On products of automata with identity

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In spite of the fascinating Krohn—Rhodes theory the homomorphically complete classes of automata have not yet been satisfactorily characterized for the α_0-product. Recently there has been keen activity in finding nice homomorphically complete classes. Continuing the work which was begun by N. V. Evtusenko [6], P. Dömösi gave a very interesting homomorphically complete class for the α_0-product consisting of automata having 3 input signs (cf. [3]). His idea was to use not only permutation automata for the homomorphic realization of permutation automata. He applied a technique combining shiftregisters with permutation automata, and in a sense his use of shiftregisters originates in [4]. It was apparent for us that Dömösi did not completely exploit the advantages of this method. The present paper is a collection of a few remarks immediately obtainable just by simple generalization.

The basic idea behind the use of shiftregisters is this. Let a part of the product automaton work in an absolutely free way by sections, and if enough information has been accumulated try to have this information govern the next move simulating the behaviour of the automaton to be realized homomorphically. Not surprisingly this has something to do with generalized products, i.e. products allowing an input sign to be coded with an input word of arbitrary length. Namely, this shiftregister technique can be used for converting generalized products to ordinary products. Unfortunately this conversion can not always be carried out. But the presence of input signs inducing the identity mapping on the state set does make the conversion possible under wide circumstances.

1. Preliminaries

We shall be using standard automata theoretic notions. An automaton is meant a system $\mathbf{A}=(A, X, \delta)$, where A and X are finite nonvoid sets, the state set and the input alphabet, and the transition function δ maps $A\times X$ into A. Denoting by X^* the free semigroup with identity λ generated by X, the transition function extends to a map $A\times X^*\to A$ as usual. Given a word $p\in X^*$, the length of p is denoted $|p|$. Every word $p\in X^*$ induces a translation $t_p^{\mathbf{A}}: A\to A$ of the state set: $t_p^{\mathbf{A}}(a)=\delta(a, p)$ for all $a\in A$. If no confusion may arise, we write t_p instead of $t_p^{\mathbf{A}}$. All translations $t_p, p\in X^*$, form a semigroup with respect to function composition. This semigroup $S(\mathbf{A})$ is called the characteristic semigroup of $\mathbf{A}$.

For every automaton $\mathbf{A}=(A, X, \delta)$, we define the automata $\mathbf{A}^\lambda$ and $\mathbf{A}^*$ as follows: $\mathbf{A}^\lambda=(A, \{t_\lambda^{\mathbf{A}}, t_x^{\mathbf{A}}|x\in X\}, \delta^\lambda)$, $\mathbf{A}^*=(A, S(\mathbf{A}), \delta^*)$, where $\delta^\lambda(a, t_\lambda^{\mathbf{A}})=t_\lambda^{\mathbf{A}}(a)=a$, $\delta^\lambda(a, t_x^{\mathbf{A}})=t_x^{\mathbf{A}}(a)$, and $\delta^*(a, t_p^{\mathbf{A}})=t_p^{\mathbf{A}}(a)$ for any $a\in A$, $x\in X$, $p\in X^*$. Notice that $S(\mathbf{A})=S(\mathbf{A}^\lambda)=S(\mathbf{A}^*)$. For a class $\mathcal{K}$ of automata put

$$\mathcal{K}^\lambda = \{\mathbf{A}^\lambda|\mathbf{A}\in\mathcal{K}\},$$

$$\mathcal{K}^* = \{\mathbf{A}^*|\mathbf{A}\in\mathcal{K}\}.$$

Let $\mathbf{A}=(A, X, \delta)$ and $\mathbf{B}=(B, Y, \delta')$ be two automata. $\mathbf{A}$ is called an X subautomaton of $\mathbf{B}$, if $A\subseteq B$, $X\subseteq Y$, and δ is the restriction of δ' to $A\times X$. If $X=Y$, we speak about a subautomaton. Take two mappings $h_1: A\to B$ and $h_2: X\to Y$. This pair of functions is said to be an X-homomorphism $\mathbf{A}\to\mathbf{B}$ if $h_1(\delta(a, x))=\delta'(h_1(a), h_2(x))$ for every $a\in A$, $x\in X$. If in addition both h_1 and h_2 are bijective, we call the pair (h_1, h_2) an X-isomorphism, and $\mathbf{A}$ X-isomorphic to $\mathbf{B}$. Letting $X=Y$ and h_2 the identity map $X\to Y$, h_1 becomes a homomorphism $\mathbf{A}\to\mathbf{B}$. $\mathbf{B}=(B, X, \delta')$ is a homomorphic image of $\mathbf{A}$ if there is a surjective homomorphism $\mathbf{A}\to\mathbf{B}$. Bijective homomorphisms are called isomorphisms.

Take a class $\mathcal{K}$ of automata. Then $\mathbf{S}(\mathcal{K})$, $\mathbf{H}(\mathcal{K})$ and $\mathbf{I}(\mathcal{K})$ will respectively denote the classes of all subautomata, homomorphic images and isomorphic images of automata from $\mathcal{K}$.

Now we recall the concept of general products of automata. Let $\mathbf{A}_j=(A_j, X_j, \delta_j)$, $j\in[n]=\{1, ..., n\}$, $n\geqq 0$ be arbitrary automata and take a system of so called feedback functions $\varphi_j: A_1\times...\times A_n\times X\to X_j$, $j\in[n]$, where X is any alphabet. The automaton $\mathbf{A}=(A_1\times...\times A_n, X, \delta)$ will be called the general product (g-product, for short) of automata $\mathbf{A}_j$ with respect to φ and X, provided that

$$\delta((a_1, ..., a_n), x) = (\delta_1(a_1, x_1), ..., \delta_n(a_n, x_n)),$$

$$x_j = \varphi_j(a_1, ..., a_n, x)$$

for every $a_1\in A_1, ..., a_n\in A_n, x\in X$ and $j\in[n]$. We use the notation $\mathbf{A}_1\times...\times\mathbf{A}_n(\varphi, X)$ for general products. If all the $\mathbf{A}_j$'s coincide, we speak about a power.

Take the general product above, and let $i\geqq 0$ be an arbitrary integer. If none of the feedback functions φ_j depends on the state variables a_k having indices $k>j+i-1$, the g-product is called an α_i-product. In case of an α_i-product we shall indicate only those variables of a feedback function on which it may depend.

