

Properties of the fuzzy connectives in the light of the general representations theorem

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Introduction

A number of papers deal with the operators of fuzzy logic. All of these publications assume associativity, isotonicity, continuity, and the validity of the permanence principle. The representation theorem of the class of such operators were examined only under assuming strict isotonicity in the literature [2], [6], [7], [8]. Here we shall consider properties of the operator class supposing isotonicity only, with conditions under which De Morgan identity holds; first for the case with Archimedean properties, and then for the general one. The properties of the limes operators of operator series will be studied, too. Finally, the distributive class is determined.

Preliminaries and basic results

Concerning the conjunction $c(x, y)$ and disjunction $d(x, y)$ featuring if fuzzy logic, we assume that the mappings $c: [0, 1] \times [0, 1] \rightarrow [0, 1]$ and $d: [0, 1] \times [0, 1] \rightarrow [0, 1]$ are:

(i) associative:

$$c(x, c(y, z)) = c(c(x, y), z), \quad (1)$$

$$d(x, d(y, z)) = d(d(x, y), z).$$

(ii) isotonic, i.e. if $x \leq x'$, then for all y :

$$c(x, y) \leq c(x', y), \quad d(x, y) \leq d(x', y), \quad (2)$$

$$c(y, x) \leq c(y, x'), \quad d(y, x) \leq d(y, x'). \quad (3)$$

(iii) the principle of permanence holds:

$$c(0, 0) = c(0, 1) = c(1, 0) = 0, \quad c(1, 1) = 1, \quad (4)$$

$$d(1, 1) = d(0, 1) = d(1, 0) = 1, \quad d(0, 0) = 0. \quad (5)$$

(iv) continuous.

Since the theorems related to the conjunctive operator can be proved in an analogous way as those related to the disjunction, the theorems below are proved only for conjunctive operators.

Theorem 1. For a conjunctive operator (disjunctive operator) satisfying conditions (i—iv) it holds that

$$c(x, 1) = c(1, x) = x, \quad d(x, 1) = d(1, x) = 1, \quad (6)$$

$$c(x, 0) = c(0, x) = 0, \quad d(x, 0) = d(0, x) = x. \quad (7)$$

Proof. [3].

Definition. The conjunctive operator $c(x, y)$ (disjunctive operator $d(x, y)$) is Archimedean if $c(x, y) < x$ and $d(x, y) > x$ for $x \in (0, 1)$.

Definition. The pseudo-inverse of a strictly monotonously decreasing function $f : [a, b] \rightarrow [f(a), f(b)]$ is

$$f^{(-1)}(x) = \begin{cases} b & \text{if } x \equiv f(b), \\ f^{-1}(x) & \text{if } f(b) \equiv x \equiv f(a), \\ a & \text{if } f(a) \equiv x. \end{cases} \quad (8)$$

Theorem 2. For an Archimedean conjunctive operator (disjunctive operator) with properties (i—iv) there always exists a strictly monotonous function f_c (f_d), for which

$$f_c(1) = 0, \quad f_d(0) = 0, \quad (9)$$

$$f_c(0) = r_c, \quad f_d(1) = r_d, \quad (10)$$

where r_c and r_d may be $+\infty$ or $-\infty$, and is such that

$$c(x, y) = f_c^{(-1)}(f_c(x) + f_c(y)), \quad d(x, y) = f_d^{(-1)}(f_d(x) + f_d(y))$$

where, apart from a factor $\alpha \neq 0$, $f_c(x)$ ($f_d(x)$) is uniquely defined.

The function $f_c(x)$ ($f_d(x)$) will be called "the additive generator of the function $c(x, y)$ ($d(x, y)$)".

Proof. [9].

Consequence. All such operators are commutative.

Definition. The operator $c(x, y)$ ($d(x, y)$) is reducible with respect to both sides if, in the case $(t, w) \in (0, 1]$ ($(t, w) \in [0, 1)$) $c(t, u) = c(t, v)$ or $c(u, w) = c(v, w)$ ($d(t, u) = d(t, v)$ or $d(u, w) = d(v, w)$) holds if and only if, $u = v$. If $c(x, y)$ ($d(x, y)$) is continuous, then the reducibility is equivalent to the strict isotonicity.

In the particular case when the operation $c(x, y)$ ($d(x, y)$) is strictly isotonic, we may obtain from Theorem 2 the theorem of Aczél [1], and then $r_c = -\infty$ or $r_c = \infty$.

A consideration of this special case is to be found in [3]. Let us subsequently assume that $r_c \neq \pm\infty$, $r_d \neq \pm\infty$.

