α_0-product we shall indicate only those variables of φ_i on which it may depend. Index sets $[n]$ are supposed to have the natural ordering.

The concept of the direct product easily comes by specialization, too. A general product $\mathfrak{A}=(A, F)=\Pi(\mathfrak{A}_i|i\in I, \varphi, F)$ is a direct product if all factors $\mathfrak{A}_i$ are top-down F-algebras and $(a, f)\varphi_i=f, i\in I, f\in F, a\in A$.

Take a class $\mathbf{K}$ of top-down algebras. The operators $\mathcal{P}_g$, $\mathcal{P}_{\alpha_0}$, $\mathcal{P}$ and $\mathcal{P}_{f\alpha_0}$ are defined by the following list:

$\mathcal{P}_g(\mathbf{K})$: all general products of factors from $\mathbf{K}$,
$\mathcal{P}_{\alpha_0}(\mathbf{K})$: all α_0-products of factors from $\mathbf{K}$,
$\mathcal{P}(\mathbf{K})$: all direct products of factors from $\mathbf{K}$,
$\mathcal{P}_{f\alpha_0}(\mathbf{K})$: all α_0-product of finitely many factors from $\mathbf{K}$.

According to the universal algebraic analogy (see also the next section), classes $\mathbf{K}\subseteq\mathbf{K}_F$ closed under the operators $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{P}$ are called varieties. However, the main interest will be in type-independent varieties. By definition, a type-independent variety is a class $\mathbf{K}\subseteq\mathbf{K}_R$ closed under the operators $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{P}_g$.

2. Varieties of top-down algebras

Top-down algebras of rank type $R=\{1\}$ will be called unoids. Since in a unoid (A, F) every operation is a function $f: A\to A$, unoids are ordinary algebras.

Let $\mathfrak{A}=(A, F)$ be a top-down F-algebra. There is a simple way to associate a unoid $\mathfrak{A}^u=(A, F^u)$ with $\mathfrak{A}$: put $a(f,i)=afpr_i$ for every $a\in A$ and $(f, i)\in F^u$. Here the unary type F^u consists of all pairs (f, i) with $f\in F_n$ and $i\in[n]$. It is obvious that every homomorphism from a top-down F-algebra $\mathfrak{A}$ into a top-down F-algebra

$\mathfrak{B}$ becomes a homomorphism $\mathfrak{A}^u \to \mathfrak{B}^u$, and the resulting functor is an isomorphism of the category of all top-down F-algebras onto the category of all F^u-unoids. Varieties are preserved under this transition, if $\mathbf{V} \subseteq \mathbf{K}_F$ is a variety, then so is $\mathbf{V}^u = \{\mathfrak{A}^u | \mathfrak{A} \in \mathbf{V}\}$, and conversely. Anyway, this simple transition allows us to adapt well-known concepts and facts from universal algebra to our top-down algebras, e.g., for any class $\mathbf{K} \subseteq \mathbf{K}_F$, $\mathscr{HSP}(\mathbf{K})$ is least variety containing $\mathbf{K}$.

The concept of an identity also extends to our case in an obvious way. An F-identity is either a formal equation $xp=xq$ or $xp=yq$, where x and y are different variables, p and q are words over the alphabet F^u, i.e., $p, q \in (F^u)^*$. Expressions zp with $z \in \{x, y\}$ and $p \in (F^u)^*$ are called polynomial symbols. The number of letters appearing in zp is called the length of zp and is denoted $|zp|$. The set $\mathrm{pre}(zp)$ is defined by $\mathrm{pre}(zp) = \{zq \mid |zq| < |zp|,\ \exists r\ qr = p\}$. An F-identity is satisfied by an F-algebra $\mathfrak{A}$ if it is satisfied by the unoid $\mathfrak{A}^u$ in the ordinary sense. In this case we also say the F-identity holds in $\mathfrak{A}$.

For a class $\mathbf{K} \subseteq \mathbf{K}_F$, $\mathrm{Id}(\mathbf{K})$ denotes the set of all identities satisfied by every $\mathfrak{A} \in \mathbf{K}$. Further, if Σ is a set of F-identities, then $\mathrm{Mod}(\Sigma)$ is the class of all top-down F-algebras satisfying every F-identity in Σ. We write $\Sigma \models \Delta$ to mean that $\mathrm{Mod}(\Sigma) \subseteq \mathrm{Mod}(\Delta)$.