We shall make use of an interesting generalization of g-products. Take the automata $\mathbf{A}_j$ as in the definition of a g-product but now let $\varphi_j: A_1\times...\times A_n\times X\to X_j^*$, $j\in[n]$. The g^*-product $\mathbf{A}_1\times...\times\mathbf{A}_n(X, \varphi)$ is defined on exact analogy of the g-product with the exception that

$$\delta(a_1, ..., a_n, x) = (\delta_1(a_1, p_1), ..., \delta_n(a_n, p_n)),$$

where $p_j=\varphi_j(a_1, ..., a_n, x)$, $j\in[n]$. Allowing only words of length not exceeding 1 in the ranges of the feedback functions, we get the notion of a g^λ-product, or general λ-product. Note that g-products are special g^λ-products, and g^λ-products are special cases of the g^*-product. The concept of an α_i^*-product or that of an α_i^λ-product is derived in the same way as α_i-products were obtained.

Take a class $\mathcal{K}$ of automata. We put

$\mathbf{P}_g(\mathcal{K})$: all g-products of automata from $\mathcal{K}$,

$\mathbf{P}_{\alpha_i}(\mathscr{K})$: all α_i-products of automata from $\mathscr{K}$,
$\mathbf{P}^*_g(\mathscr{K})$: all g^*-products of automata from $\mathscr{K}$,
$\mathbf{P}^*_{\alpha_i}(\mathscr{K})$: all α^*_i-products of automata from $\mathscr{K}$,
$\mathbf{P}^\lambda_g(\mathscr{K})$: all g^λ-products of automata from $\mathscr{K}$,
$\mathbf{P}^\lambda_{\alpha_i}(\mathscr{K})$: all α^i-products of automata from $\mathscr{K}$.

Observe that the following are identities:

$$P^*_g(\mathscr{K}) = P_g(\mathscr{K}^*), P^*_{\alpha_i}(K) = P_{\alpha_i}(\mathscr{K}^*),$$

$$P^\lambda_g(\mathscr{K}) = P_g(\mathscr{K}^\lambda), P^\lambda_{\alpha_i}(K) = P_{\alpha_i}(\mathscr{K}^\lambda).$$

Our principal interest will be in operators **HSP** where **P** is any of the product operators above. We shall give a sufficient condition for having $\mathbf{HSP}^*_{\alpha_0}(\mathscr{K}) = \mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$, as well as a necessary and sufficient condition assuring $\mathbf{HSP}^*_{\alpha_1}(\mathscr{K}) = \mathbf{HSP}^\lambda_{\alpha_1}(\mathscr{K})$. As regards α_i-products with $i \geqq 2$, we show that $\mathbf{HSP}^\lambda_{\alpha_i}(\mathscr{K}) = \mathbf{HSP}^*_g(\mathscr{K})$ is identically valid. These are the main results. In addition, we shall discuss homomorphically complete classes. Recall that a class $\mathscr{K}$ is homomorphically complete for the g-product if $\mathbf{HSP}_g(\mathscr{K})$ is the class of all automata. Isomorphic completeness and homomorphic completeness with respect to other types of the product are similarly defined. We end the paper by presenting a class of automata which is homomorphically complete for the α_0-product and contains automata having only 2 input signs.

The concept of g-products was introduced by V. M. Gluskov in [10]. The hierarchy of α_i-products is due to F. Gécseg [8]. The α_0-product was called loop-free product or R-product earlier. Or even, the formation of α_0-products is equivalent to the iterated quasi-superposition. Generalized products appear in F. Gécseg [7]. Some elementary properties of the products will be used in the sequel without any reference.

We are indebted to Prof. F. Gécseg for inspiring conversations. His new book [9] is an excellent summary of recent results on products of automata.

2. Homomorphic realization

The reason for introducing the α_i-products was to decrease the complexity of the general product. On the other hand, it made possible the investigation of deeper structural properties of automata and, at the same time, gave a framework for achieving deep results. The cruical example is the Krohn—Rhodes theory. F. Gécseg observed how to translate this theory into the scope of α^*_0-products. His achievements will be summarized in Theorem 1. In this theorem, as well as throughout the paper, $\mathbf{A}_0$ denotes the two-state reset automaton $([2], \{x, y\}, \delta_0)$, $\delta_0(1, x) = \delta_0(2, x) = 1$, $\delta_0(1, y) = \delta_0(2, y) = 2$. The automaton $\mathbf{A}^\lambda_0$ can be identified with $([2], \{x_0, x, y\}, \delta'_0)$, where δ'_0 coincides with δ_0 on $[2] \times \{x, y\}$, and x_0 induces the identity.

Theorem 1. A class $\mathscr{K}$ of automata is homomorphically complete for the α^*_0-product if and only if the following are valid:

(i) There is an automaton in $\mathscr{K}$ whose characteristic semigroup contains a subsemigroup isomorphic to $S(\mathbf{A}_0)$.

(ii) For every finite simple group G, there exists an automaton $\mathbf{A}\in\mathcal{K}$ such that G is a homomorphic image of a subgroup of $S(\mathbf{A})$.

Consequently, there exists no minimal homomorphically complete class of automata for the α_0^*-product.

Combining the proof with the Krohn—Rhodes theory one gets:

Corollary 1. Let $\mathcal{K}$ be a class satisfying (i) above, and take an automaton $\mathbf{A}$. Then $\mathbf{A}\in\mathbf{HSP}^*_{\alpha_0}(\mathcal{K})$ if and only if whenever a simple group G is a homomorphic image of a subgroup of $S(\mathbf{A})$, there is an automaton $\mathbf{B}\in\mathcal{K}$, for which a subgroup of $S(\mathbf{B})$ can be mapped homomorphically onto G. A part of this holds for any class $\mathcal{K}$. Namely, whenever a simple group G is a homomorphic image of a subgroup of $S(\mathbf{A})$ and $\mathbf{A}\in\mathbf{HSP}^*_{\alpha_0}(\mathcal{K})$, then a subgroup of $S(\mathbf{B})$ can be mapped homomorphically onto G for an automaton $B\in\mathcal{K}$.

We think the above theorem clearly justifies the importance of generalized products. Our present purpose is to show that generalized products can be replaced by λ-products in most cases as far as homomorphic realization is concerned with. Theorem 1 will be our starting point for α_0^λ-products, and we shall make an attempt to combine it with a technique used by P. Dömösi in [3].

First of all we need a few concepts. Automata $\mathbf{C}_n=(\{a_1, \dots, a_n\}, \{x\}, \delta)$ satisfying $\delta(a_i, x)=a_{i+1}$ $(i=1, \dots, n-1)$, $\delta(a_n, x)=a_1$ will be called counters. Counters of one state are said to be trivial. An automaton $\mathbf{A}=(A, X, \delta)$ is called counter-free if and only if, whenever a counter $\mathbf{C}$ is an X-subautomaton of $\mathbf{A}$, it follows that $\mathbf{C}$ is trivial. In other words this means that $\delta(a_1, x)=a_2, \dots, \delta(a_{n-1}, x)=a_n$, $\delta(a_n, x)=a_1$ implies $n=1$ for all $x\in X$ and different states $a_1, \dots, a_n\in\mathbf{A}$. A class $\mathcal{K}$ of automata is counter-free if every $\mathbf{A}\in\mathcal{K}$ is counter-free.