Theorem 3. For the monotonously decreasing additive generator $f_c(x)$ ($f_d(x)$) of $c(x, y)$ ($d(x, y)$), there always exists a strictly monotonously increasing additive generator of $c(x, y)$.

Proof. Since the additive generator $f_c(x)$ is determined up to a constant multiple factor $\alpha \neq 0$, let us select $\alpha = -1$ and define

$$\tilde{f}_c(x) = -f_c(x);$$

this \tilde{f}_c will then satisfy the statement of the theorem.

We restrict our considerations below to monotonously decreasing generator when we are speaking on conjunctions, and to monotonously increasing additive one in the case of disjunctions, that is, we shall suppose that

$$0 < r_c < \infty \quad (0 < r_d < \infty)$$

Theorem 4. For the additive generator $f_c(x)$ ($f_d(x)$) of the operation $c(x, y)$ ($d(x, y)$) we can give a multiplicative generator $\tilde{f}_c(x)$ ($\tilde{f}_d(x)$), where $f_c, \tilde{f}_c: [0, 1] \rightarrow$

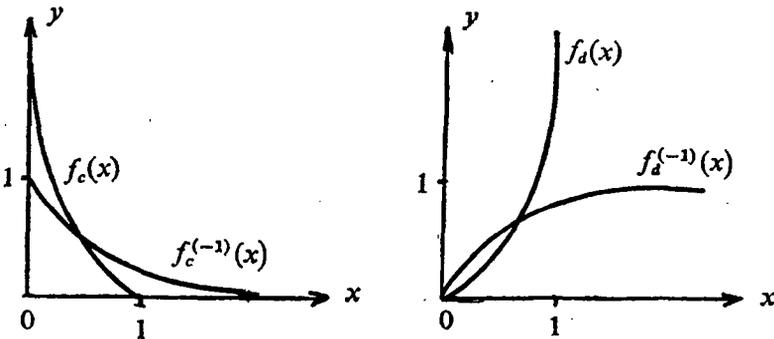


Fig. 1

Generator functions of conjunctive and disjunctive operators for the additive representation case

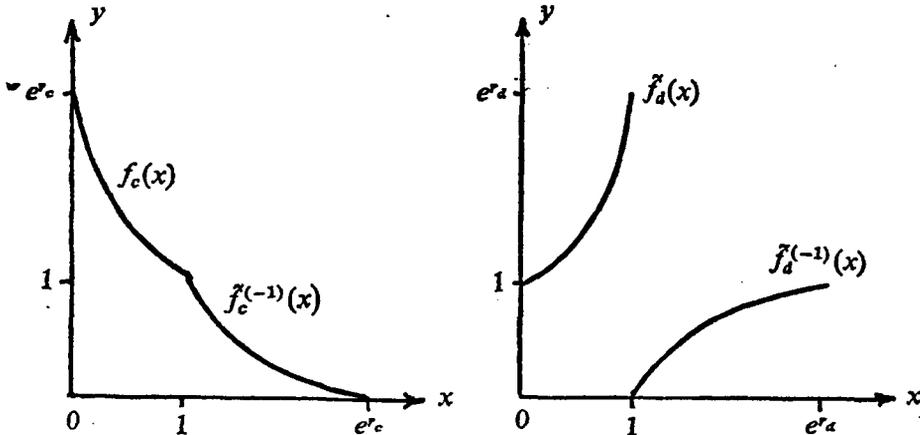


Fig. 2

Generator functions of conjunctive and disjunctive operators for the multiplicative representation case

$$\rightarrow [e^c, 1], \quad f_d(x): [0, 1] \rightarrow [1, e^a],$$

$$c(x, y) = f_c^{(-1)}(f_c(x) f_c(y)), \quad d(x, y) = f_d^{(-1)}(f_d(x) f_d(y)). \quad (11)$$

Proof. The proof is obvious if we put

$$f_c(x) = \exp(f_c(x)), \quad \text{so: } f_c^{(-1)}(x) = f_c^{(-1)}(\ln(x))$$

and

$$f_c^{(-1)}(f_c(x) f_c(y)) = f_c^{(-1)}(\ln(\exp(f_c(x))) \cdot \exp(f_c(y))) = f_c^{(-1)}(f_c(x) + f_c(y)).$$

3. The De Morgan class of fuzzy operators

For investigation of De Morgan identity we need a mapping, the so called negation $n: [0, 1] \rightarrow [0, 1]$. We shall assume the usual properties:

- (1) n decreases strictly monotonously,
- (2) n is continuous,
- (3) the principle of permanence holds for n

$$n(1) = 0 \quad \text{and} \quad n(0) = 1.$$

Below, we distinguish two forms of De Morgan identity, the conjunctive form

$$n(c(x, y)) = d(n(x), n(y)) \quad (12)$$

and the disjunctive form

$$n(d(x, y)) = c(n(x), n(y)). \quad (13)$$

The theorems will be stated and proved only for the conjunctive form.