With these concepts in mind one can easily reformulate Birkhoff's Theorem for top-down algebras. A class $\mathbf{K} \subseteq \mathbf{K}_F$ is a variety if and only if $\mathbf{K} = \mathrm{Mod}(\Sigma)$ for a set of F-identities Σ. Σ can be chosen $\mathrm{Id}(\mathbf{K})$. Consequently, $\mathscr{HSP}(\mathbf{K}) = \mathrm{Mod}(\mathrm{Id}(\mathbf{K}))$ for any class $\mathbf{K}$.

A crucial point in the universal algebraic proof of Birkhoff's Theorem is the existence of all free algebras in a variety. Free algebras exist in varieties of top-down algebras, too. If $\mathbf{V} \subseteq \mathbf{K}_F$ is a variety of top-down algebras then a free algebra $\mathfrak{A} = (A, F) \in \mathbf{V}$ with free generator $a \in A$ has the following property. An F-identity $xp = xq$ holds in $\mathbf{V}$ if and only if $ap = aq$. Similarly, if $\mathfrak{A} \in \mathbf{V}$ is freely generated by $a_1, a_2 \in A$, an F-identity $xp = yq$ belongs to $\mathrm{Id}(\mathbf{V})$ if and only if $a_1 p = a_2 q$.

3. Type-independent varieties

In this section we are going to develop a theory of type-independent varieties of top-down algebras similar to the theory of varieties exhibited in the previous one. To start with notice

Statement 1. For every class $\mathbf{K}$, $\mathscr{HSP}_g(\mathbf{K})$ is the least type-independent variety containing $\mathbf{K}$.

To show that type-independent equational classes also have equational characterizations we now introduce the notion of a p-identity. There are 3 types of p-identities, namely

(i) $(u, v) = (u, w)$,

(ii) $(u, z_1, v) = (u, z_2, w)$,

(iii) $v = w$

where u, v, w are possibly void words in $\{(n, i) | n \in R, i \in [n]\}^*$, and $z_1, z_2 \in \{(n, i) | n \in R, i \in [n]\}$. In more detail, say $u = (l_1, i_1) \dots (l_r, i_r)$, $v = (m_1, j_1) \dots (m_s, j_s)$, $w = (n_1, k_1) \dots$

$\dots(n_t, k_t)$, $z_1=(d,i)$ and $z_2=(d,j)$. It is required that $i\neq j$. Given a type F, each of these p-identities induces a set of F-identities. These are given by the formulae below:

(i') $xpq_1 = xpq_2$,

(ii') $xp(f, i)q_1 = xp(f, j)q_2$,

(iii') $xq_1 = yq_2$

where $p=(f_1, i_1)\dots(f_r, i_r)$, $q_1=(g_1, j_1)\dots(g_s, j_s)$, $q_2=(h_1, k_1)\dots(h_t, k_t)$, further, $f_1\in F_{l_1}, \dots, f_r\in F_{l_r}$, $g_1\in F_{m_1}, \dots, g_s\in F_{m_s}$, $h_1\in F_{n_1}, \dots, h_t\in F_{n_t}$, and finally, $f\in F_d$. A p-identity is said to be satisfied by a top-down F-algebra $\mathfrak{A}$ if all its induced F-identities are satisfied by $\mathfrak{A}$. Alternatively, this is expressed by saying the p-identity holds in $\mathfrak{A}$.

Let Ω be a set of p-identities. The set of F-identities induced by p-identities from Ω is denoted Ω_F. Ω^* denotes the class of all top-down algebras satisfying every member of Ω. Further, if $\mathbf{K}$ is an arbitrary class of top-down algebras, $\mathbf{K}^*$ is the set of all p-identities which hold in every $\mathfrak{A}\in\mathbf{K}$.

The following proposition easily comes from the definitions.

Statement 2. $(\mathscr{H}\mathscr{S}\mathscr{P}_g(\mathbf{K}))^*=\mathbf{K}^*$ holds for every class $\mathbf{K}$.

The clue in our treatment is

Lemma 1. If $\mathbf{K}$ is a type-independent variety then $\mathrm{Id}\,\mathbf{K}_F^*\models\mathrm{Id}\,(\mathbf{K}\cap\mathbf{K}_F)$.