Besides counters we shall be using shiftregisters. Let X be an alphabet. A shift-register over X of length $n\geqq 1$ is an automaton (X^n, X, δ) with transitions $\delta(x_1\dots x_n, x)=x_2\dots x_n x$, $x_1\dots x_n\in X^n$, $x\in X$.

Let X and Y be arbitrary alphabets and take a mapping $\tau: X^n\to Y^n$, $n\geqq 1$. Following the ideas of P. Dömösi we put $R_\tau=\{(p, q)\in X^*\times Y^* \mid 1\leqq |p|, |q|\leqq n, |p|+|q|=n+1\}$ and define the automaton $\mathbf{R}_\tau=(R_\tau, X, \delta_\tau)$ as follows:

$$\delta_\tau((p, yq), x) = \begin{cases} (px, q) & \text{if } |p|\neq n, \\ (x, \tau(p)) & \text{if } |p| = n, \end{cases}$$

where $x\in X$, $(p, yq)\in R_\tau$ with $y\in Y$.

Lemma 1. Let $\mathbf{C}_n$ be an n-state counter. Then $\mathbf{R}_\tau\in\mathbf{HSP}_{\alpha_0}(\{\mathbf{C}_n, \mathbf{A}_0\})$.

Proof. The proof is a slight modification of Dömösi's construction.

Let $\mathbf{A}_1=\mathbf{C}_n=([n], \{x_0\}, \delta_1)$ be an n-state counter, $\mathbf{A}_2=(X^n, X, \delta_2)$ a shiftregister, and set $\mathbf{A}_3=(Y^n, \bar{Y}^n\cup Y, \delta_3)$, where $\bar{Y}=\{\bar{y}\mid y\in Y\}$ and

$$\delta_3(y_1\dots y_n, y) = y_2\dots y_n y,$$

$$\delta_3(y_1\dots y_n, \bar{z}_1\dots\bar{z}_n) = z_1\dots z_n,$$

all $y_1 \dots y_n \in Y^n$, $\bar{z}_1 \dots \bar{z}_n \in \bar{Y}^n$, $y \in Y$. Form the α_0-product $\mathbf{A} = \mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3(\varphi, X)$ with

$$\varphi_1(x) = x_0,$$

$$\varphi_2(i, x) = x,$$

$$\varphi_3(i, x_1 \dots x_n, x) = \begin{cases} \overline{\tau(x_1 \dots x_n)} & \text{if } i = n \text{ [1]} \\ \text{arbitrary } y \in Y & \text{if } i \neq n, \end{cases}$$

$x \in X$, $i \in [n]$, $x_1 \dots x_n \in X^n$.

It is easy to check that the assignment $(i, x_1 \dots x_n, y_1 \dots y_n) \to (x_{n-i+1} \dots x_n, y_1 \dots y_{n-i+1})$ gives a homomorphism $\mathbf{A} \to \mathbf{R}_\tau$. On the other hand, both $\mathbf{A}_2$ and $\mathbf{A}_3$ are definite automata of degree n. Recall that an automaton (B, Z, δ) is called definite of degree n, if and only if $\delta(b, w) = \delta(c, w)$ holds for every $b, c \in B$ and $w \in Z^n$. Thus, $\mathbf{A}_2, \mathbf{A}_3 \in \mathbf{ISP}_{\alpha_0}(\{\mathbf{A}_0\})$ by a result of *B*. Imreh (cf. [11]). (Note that also the Krohn—Rhodes theorem helps in establishing $\mathbf{A}_2, \mathbf{A}_3 \in \mathbf{HSP}_{\alpha_0}(\{\mathbf{A}_0^\lambda\})$ what would be enough for our purposes in this section.) Since $\mathbf{A}_2, \mathbf{A}_3 \in \mathbf{ISP}_{\alpha_0}(\{\mathbf{A}_0\})$ and $\mathbf{R}_\tau \in \mathbf{HSP}_{\alpha_0}(\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_2\})$, it follows that $\mathbf{R}_\tau \in \mathbf{HSP}_{\alpha_0}(\{\mathbf{C}_n, \mathbf{A}_0\})$.

Lemma 2. If $\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$ contains a nontrivial counter then $\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$ contains an infinite number of counters of different lengths.

Proof. This statement was proved in [3].

The following theorem will bear fundamental importance in our discussions.

Theorem 2. Suppose that $\mathscr{K}$ is not counter-free and $\mathbf{A}_0 \in \mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$. Then $\mathscr{K}^* \subseteq \mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$.

Proof. Take an automaton $\mathbf{A} = (A, X, \delta) \in \mathscr{K}$. Then $\mathbf{A}^\lambda \in \mathbf{P}^\lambda_{\alpha_0}(\mathscr{K})$, whence we may assume that there is a sign $x_0 \in X$ inducing the identity mapping $A \to A$. We are going to show that $\mathbf{A}^* = (A, S(\mathbf{A}), \delta^*) \in \mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$. Let $S(\mathbf{A}) = \{t^{\mathbf{A}}_{p_1}, \dots, t^{\mathbf{A}}_{p_k}\} = Y$, where $p_1, \dots p_k$ are words in X^*. Since x_0 induces the identity mapping $A \to A$, the words p_i can be picked out so that $|p_1| = \dots = |p_k| = n$. Or even, the previous lemma makes possible to choose n in such a manner that an n-state counter is in $\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$. Obviously, there exists a mapping $\tau \colon Y^n \to X^n$ satisfying the equation $t^{\mathbf{A}^*}_w = t^{\mathbf{A}}_{\tau(w)}$ for every $w \in Y^n$. We form an α_0-product of R_τ and $\mathbf{A}$ and show that $\mathbf{A}^*$ is a homomorphic image of this product. Since $\mathbf{R}_\tau \in \mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$, this yields $\mathbf{A}^* \in \mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$.

Take the α_0-product $\mathbf{R}_\tau \times \mathbf{A}(\varphi, Y)$ with $\varphi_1(y) = y$ and $\varphi_2((p, xq), y) = x$, and define the mapping $h \colon R_\tau \times A \to A$ by $h((p, q), a) = \delta^*(\delta(a, q), p)$. Then h is a homomorphism of the product onto $\mathbf{A}^*$, ending the proof of Theorem 2.