Theorem 5. For a given conjunctive (disjunctive) operator $c(x, y)$ ($d(x, y)$) and a negation operator $n(x)$, it is possible to construct a disjunctive (conjunctive) operator satisfying the conjunctive (disjunctive) De Morgan identity.

Proof. Let $f_c(x)$ be the generator function of $c(x, y)$. Let us define

$$f_d(x) = \frac{f_c(n^{-1}(x))}{a} \quad (14)$$

where $a > 0$ is an optional real number. The function $d(x, y)$ generated by $f_d(x)$ will in fact be a disjunction, since $f_d(x)$ satisfies the conditions which ensure that $f_d(x)$ is a generator function of a disjunctive operator. Furthermore,

$$\begin{aligned} n(c(x, y)) &= n(f_c^{(-1)}(f_c(x) f_c(y))) = n\left(f_c^{(-1)} a \left(\frac{f_c(n^{-1}(n(x)))}{a} + \frac{f_c(n^{-1}(n(y)))}{a} \right)\right) = \\ &= n(f_c^{(-1)} a (f_d(n(x)) + f_d(n(y)))) = f_d^{(-1)}(f_d(n(x)) + f_d(n(y))) = d(n(x), n(y)) \end{aligned}$$

where we have made use of the fact that $n(f_c^{(-1)}(ax)) = f_d^{(-1)}(x)$, and thus the theorem is proved.

Theorem 6. Let $c(x, y)$ be a conjunction, and $d(x, y)$ a disjunction. A negation $n(x)$ can then always be given such that the conjunctive (disjunctive) De Morgan identity holds.

Proof. Let

$$n(x) = f_d^{(-1)} \left(\frac{f_d(1)}{f_c(0)} f_c(x) \right) = f_d^{(-1)} \left(\frac{r_d}{r_c} f_c(x) \right) = f_d^{(-1)}(af_c(x)).$$

As concerns the $n(x)$ defined in this way it can be seen that

- a) it is bimorphous, since $\frac{r_d}{r_c} f_c(x) \in [0, r_d]$,
- b) it decreases strictly monotonously, because of the monotonicity of $f_c(x)$ and $f_d^{(-1)}(x)$,
- c) $n(x)$ is continuous, since $f_c(x)$ and $f_d^{(-1)}(x)$ are such too,
- d) it satisfies the conjunctive De Morgan identity.

From the substitution $x = n^{-1}(y)$ we obtain

$$f_d(y) = \frac{f_c(n^{-1}(y))}{a}$$

from which the statement follows.

Theorem 7. Let $c(x, y)$ be a conjunctive, $d(x, y)$ a disjunctive operator and $n(x)$ a negation. The conjunctive De Morgan identity holds if and only if

$$f_d(x) = \frac{f_c(n^{-1}(x))}{a}.$$

Proof. The sufficiency has already been seen. From the conjunctive De Morgan identity we have

$$d(x, y) = n^{-1}(c(n^{-1}(x), n^{-1}(y))). \tag{15}$$

The conjunctive operator is associative

$$c(c(x, y), z) = c(x, c(y, z)),$$

and so

$$n(c(c(x, y), z)) = n(c(x, c(y, z))).$$

Using the conjunctive De Morgan identity

$$d(n(c(x, y)), n(z)) = d(n(x), n(c(y, z))).$$

Using again the conjunctive De Morgan identity

$$d(d(n(x), n(y)), n(z)) = d(n(x), d(n(y), n(z))).$$

Replacing $n(x), n(y), n(z)$ by x, y, z , we find that $d(x, y)$ is associative.