Proof. Assume to the contrary there exists an F-identity in $\mathrm{Id}\,(\mathbf{K}\cap\mathbf{K}_F)$ which is not a consequence of $\mathbf{K}_F^*$. Among these there is one having minimum weight. The weight of on F-identity $xp=yq$ is defined $\mathrm{card}\,(\mathrm{pre}\,(xp)\cup\mathrm{pre}\,(xq))$. Similarly, the weight of $xp=yq$ is just $\mathrm{card}\,(\mathrm{pre}\,(xp)\cup\mathrm{pre}\,(yq))$. We shall restrict ourselves to the case this minimum weight F-identity is $xp=xq$. The other case can be handled likewise.

Take a free algebra $\mathfrak{A}=(A, F)$ in the variety $\mathbf{K}\cap\mathbf{K}_F$ with free generator a. We are going to show that whenever $xr, xs\in\mathrm{pre}\,(xp)\cup\mathrm{pre}\,(xq)$ and $ar=as$, then xr and xs coincide. Thus, suppose $xr, xs\in\mathrm{pre}\,(xp)\cup\mathrm{pre}\,(xq)$ and $ar=as$. We may choose xr and xs so that $|xr|\leq|xt|\leq|xs|$ provided that $xt\in\mathrm{pre}\,(xp)\cup\mathrm{pre}\,(xq)$ and $ar=at\ (=as)$. Let us substitute xr for xs if $xs\in\mathrm{pre}\,(xp)$, and apply the same substitution for xq. Denote the resulting polynomial symbols by $x\bar{p}$ and $x\bar{q}$, respectively. If xr is different from xs then both F-identities $x\bar{p}=x\bar{q}$ and $xr=xs$ have weight strictly less than that of $xp=xq$. On the other hand, $ar=as$ and $a\bar{p}=ap=$ $=aq=a\bar{q}$ yield $xr=xs$, $x\bar{p}=x\bar{q}\in\mathrm{Id}\,(\mathbf{K}\cap\mathbf{K}_F)$. By the choice of $xp=xq$ we have $\mathbf{K}_F^*\models\{xr=xs,\ x\bar{p}=x\bar{q}\}$, while $\{xr=xs,\ x\bar{p}=x\bar{q}\}\models\{xp=xq\}$ follows via the construction. Therefore, $\mathbf{K}_F^*\models\{xp=xq\}$. This contradiction arose from the assumption xr is different from xs, hence, xr and xs coincide.

Write xp and xq in more detail as

$$xp = x(f_1, i_1)\dots(f_r, i_r)(g_1, j_1)\dots(g_s, j_s),$$

$$xq = x(f_1, i_1)\dots(f_r, i_r)(h_1, k_1)\dots(h_t, k_t),$$

where $f_1\in F_{l_1}, \dots, f_r\in F_{l_r}$, $g_1\in F_{m_1}, \dots, g_s\in F_{m_s}$, $h_1\in F_{n_1}, \dots, h_t\in F_{n_t}$ and $(g_1, j_1)\neq$

$\neq (h_1, k_1)$ if $s, t>0$. First suppose that $g_1 \neq h_1$ if $s, t>0$. Since $\mathbf{K}_F^* \models \{xp=xq\}$, $xp=xq \notin \mathbf{K}_F^*$. Consequently, the p-identity

$$((l_1, i_1) \dots (l_r, i_r), (m_1, j_1) \dots (m_s\, j_s)) = ((l_1, i_1) \dots (l_r, i_r), (n_1, k_1) \dots (n_t, k_t))$$

is not in $\mathbf{K}^*$. This means that there exist a top-down F'-algebra $\mathfrak{B}=(B, F')\in\mathbf{K}$, operational symbols $f_1'\in F_{l_1}', \dots, f_r'\in F_{l_r}'$, $g_1'\in F_{m_1}', \dots, g_s'\in F_{m_s}'$, $h_1'\in F_{n_1}', \dots, h_t'\in F_{n_t}'$, and an element $b\in B$ with

$$bp' = b(f_1', i_1) \dots (f_r', i_r)(g_1', j_1) \dots (g_s', j_s) \neq$$
$$\neq b(f_1', i_1) \dots (f_r', i_r)(h_1', k_1) \dots (h_t', k_t) = bq'.$$