Theorem 3. Suppose that a class $\mathscr{K}$ of automata is not counter-free and the reset automaton $\mathbf{A}_0$ is in $\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$. Then $\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K}) = \mathbf{HSP}^*_{\alpha_0}(\mathscr{K})$. Further, an automaton $\mathbf{A}$ is in $\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$ if and only if, whenever a simple group G is a homomorphic image of a subgroup of $S(\mathbf{A})$, then G is a homomorphic image of a subgroup of $S(\mathbf{B})$ for an automaton $\mathbf{B} \in \mathscr{K}$.

Proof. The inclusion $\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K}) \subseteq \mathbf{HSP}^*_{\alpha_0}(\mathscr{K})$ is obviously valid. Con-

[1] For a word $y_1 \dots y_n \in Y^n$, $\overline{y_1 \dots y_n} = \bar{y}_1 \dots \bar{y}_n$.

versely, $\mathbf{HSP}^*_{\alpha_0}(\mathscr{K})=\mathbf{HSP}_{\alpha_0}(\mathscr{K}^*)\subseteq\mathbf{HSP}_{\alpha_0}\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})=\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$ follows by Theorem 2. The second statement is a consequence of the first one and of Corollary 1.

Corollary 2. A class $\mathscr{K}$ of automata is homomorphically complete for the α_0^λ-product if and only if the following conditions hold:

(i) $\mathscr{K}$ is not counter-free,
(ii) $\mathbf{A}_0\in\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K})$,
(iii) for every finite simple group G, there exists an automaton $\mathbf{A}\in\mathscr{K}$ such that G is a homomorphic image of a subgroup of $S(\mathbf{A})$.

Proof. The sufficiency follows by Theorem 3. The necessity of condition (ii) is trivial, while the necessity of (iii) comes from Theorem 3. P. Dömösi proved in [2] that no counter-free class can be homomorphically complete for the α_0-product. The reason is that only the trivial counters are in $\mathbf{HSP}_{\alpha_0}(\mathscr{K})$ if $\mathscr{K}$ is counter-free.

Example 1. For every $n\geqq 1$, let $\mathbf{A}_n$ be an automaton whose characteristic semigroup is isomorphic to the symmetric group S_n of all permutations $[n]\to[n]$. The class consisting of $\mathbf{A}_0$ and these automata $\mathbf{A}_n$ $(n\geqq 1)$ is homomorphically complete for the α_0^λ-product. Consequently, $\mathscr{K}^\lambda$ is homomorphically complete for the α_0-product. Since S_n can be generated by 2 permutations, there exists a homomorphically complete class of automata for the α_0^λ-product which contains automata having 2 input signs. On the other hand no class $\mathscr{K}$ consisting of automata having a single input sign can be homomorphically complete for the α_0^λ-product since every automaton in $\mathscr{K}$ would be commutative. Consequently, $S(\mathbf{A})$ would be commutative for each $\mathbf{A}\in\mathscr{K}$, henceforth neither condition (ii) nor (iii) of Corollary 2 could be satisfied by $\mathscr{K}$. Or even, every homomorphically complete class for the α_0^λ-product must contain an infinite number of automata having at least 2 input signs.

Corollary 3. There exists no minimal homomorphically complete class of automata for the α_0^λ-product.

Proof. Suppose that $\mathscr{K}$ is homomorphically complete for the α_0^λ-product. Then $\mathscr{K}$ contains an automaton $\mathbf{B}_0$ which is not counter-free, and there are $\mathbf{B}_1, \dots$ $\dots, \mathbf{B}_n\in\mathscr{K}$ such that $\mathbf{A}_0\in\mathbf{HSP}^\lambda_{\alpha_0}(\{\mathbf{B}_1, \dots, \mathbf{B}_n\})$. Since every simple group is isomorphic to a subgroup of a larger simple group, also $\mathscr{K}-\{\mathbf{B}\}$ is homomorphically complete for the α_0^λ-product for any $\mathbf{B}\in\mathscr{K}-\{\mathbf{B}_0, \dots, \mathbf{B}_n\}$.

Corollary 4. There exists a class of automata which is homomorphically complete for the α_0^λ-product but not homomorphically complete for the α_0-product. Similarly, there is a homomorphically complete class for the α_0^*-product which is not homomorphically complete for the α_0^λ-product.

Proof. By a result of P. Dömösi, there exists a minimal homomorphically complete class of automata for the α_0-product (cf. [1]). Thus, the first statement follows by comparing this result with the previous corollary. To prove the second statement, we give a class $\mathscr{K}$ homomorphically complete for the α_0^*-product but not homomorphically complete for the α_0^λ-product.

For every integer $n\geqq 2$, let $\mathbf{A}_n=([2n]\cup\{2'\}, \{x_1, x_2, x_3, x_4\}, \delta_n)$ be the automaton with transitions $\delta_n(i, x_1)=i+1$ if i is odd, $\delta_n(i, x_2)=i+1 \bmod 2n$ if i is even, $\delta_n(1, x_3)=2$, $\delta_n(2, x_4)=3$, $\delta_n(3, x_3)=2'$, $\delta_n(2', x_4)=1$, and finally, $\delta_n(i, x)=i$,

$\delta_n(2', x)=2'$ in all remaining cases. Put $\mathscr{K}=\{A_0\}\cup\{A_n|n\geqq 1\}$. To show that $\mathscr{K}$ is homomorphically complete for the α_0^*-product observe that all automata $\mathbf{B}_n=([n], \{x_1, x_2, x_3\}, \delta_n')$ $(n\geqq 1)$ are in $\mathbf{ISP}^*_{1\alpha_0}(\mathscr{K})$ [2] where δ_n' is defined so that x_1 induces the cyclic permutation $(12...n)$, y the transposition (12), while x_3 induces the identity permutation (1). Thus, $\mathbf{HSP}^*_{\alpha_0}(\mathscr{K})=\mathbf{HSP}^*_{\alpha_0}(\{A_0, B_1, B_2, ...\})$ is the class of all automata. On the other hand $\mathscr{K}$ is counter-free, hence $\mathscr{K}$ is not homomorphically complete for the α_0^λ-product.

Before turning to α_1^λ-products we need a few definitions.

A cycle in an automaton (A, X, δ) is a sequence of pairwise distinct states $a_1, ..., a_n$ so that $\delta(a_i, x_i)=a_{i+1}$ $(i=1, ..., n-1)$ and $\delta(a_n, x_n)=a_1$ for some $x_1, ..., x_n\in X$. The integer n is called the length of the cycle. Cycles of length 1 are called trivial, and an automaton is said to be monotone if and only if it contains only trivial cycles. An automaton (A, X, δ) will be called discrete if $\delta(a, x)=a$ for every $a\in A$, $x\in X$. Finally, one-state automata will be referred to as trivial automata.