$d(x, y)$ is isotonic. If $y \leq y'$, then $n(y) \geq n(y')$, $c(x, y)$ is isotonic, hence

$$c(n(x), n(y)) \geq c(n(x), n(y')).$$

As $n^{-1}(x)$ is strictly monotonous

$$d(x, y) = n^{-1}(c(n(x), n(y))) \leq n^{-1}(c(n(x), n(y'))) = d(x, y').$$

The continuity of $d(x, y)$ is a consequence of those of $c(x, y)$ and $n(x)$.
 $d(x, y)$ is Archimedean since

$$c(n(x), n(x)) < n(x) \quad x \in (0, 1),$$

and so

$$d(x, x) = n^{-1}(c(n(x), n(x))), \quad n^{-1}(n(x)) = x.$$

$d(x, y)$ satisfies the permanence principle:

$$c(x, 1) = c(1, x) = x,$$

$$c(x, 0) = c(0, x) = 0,$$

and so

$$d(x, 0) = n^{-1}(c(n(x), n(0))) = n^{-1}(n(x)) = x,$$

$$d(x, 1) = n^{-1}(c(n(x), n(1))) = n^{-1}(0) = 1.$$

Thus, for $d(x, y)$ there exists a generator function $f_d(x)$ which is determined uniquely, apart from a constant factor. We have seen that (14) is a generator function, and since its constant factor too is of such a form, we have proved the theorem.

If $n(n(x))=x$ holds, then the conjunctive and disjunctive De Morgan identities are the same.

Let us now consider the non-Archimedean case. On the basis of the representative theorem of [9], there exists a finite or (uncountably) infinite series of discrete intervals $M_i=(a_i, b_i)$ such that

$$c(x, y) = \begin{cases} f_{c,i}^{(-1)}(f_{c,i}(x) + f_{c,i}(y)) & x, y \in (a_i, b_i), \\ \min(x, y) & x, y \in [0, 1] \setminus \bigcup_i M_i. \end{cases} \quad (16)$$

The De Morgan class is constructed so that the conditions required in the Archimedean case hold for the generator functions in the corresponding (a_i, b_i) .

4. Limes operators of the operator class

An important role is played below by the following definition [6].

Definition: Let

$$t_c(x, y) = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{otherwise,} \end{cases} \quad t_d(x, y) = \begin{cases} x & \text{if } y = 0, \\ y & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \quad (17)$$

be a non-continuous conjunctive or disjunctive operator.

Theorem 8. All conjunctive (disjunctive) operators satisfying conditions (i–iv) have the following properties:

$$t_c(x, y) \equiv c(x, y) \equiv \min(x, y); \quad \max(x, y) \equiv d(x, y) \equiv t_d(x, y). \quad (18)$$

Proof. By definition we have $t_c(x, y)=0$, $c(x, y) \geq 0$ provided that $x, y \in [0, 1]$ and $t_c(1, x)=t_c(x, 1)=c(1, x)=c(x, 1)=x$, thus

$$t_c(x, y) \equiv c(x, y).$$

Moreover, let us assume that $x \leq y$. Utilizing the fact that $f_c(x)$ and $f_c^{(-1)}(x)$ decrease strictly monotonously, we have

$$c(x, y) = f_c^{(-1)}(f_c(x) + f_c(y)) \leq f_c^{(-1)}(f_c(x)) = x = \min(x, y).$$

Theorem 9. Let $\lambda > 0$ be a real. If $f_c(x)$ ($f_d(x)$) is the generator function of a conjunctive (disjunctive) operator $c(x, y)$ ($d(x, y)$), then

$$f_c^\lambda(x) \quad (f_d^\lambda(x)) \tag{19}$$

are also the generator functions of some conjunction and disjunction denoted by $c_\lambda(x, y)$ ($d_\lambda(x, y)$).

Proof. If $f_c(x)$ is the generator function of the conjunctive operator, then $f_c(x)$ too satisfies the properties of the generator function. Thus

$$c_\lambda(x, y) = f_c^{(-1)}((f_c^\lambda(x) + f_c^\lambda(y))^{1/\lambda});$$

$$d_\lambda(x, y) = f_d^{(-1)}((f_d^\lambda(x) + f_d^\lambda(y))^{1/\lambda}).$$

Theorem 10. Let $\lambda, c_\lambda(x, y), d_\lambda(x, y)$ be the same as above. Then the following relations are true.

$$1. \quad \lim_{\lambda \rightarrow \infty} c_\lambda(x, y) = \min(x, y), \quad \lim_{\lambda \rightarrow \infty} d_\lambda(x, y) = \max(x, y), \tag{20}$$

$$2. \quad \lim_{\lambda \rightarrow 0} c_\lambda(x, y) = t_c(x, y), \quad \lim_{\lambda \rightarrow 0} d_\lambda(x, y) = t_d(x, y), \tag{21}$$

Proof. 1. Let us assume that $x \leq y$; then $\min(x, y) = x$. Since it holds that $c_\lambda(x, y) \leq \min(x, y)$, we have

$$\begin{aligned} \min(x, y) &\leq \lim_{\lambda \rightarrow \infty} c_\lambda(x, y) \leq \lim_{\lambda \rightarrow \infty} f_c^{(-1)}((2f_c^\lambda(x))^{1/\lambda}) = \lim_{\lambda \rightarrow \infty} f_c^{(-1)}(2^{1/\lambda} f_c(x)) = \\ &= x = \min(x, y). \end{aligned}$$