Now define an α_0-product $\mathfrak{L}=\Pi(\mathfrak{A}, \mathfrak{B}|\varphi, F)$ so that $\varphi_1\colon F\to F$ is the identity function and $\varphi_2\colon A\times F\to F'$ is any function with

$$(a(f_1, i_1) \dots (f_u, i_u), f_{u+1})\varphi_2 = f_{u+1}', \quad u = 0, \dots, r-1,$$
$$(a(f_1, i_1) \dots (f_r, i_r)(g_1, j_1) \dots (g_u, j_u), g_{u+1})\varphi_2 = g_{u+1}',$$
$$u = 0, \dots, s-1,$$
$$(a(f_1, i_1) \dots (f_r, i_r)(h_1, k_1) \dots (h_u, k_u), h_{u+1})\varphi_2 = h_{u+1}',$$
$$u = 0, \dots, t-1.$$

The first part of the proof garantees the existence of such an α_0-product. It is easy to check that

$$(a, b)ppr_2 = bp' \neq bq' = (a, b)qpr_2,$$

thus $xp=xq\notin \mathrm{Id}(\{\mathscr{L}\})$. Since $\mathscr{L}\in\mathbf{K}\cap\mathbf{K}_F$, this gives a contradiction.

The second case, i.e. when $s, t>0$ and $g_1=h_1$ yields a similar contradiction just take the p-identity

$$((l_1, i_1) \dots (l_r, i_r), (m_1, j_1), (m_2, j_2) \dots (m_s, j_s)) =$$
$$= ((l_1, i_1) \dots (l_r, j_r), (n_1, k_1), (n_2, k_2) \dots (n_t, k_t)).$$

Remark. Since only α_0-products were used in the previous proof, the statement of Lemma 1 holds even if closure under $\mathscr{P}_{\alpha_0}$ is supposed instead of closure under $\mathscr{P}_g$. Furthermore, closure under $\mathscr{H}$ and $\mathscr{S}$ is not strictly required.

Lemma 2. $\mathrm{Id}(\mathscr{P}_{\alpha_0}(\mathbf{K})\cap\mathbf{K}_F)=\mathrm{Id}(\mathscr{P}_{f\alpha_0}(\mathbf{K})\cap\mathbf{K}_F)$ holds for every class $\mathbf{K}$ and type F.

Proof. This statement has been proved for ordinary algebras in [1]. The same idea applies here. However, it should be noted that [4] also contains the proof.

We are ready to prove the main result:

Theorem. For any class $\mathbf{K}$ of top-down algebras, $\mathbf{K}^{**}=\mathscr{H}\mathscr{S}\mathscr{P}_g(\mathbf{K})=\mathscr{H}\mathscr{S}\mathscr{P}_{\alpha_0}(\mathbf{K})=$ $=\mathscr{H}\mathscr{S}\mathscr{P}\mathscr{P}_{f\alpha_0}(\mathbf{K})$.

Proof. $\mathbf{K}^{**}\supseteq\mathscr{H}\mathscr{S}\mathscr{P}_g(\mathbf{K})$ is valid by Statement 2. Inclusions $\mathscr{H}\mathscr{S}\mathscr{P}_g(\mathbf{K})\supseteq$ $\supseteq\mathscr{H}\mathscr{S}\mathscr{P}_{\alpha_0}(\mathbf{K})\supseteq\mathscr{H}\mathscr{S}\mathscr{P}\mathscr{P}_{f\alpha_0}(\mathbf{K})$ are trivial. $\mathscr{H}\mathscr{S}\mathscr{P}_{\alpha_0}(\mathbf{K})\supseteq\mathbf{K}^{**}$ follows by Lemma 1 and the Remark. Finally, $\mathscr{H}\mathscr{S}\mathscr{P}\mathscr{P}_{f\alpha_0}(\mathbf{K})\supseteq\mathscr{H}\mathscr{S}\mathscr{P}_{\alpha_0}(\mathbf{K})$ is valid by Lemma 2.

Corollary. The following three statements are equivalent for every class **K**:

(i) **K** is a type independent equational class,
(ii) $\mathbf{K}=\Omega^*$ for a set Ω of p-identities,
(iii) $\mathbf{K}=\mathbf{K}^{**}$.

Note. Equations $\mathcal{HSP}_{\theta}(\mathbf{K})=\mathcal{HSP}_{\alpha_0}(\mathbf{K})=\mathcal{HSPP}_{f\alpha_0}(\mathbf{K})$ have already been established in [4].

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