In the sequel we shall need

Lemma 3. Suppose that an automaton $\mathbf{A}=(A, X, \delta)$ contains a cycle of length at least 2. Then $\mathbf{A}_0\in\mathbf{HSP}^\lambda_{\alpha_1}(\{\mathbf{A}\})$.

Proof. Let us assume that $\mathbf{A}$ contains the nontrivial cycle $a_1, ..., a_n$ so that $\delta(a_i, x_i)=a_{i+1}$ $(i=1, ..., n-1)$ and $\delta(a_n, x_n)=a_1$ for some $x_1, ..., x_n\in X$.

Construct the α_1^λ-product $\mathbf{B}=\mathbf{A}^{n+2}(\varphi, \{x, y\})$, where

$$\varphi_i(c_1, ..., c_i, x)=\begin{cases} x_j & \text{if} \quad c_i=a_j\neq a_1,\\ x_1 & \text{if} \quad c_i=a_1 \quad \text{and} \quad c_m\neq a_1 \quad \text{when} \quad 1\leqq m<i,\\ \lambda & \text{in all other cases,}\end{cases}$$

$$\varphi_i(c_1, ..., c_i, y)=\begin{cases} x_j & \text{if} \quad c_i=a_j\neq a_1,\\ x_1 & \text{if} \quad c_i=a_1 \quad \text{and} \quad c_m=c_l=a_1 \quad \text{for some} \quad 1\leqq m<l<i,\\ \lambda & \text{in all other cases.}\end{cases}$$

Taking the subset
$C=\{(c_1, ..., c_{n+2})|\{a_2, ..., a_n\}\subset\{c_1, ..., c_{n+2}\}$ and a_1 is contained exactly 3 times in the system $\{c_1, ..., c_{n+2}\}\}$, the automaton $\mathbf{C}=(C, \{x, y\}, \delta_B)$ is a subautomaton of $\mathbf{B}$. Lastly, it can easily be verified that the reset automaton $\mathbf{A}_0$ is a homomorphic image of $\mathbf{C}$ under the mapping $h: C\to[2]$ defined by

$$h(c_1, ..., c_{n+2})=$$

$$=\begin{cases} 1 & \text{if } a_2 \text{ preceeds at least two occurrences of } a_1 \text{ in } (c_1, ..., c_{n+2}),\\ 2 & \text{in all other cases.}\end{cases}$$

Theorem 4. Suppose that $\mathscr{K}$ contains an automaton which is not monotone, and let $\mathbf{A}$ be an arbitrary automaton. Then $\mathbf{A}\in\mathbf{HSP}^\lambda_{\alpha_1}(\mathscr{K})$ $(\mathbf{A}\in\mathbf{HSP}^*_{\alpha_1}(\mathscr{K}))$ if and only if, whenever a simple group G is a homomorphic image of a subgroup of $S(\mathbf{A})$, there exists an automaton $\mathbf{B}\in\mathbf{P}^\lambda_{1\alpha_1}(\mathscr{K})$ $(\mathbf{B}\in\mathbf{P}^*_{1\alpha_1}(\mathscr{K}))$ such that a subgroup of $S(\mathbf{B})$

[2] $P^*_{1\alpha_i}(\mathscr{K})$ denotes the class of all single factor α_i^*-products of automata from $\mathscr{K}$. The operators $P^\lambda_{1\alpha_i}$ and $P_{1\alpha_i}$ are defined similarly.

can be mapped homomorphically onto G. Otherwise, i.e. if $\mathcal{K}$ consists of monotone automata, equation $\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathcal{K})=\mathbf{HSP}^{*}_{\alpha_1}(\mathcal{K})$ is universally valid, and 3 cases arise.

(i) If there is a nondiscrete automaton in $\mathcal{K}$, then $\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathcal{K})$ is the class of all monotone automata.

(ii) If every automaton from $\mathcal{K}$ is discrete but $\mathcal{K}$ contains a nontrivial automaton, $\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathcal{K})$ is the class of all discrete automata.

(iii) Finally, if $\mathcal{K}$ contains only trivial automata, then $\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathcal{K})$ is the class of all trivial automata.

Proof. Assume that $\mathcal{K}$ contains a nonmonotone automaton. Then $\mathbf{P}^{\lambda}_{1\alpha_1}(\mathcal{K})$ is not counter-free and $\mathbf{A}_0\in\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathcal{K})$. Since $\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathcal{K})=\mathbf{HSP}_{\alpha_0}\mathbf{P}^{\lambda}_{1\alpha_1}(\mathcal{K})=$ $=\mathbf{HSP}^{\lambda}_{\alpha_0}\mathbf{P}^{\lambda}_{1\alpha_1}(\mathcal{K})$, the first statement of Theorem 4 follows by Theorem 3 for α_0^{λ}-products. As regards α_1^{*}-products, the proof is similar just use equation $\mathbf{HSP}^{*}_{\alpha_1}(\mathcal{K})=$ $=\mathbf{HSP}^{\lambda}_{\alpha_0}\mathbf{P}^{*}_{1\alpha_1}(\mathcal{K})$.

Now suppose that $\mathcal{K}$ contains only monotone automata. Then the same holds for $\mathcal{K}^*$, and by $\mathbf{HSP}^{*}_{g}(\mathcal{K})=\mathbf{HSP}_{g}(\mathcal{K}^*)$, even for $\mathbf{HSP}^{*}_{g}(\mathcal{K})$.

If there is a nondiscrete automaton in $\mathcal{K}$, then the elevator $\mathbf{E}=([2], \{x, y\}, \delta)$ having transitions $\delta(1,x)=1$, $\delta(1,y)=\delta(2,x)=\delta(2,y)=2$ is in $\mathbf{IP}^{\lambda}_{1\alpha_0}(\mathcal{K})$. By a result in [7], every monotone automaton is already in $\mathbf{ISP}_{\alpha_0}(\{\mathbf{E}\})$. Hence we have $\mathbf{HSP}^{*}_{g}(\mathcal{K})=\mathbf{HSP}^{*}_{\alpha_1}(\mathcal{K})=\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathcal{K})=\mathbf{ISP}_{\alpha_0}\mathbf{P}^{\lambda}_{1\alpha_0}(\mathcal{K})=\mathbf{ISP}^{\lambda}_{\alpha_0}(\mathcal{K})$ is the class of all monotone automata.

The proof in the remaining two cases is obvious. We have $\mathbf{HSP}^{*}_{g}(\mathcal{K})=\mathbf{ISP}_{\alpha_0}(\mathcal{K})$.