2. If $y = 1$, then $c_\lambda(x, 1) = x$. Similarly, if $x = 1$, then $c(1, y) = y$.

Let us assume that $x < 1, y < 1$ and $x \leq y$. Since

$$0 \leq \lim_{\lambda \rightarrow 0} c_\lambda(x, y) = \lim_{\lambda \rightarrow 0} f_c^{(-1)}((f_c^\lambda(x) + f_c^\lambda(y))^{1/\lambda}) = \lim_{\lambda \rightarrow 0} f_c^{(-1)}(2^{1/\lambda} f_c(y)) = 0,$$

we have

$$\begin{aligned} c_0(x, y) &= t_c(x, y), & d_0(x, y) &= t_d(x, y), \\ c_\infty(x, y) &= \min(x, y), & d_\infty(x, y) &= \max(x, y). \end{aligned}$$

Example. Let $f_c(x) = 1 - x$. We then obtain the operator of Yager [10] by constructing $c(x, y)$.

Definition. $c(x, y)$ ($d(x, y)$) has the classification property, if $c(x, n(x)) = 0$ ($d(x, n(n)) = 1$) for every negation $n(x)$.

Theorem 11. Let $c(x, y)$ ($d(x, y)$) be any mapping such that it is isotonic, and satisfies $c(1, x) = c(x, 1) = x$ ($d(0, x) = d(x, 0) = x$).

(i) $c(x, y)$ ($d(x, y)$) is idempotent if and only if

$$c(x, y) = \min(x, y) \quad (d(x, y) = \max(x, y)),$$

(ii) $c(x, y) (d(x, y))$ has the classification property if and only if

$$c(x, y) = t_c(x, y), \quad d(x, y) = t_d(x, y).$$

Proof. The sufficiency is obvious.

(i) Let us assume that $c(x, y)$ is idempotent and $x \leq y$. Then

$$x = c(x, x) \leq c(x, y) \leq c(x, 1) = x \quad \text{i.e.}$$

$$c(x, y) = x = \min(x, y).$$

(ii) If $x=1$ or $y=1$, then the statement is obvious. If $x_0 < 1$, $y_0 < 1$, then there is a negation $n(x)$ for which $x_0 = n(y_0)$. Since $c(x, y)$ has the classification property we have $c(x_0, y_0) = c(x_0, n(x_0)) = t_c(x_0, y_0)$.

Hence, we have characterized the operators $c_0(x, y)$, $c_\infty(x, y)$ algebraically, too.

5. The distributive operator class

We wish to describe operators satisfying properties (i)—(iv) which are distributive with respect to each other, i.e.

$$c(x, d(y, z)) = d(c(x, y), c(x, z)), \quad (22)$$

$$d(x, c(y, z)) = c(d(x, y), d(x, z)). \quad (23)$$

Theorem 12. The operators $c(x, y)$ and $d(x, y)$ are distributive with respect to each other if and only if:

$$c(x, y) = \min(x, y), \quad d(x, y) = \max(x, y). \quad (24)$$

Proof. On the basis of theorem 1, boundary conditions (6) and (7) hold. Utilizing the distributivity:

$$x = d(x, 0) = d(x, c(0, 0)) = c(d(x, 0), d(x, 0)) = c(x, x)$$

we have that $c(x, y)$ is idempotent. On the basis of theorem 11, for an idempotent, isotonic conjunctive (disjunctive) operator satisfying the boundary conditions

$$c(x, y) = \min(x, y).$$

6. Discussion

The properties of monotonous fuzzy operators (interpreted on sets) satisfying associativity have been examined.

- A₁ Strictly monotonous operators,
- A₂ Archimedean operators,
- A₃ Monotonous operators

A necessary and sufficient condition on generator functions was formulated which ensures that De Morgan identity is true. The distributivity holds if and only

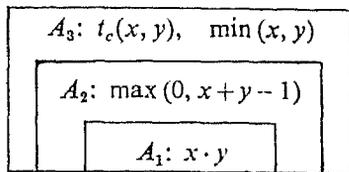


Fig. 3
Hierarchic order of operators

if the operator is the min or max. By means of operator series associated to the operators we obtained lower and upper limits of the operators, as limes. The algebraic characterization of these operators was also given.

A further examination is necessary to establish the classification property of the operators and the relation of the (not strictly monotonous) negations.

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(Received Nov. 17, 1984)