Corollary 5. There exists an algorithm to decide for a finite class $\mathcal{K}$ and an automaton $\mathbf{A}$ whether $\mathbf{A}\in\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathcal{K})$ $(\mathbf{A}\in\mathbf{HSP}^{*}_{\alpha_1}(\mathcal{K}))$.

Corollary 6. Since $\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathcal{K})\subseteq\mathbf{HSP}^{*}_{\alpha_1}(\mathcal{K})$ always holds, $\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathcal{K})=$ $=\mathbf{HSP}^{*}_{\alpha_1}(\mathcal{K})$ if and only if one of the following 2 conditions is valid.

(i) $\mathcal{K}$ consists of monotone automata.

(ii) There is a nonmonotone automaton in $\mathcal{K}$, and whenever a simple group G is a homomorphic image of a subgroup of $S(\mathbf{A})$ for an automaton $\mathbf{A}\in\mathbf{P}^{*}_{1\alpha_1}(\mathcal{K})$, there is an automaton $\mathbf{B}\in\mathbf{P}^{\lambda}_{1\alpha_1}(\mathcal{K})$ such that a subgroup of $S(\mathbf{B})$ can be mapped homomorphically onto G.

Corollary 7. A class $\mathcal{K}$ of automata is homomorphically complete for the α_1^{λ}-product (α_1^{*}-product) if and only if, for every simple group G, there exists an automaton $\mathbf{A}\in\mathbf{P}^{\lambda}_{1\alpha_1}(\mathcal{K})$ $(\mathbf{A}\in\mathbf{P}^{*}_{1\alpha_1}(\mathcal{K}))$ so that a subgroup of $S(\mathbf{A})$ can be mapped homomorphically onto G.

Corollary 8. There exists no minimal homomorphically complete class for the α_1^{λ}-product (α_1^{*}-product).

Now we present a new proof for a part of a nice result of F. Gécseg [7].

Theorem 5. The following 3 statements are equivalent for every class $\mathcal{K}$ of automata.

(i) $\mathcal{K}$ is homomorphically complete for the α_1^{*}-product.

(ii) For every integer $n\geqq 1$, there exists an automaton $\mathbf{A}_n=(A, X, \delta)\in\mathcal{K}$

having at least n different states $a_1, \ldots, a_n \in A$ such that for every $i, j \in [n]$, there is a word $p \in X^*$ satisfying $\delta(a_i, p) = a_j$.

(iii) $\mathcal{K}$ is isomorphically complete for the α_1^*-product.

Proof. We prove that (i) implies (ii). Suppose that $\mathcal{K}$ is homomorphically complete for the α_1^*-product. It is enough to prove (ii) for n prime. Take the cyclic group Z_n. Since Z_n is simple, there are an automaton $\mathbf{A}_n' = (A, X', \delta') \in \mathbf{P}_{1\alpha_1}^*(\mathcal{K})$ and a subgroup H of $S(\mathbf{A}_n')$ such that Z_n is a homomorphic image of H. Note that H is isomorphic to a permutation group of a subset $A' \subseteq A$. Since Z_n has an element of order n, there must be a translation $t_p \in H$ of order kn for an integer $k \geqq 1$. Henceforth, there are different states $a_1, \ldots, a_{ln} \in A'$ $(l \geqq 1)$ for which $\delta(a_i, p) = a_{i+1 \bmod ln}$ $(i \in [kn])$. Taking $a_1, \ldots, a_n$ we see that $\mathbf{A}_n'$ satisfies condition (ii). Let $\mathbf{A}_n = (A, X, \delta) \in \mathcal{K}$ be an automaton such that $\mathbf{A}_n' \in \mathbf{P}_{1\alpha_1}^*(\mathbf{A}_n)$. Clearly, also $\mathbf{A}_n$ satisfies (ii) with $a_1, \ldots, a_n \in A$.

For the sake of completeness we recall from [7] that every n-state automaton is already in $\mathbf{ISP}_{1\alpha_1}^*(\{\mathbf{A}_n\})$, while (iii)$\Rightarrow$(i) is trivial.

Suppose we are given $n \geqq 1$ boxes $B_1, \ldots, B_n$ and $k \leqq n$ pebbles numbered from 1 to k. In addition, k boxes, say $B_{i_1}, \ldots, B_{i_k}$, are distinguished so that $i_1 < \ldots < i_k$. Initially B_{i_j} contains the pebble numbered j, $j = 1, \ldots, k$. The game goes on as follows. At each step we take out the pebbles from the boxes and put all pebbles which were in B_i back into B_i or put all of them into box B_{i+1}. The pebbles from B_n go into B_n or B_1. After a number of steps the pebbles get back into the distinguished boxes, each distinguished box B_{i_t} containing a pebble numbered j_t, $t \in [k]$. Clearly, $(j_1 \ldots j_k)$ is a power of the cyclic permutation $(1 \ldots k)$. This proves our

Observation. Let $\mathbf{C}_n$ be a counter, $\mathbf{A} \in \mathbf{P}_{1\alpha_1}^{\lambda}(\mathbf{C}_n)$. Then every subgroup of $S(\mathbf{A})$ is isomorphic to a subgroup of a cyclic group Z_k with $k \leqq n$, whence cyclic.

Corollary 9. There exists a class $\mathcal{K}$ which is homomorphically complete for the α_i^*-product but not homomorphically complete for the α_1^{λ}-product.

Proof. Take a class $\mathcal{K}$ consisting of a counter $\mathbf{C}_n$ for each $n \geqq 1$. $\mathcal{K}$ is homomorphically complete for the α_1^*-product by Theorem 5. Since every subgroup of $S(\mathbf{C}_n)$ $(n \geqq 1)$ is cyclic, but there are noncyclic finite simple groups, $\mathcal{K}$ is not homomorphically complete for the α_1^{λ}-product.

We do not know whether there exists a class $\mathcal{K}$ which is homomorphically complete for the α_1^{λ}-product but not homomorphically complete for the α_1-product[3]. It is clear that there exists a class $\mathcal{K}$ such that $\mathbf{HSP}_{\alpha_1}(\mathcal{K})$ is a proper subclass of $\mathbf{HSP}_{\alpha_1}^{\lambda}(\mathcal{K})$, take e.g. $\mathcal{K} = \{([2], \{x\}, \delta)\}$, $\delta(1, x) = \delta(2, x) = 2$.

Now we turn our attention to the α_2^{λ}-product and the g^{λ}-product.

Theorem 6. $\mathbf{HSP}_{\alpha_2}^{\lambda}(\mathcal{K}) = \mathbf{HSP}_g^*(\mathcal{K})$ for every class $\mathcal{K}$. Furthermore, four cases arise. If $\mathcal{K}$ contains a nonmonotone automaton, then $\mathbf{HSP}_{\alpha_2}^{\lambda}(\mathcal{K})$ is the class of all automata. If $\mathcal{K}$ consists of monotone automata one of which is not discrete, then $\mathbf{HSP}_{\alpha_2}^{\lambda}(\mathcal{K})$ is the class of all monotone automata. If $\mathcal{K}$ consists of discrete automata and contains a nontrivial automaton, then $\mathbf{HSP}_{\alpha_2}^{\lambda}(\mathcal{K})$ is the class of all

[3] Recently Ésik has shown the existence of such a class.

discrete automata. Finally, if every automaton contained by $\mathscr{K}$ is trivial, then $\mathbf{HSP}^{\lambda}_{\alpha_2}(\mathscr{K})$ is the class of all trivial automata.

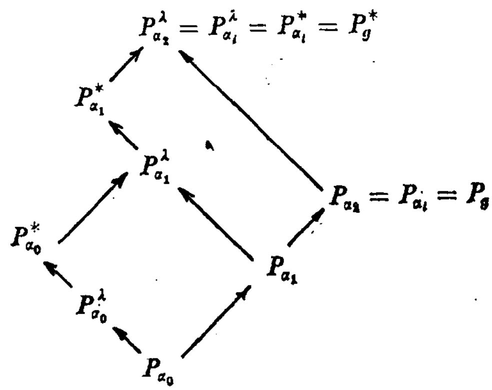
Proof. First we recall a result proved in [4]. If $\mathscr{K}$ is a class of automata such that there is an automaton $(A, X, \delta)\in\mathscr{K}$ having a state $a\in A$, signs $x, y\in X$ and words $p, q\in X^*$ with $\delta(a, x)\neq\delta(a, y)$ and $\delta(a, xp)=\delta(a, yq)=a$, then $\mathbf{HSP}_{\alpha_2}(\mathscr{K})$ is the class of all automata.

Now suppose that $\mathscr{K}$ contains an automaton which is not monotone. Then $\mathscr{K}^{\lambda}$ is homomorphically complete for the α_2-product. Hence, $\mathbf{HSP}^{\lambda}_{\alpha_2}(\mathscr{K})=\mathbf{HSP}_{\alpha_2}(\mathscr{K}^{\lambda})$ is the class of all automata, and since $\mathbf{HSP}^{\lambda}_{\alpha_2}(\mathscr{K})\subseteq\mathbf{HSP}^*_g(\mathscr{K})$, the same is true for $\mathbf{HSP}^*_g(\mathscr{K})$.

For the proof of the remaining cases see Theorem 4.

Corollary 10. There exists an algorithm to decide for a finite class $\mathscr{K}$ and an automaton $\mathbf{A}$ if $\mathbf{A}\in\mathbf{HSP}^{\lambda}_{\alpha_2}(\mathscr{K})$.

Now we come to the point of comparing the strengths of our various products with respect to homomorphic realization. The following figure gives a summary. The figure is to be interpreted as follows. If two operators, say $\mathbf{P}$ and $\mathbf{Q}$ label the same node, then this expresses that $\mathbf{P}$ and $\mathbf{Q}$ are homomorphically equivalent, i.e. $\mathbf{HSP}(\mathscr{K})=\mathbf{HSQ}(\mathscr{K})$ for every class $\mathscr{K}$. If there is a directed path from a node labeled $\mathbf{P}$ to a node labeled $\mathbf{Q}$ then $\mathbf{Q}$ is homomorphically more general than $\mathbf{P}$. This means that $\mathbf{HSP}(\mathscr{K})\subseteq\mathbf{HSQ}(\mathscr{K})$ for every $\mathscr{K}$, but there exists a class for which the inclusion is proper. Further on this situation will be denoted by $\mathbf{P}<\mathbf{Q}$. The index i denotes an arbitrary integer exceeding 2.



To justify the correctness of this figure first observe that all equivalences have been proved previously except for that the α_2-product is homomorphically equivalent to the general product. But this is the main result of [5]. On the other hand, all relations $\mathbf{P}<\mathbf{Q}$ appearing in the diagram have been established in this paper or in several papers earlier (cf. e.g. [7], [8], [9]), the only exception being $\mathbf{P}^*_{\alpha_0}<\mathbf{P}^{\lambda}_{\alpha_1}$.

To prove $\mathbf{HSP}^*_{\alpha_0}(\mathscr{K})\subseteq\mathbf{HSP}^{\lambda}_{\alpha_1}(\mathscr{K})$ for arbitrary $\mathscr{K}$, let us distinguish 3 cases.

Case 1. $\mathscr{K}$ contains a nonmonotone automaton. Then the inclusion follows by Corollary 1 and Theorem 4.

Case 2. $\mathscr{K}$ consists of monotone automata, one of which is not discrete. In this case we have $\mathbf{HSP}^*_{\alpha_0}(\mathscr{K})=\mathbf{HSP}^\lambda_{\alpha_1}(\mathscr{K})$ equals the class of all monotone automata. (Hint: an automaton in $\mathbf{IS}(\mathscr{K}^\lambda)$ is X-isomorphic to $\mathbf{E}$.)

Case 3. $\mathscr{K}$ consists of discrete automata. Now we have $\mathbf{ISP}_{\alpha_0}(\mathscr{K})=\mathbf{HSP}^*_g(\mathscr{K})$, thus, $\mathbf{HSP}^*_{\alpha_0}(\mathscr{K})=\mathbf{HSP}^\lambda_{\alpha_1}(\mathscr{K})$.

On the other hand, $\mathbf{HSP}^*_{\alpha_0}(\mathscr{K})$ is properly contained by $\mathbf{HSP}^\lambda_{\alpha_1}(\mathscr{K})$ e.g. for $\mathscr{K}=\{\mathbf{C}^\lambda_2\}$.

It should be noted that no more arrows could be added to the diagram.

3. A homomorphically complete class for the α_0-product

It was pointed out in Example 1 that there exists a class of automata having 2 input signs homomorphically complete for the α_0^λ-product. Our principal goal in this section is to show that this result can be strengthened. Such a class does exist for the α_0-product as well. This is interesting because we do not know any direct way for proving that the class of all automata with 2 input signs is homomorphically complete for the α_0-product.

Let $\mathbf{A}=(A, X, \delta)$ be an arbitrary automaton, and take a subsemigroup S of $S(\mathbf{A})$ containing an identity element. Put $\mathbf{A}^S=(A^S, S, \delta^S)$, where $A^S=\{b\in A|\ \exists a\in A, t\in S\ b=t(a)\}$ and $\delta^S(a, t)=t(a)$ for any $a\in A^S$, $t\in S$. Observe that letting $S=S(\mathbf{A})$ we get back the definition of $\mathbf{A}^*$.

The following generalization of Theorem 2 is straightforward.

Theorem 7. Let $\mathbf{A}=(A, X, \delta)$ be an automaton, S a subsemigroup of $S(\mathbf{A})$ containing identity element. Assume that there exists an integer $n\geqq 1$ satisfying $S\subseteqq\{t^{\mathbf{A}}_p|p\in X^n\}$. Then $\mathbf{A}^S\in\mathbf{HSP}_{\alpha_0}(\{\mathbf{C}_n, \mathbf{A}_0, \mathbf{A}\})$.

The characteristic semigroup of $\mathbf{A}^S$ is isomorphic to S. Let $\mathbf{B}=(B, Y, \delta)$ be an arbitrary automaton. We may construct the automaton $\mathbf{B}'=(S(B), Y, \delta')$ with transitions $\delta'(t^{\mathbf{B}}_p, y)=t^{\mathbf{B}}_{py}$, $p\in Y^*$, $y\in Y$. It is well-known that $\mathbf{B}'$ is isomorphic to a subautomaton of a direct power of $\mathbf{B}$. Henceforth $\mathbf{B}'\in\mathbf{HSP}_{\alpha_0}(\{\mathbf{B}\})$, and we have

Corollary 11. Under the assumptions of Theorem 7 it follows that $\mathbf{A}_S=(S, S, \delta_S)\in\mathbf{HSP}_{\alpha_0}(\{\mathbf{C}_n, \mathbf{A}_0, \mathbf{A}\})$ where $\delta_S(s_1, s_2)=s_1s_2$, $s_1, s_2\in S$.

Suppose now a class $\mathscr{K}$ of automata satisfies the following list of conditions.

(i) $\mathbf{A}_0\in\mathbf{HSP}_{\alpha_0}(\mathscr{K})$.

(ii) There exist an automaton $\mathbf{B}_0\in\mathscr{K}$, a subsemigroup $S_0\subseteqq S(\mathbf{B}_0)$ isomorphic to $S(\mathbf{A}^\lambda_0)$ so that for some n, an n-state counter $\mathbf{C}_n$ is in $\mathbf{HSP}_{\alpha_0}(\mathscr{K})$ and, at the same time, all elements of S_0 are induced by words of length n.

(iii) For every finite simple group G there exist an automaton $\mathbf{B}_G\in\mathscr{K}$, a subgroup $H_G\subseteqq S(\mathbf{B}_G)$, and an integer $n\geqq 1$ satisfying

(iii_1) H_G can be mapped homomorphically onto G,

(iii_2) $\mathbf{C}_n\in\mathbf{HSP}_{\alpha_0}(\mathscr{K})$,

(iii_3) every element of H_G is induced by a word of length n.

Set $\mathscr{K}'=\{\mathbf{A}_{S_0}, \mathbf{A}_{H_G} | G$ is a finite simple group$\}$. Since $\mathscr{K}'$ is obviously not counter-free, $\mathbf{A}_{S_0}$ is X-isomorphic to $\mathbf{A}_0^\lambda$, finally, the characteristic semigroup of $\mathbf{A}_{H_G}$ is isomorphic to H_G, Corollary 11 yields that $\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K}')$ is the class of all automata. Since every automaton belonging to $\mathscr{K}'$ has an input sign inducing the identity state-map, $\mathbf{HSP}^\lambda_{\alpha_0}(\mathscr{K}')=\mathbf{HSP}_{\alpha_0}(\mathscr{K}')$. However $\mathbf{HSP}_{\alpha_0}$ is a closure operator, thus $\mathscr{K}$ is homomorphically complete for the α_0-product. This is the basis of our last result.

Theorem 8. There exists a class of automata having 2 input signs which is homomorphically complete for the α_0-product.

Proof. Let $\mathbf{B}_0=([2], \{x, y\}, \delta_0)$ be the automaton with transitions $\delta_0(1, x)=2$, $\delta_0(1, y)=\delta_0(2, x)=\delta_0(2, y)=1$. Translations $t^{\mathbf{B}_0}_{xx}$, $t^{\mathbf{B}_0}_{xy}$, $t^{\mathbf{B}_0}_{yx}$ form a subsemigroup S_0 of $S(\mathbf{B}_0)$ isomorphic to $S(\mathbf{A}_0^\lambda)$ under the correspondence $t^{\mathbf{B}_0}_{x^2}\to t^{\mathbf{A}_0^\lambda}_{x_0}$, $t^{\mathbf{B}_0}_{xy}\to t^{\mathbf{A}_0^\lambda}_{x}$, $t^{\mathbf{B}_0}_{yx}\to t^{\mathbf{A}_0^\lambda}_{y}$. In addition, for every odd integer $n\geqq 3$, take the automaton $\mathbf{B}_n=([n], \{x, y\}, \delta_n)$ so that x induces the transposition (12) and y induces the cyclic permutation $(1 \dots n)$. Besides, since n is odd, there exists an odd integer m satisfying $S_n=\{t^{\mathbf{B}_n}_p | p\in\{x, y\}^m\}$. As a matter of fact, there is an m' such that every permutation of $[n]$ can be induced by a word of length at most m'. Put m the least odd integer not less than $m'+n$. Let $t=t^{\mathbf{B}}_{pn}$, $|p|=k\leqq m'$. If $m-k$ is even, put $q=py^{m-k}$. If $m-k$ is odd, take $q=px^ny^{m-(k+n)}$. We have $t=t^{\mathbf{B}_n}$ in both cases.

Now set

$$\mathscr{K}=\{\mathbf{A}_0, \mathbf{B}_0, \mathbf{B}_n | n\geqq 3 \text{ is odd}\}.$$

Since all counters of odd length as well as $\mathbf{C}_2$ are obviously in $\mathbf{HSP}_{\alpha_0}(\mathscr{K})$ and every finite simple group is isomorphic to a subgroup of S_n for odd n, $\mathscr{K}$ is homomorphically complete for the α_0-product. It should be noted that from the proof of Lemma 3 we have that $\mathbf{A}_0$ can be omitted from $\mathscr{K}$.

Corollary 12. The class $\mathscr{K}$ consisting of $\mathbf{A}_0^\lambda$ and automata $\mathbf{A}_n=([n], \{x, y\}, \delta_n)$ $(n\geqq 3)$ with $t_x=(1 \dots n)$, $t_y=(12)$ is homomorphically complete for the α_0-product. Recall that the main result of Dömösi's paper is the homomorphic completeness of $\mathscr{K}^\lambda$ for the α_0-product.

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