A probability model for priority processor-shared multiprogrammed computer systems

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1. Introduction

Queueing models have been widely used in the analysis of time-shared computer systems. In these systems an arriving job competes for the attention of the single CPU. It is forced to wait in a system of queues until it is permitted a quantum Q of service time. When this quantum expires, it is then required to join the system of queues to await its required service time. By allowing Q to shrink to zero, processor-sharing (PS) models are obtained, which provides a share of the CPU to many jobs simultaneously and equally.

Following Kleinrock [6] in a priority round-robin system the jobs are divided into n separate priority groups. A program belonging to the i-th priority group gets r_iQ unit of service each time, where quantities r_i are called service weights, $r_i>0$, $i=1, \ldots, n$. In the limit as $Q\to 0$ this model reduces to a processor-shared one with priority structure wherein a job from priority group i receives a fraction f_i of the total capacity, where $f_i=r_i/\Sigma r_j n_j$, here n_j is the number of jobs from group j in the system at time t. This kind of service discipline is referred to as PPS one. The PS model is a particular case $r_i=1$ for all i, $i=1, \ldots, n$.

We observe that the two processor-shared models are ideal in the sence that the swap-time is assumed to be zero.

The present paper deals with a multiprogrammed computer system in which a number of n jobs are permitted to circulate among the peripheral devices and the CPU. The system is assumed to have enough peripheral devices, so no queueing for I/0 operations occurs. Under PPS service discipline the jobs are supposed to be stochastically different, the i-th program is characterized by exponentially distributed I/0 time with parameter λ_i, exponentially distributed processing time with rate μ_i and service weight r_i, $i=1, \ldots, n$. All random variables are assumed to be independent of each other.

The purpose of the paper is to generalize the PS model treated by Asztalos [1], Csige—Tomkó [4], Cohen [3]. In steady state the main operational characteristics, such as CPU utilization, expected CPU busy period length, mean response times, waiting ratio, throughput, average number of jobs staying at the CPU are given. Furthermore, a system of linear equations for L—S transform of response time for program i and CPU busy period length is obtained, respectively, which can be solved by the algorithm offered. For the moments of the random variables mentioned before another system of linear equations is derived which can be solved iterative.

Finally, numerical examples illustrate the problem in question and performance measures under different service disciplines, such as preemptive resume priority (PR), PS are compared with the PPS one.

For further probabilistic models for multiprogrammed computer systems the interested reader is referred to among others: Avi—Itzhak and Heyman [2], Cohen [3], Kleinrock [5], Lehoczky and Gaver [7], Sztrik [8].

The theoretical basis of the paper can be found in Tomkó [9].

2. Mathematical description of the model

Let the random variable $v(t)$ denote the number of jobs at the CPU at time t, and let $(\alpha_1(t), \dots, \alpha_{v(t)}(t))$ indicate their indexes ordered lexicographically. Introduce the process

$$\underline{x}(t) = (v(t); \alpha_1(t), \dots, \alpha_{v(t)}(t)).$$

Since all distributions are exponential the process $(\underline{x}(t), t \geqq 0)$ is a stochastically continuous, finite state, continuous time Markov chain with state space $\bigcup_{k=1}^{n} C_k^n + \{0\}$, where C_k^n denotes the set of all combinations of order k of the integers $1, \dots, n$ in lexicograpic order and $\{0\}$ indicates the state that the CPU is idle.

Let us introduce some notations

$$\Lambda_{i_1, \dots, i_k} = \sum_{j \neq i_1, \dots, i_k} \lambda_j, \quad \Lambda = \sum_{j=1}^{n} \lambda_j,$$

$$R_{i_1, \dots, i_k} = \sum_{j=1}^{k} r_{i_j}, \quad \sigma_{i_1, \dots, i_k} = \frac{1}{R_{i_1, \dots, i_k}} \sum_{j=1}^{k} r_{i_j} \mu_{i_j} + \Lambda_{i_1, \dots, i_k}.$$

For the distribution of $\underline{x}(t)$ consider the functions given below

$$P_0(t) = P(v(t) = 0),$$

$$P_{i_1, \dots, i_k}(t) = P(v(t) = k;\ \alpha_1(t) = i_1, \dots, \alpha_k(t) = i_k), \tag{1}$$

$$(1 \leqq k \leqq n, (i_1, \dots, i_k) \in C_k^n).$$

It is easy to see that functions (1) satisfy the Kolmogorov-differential equations

$$P_0'(t) = -\Lambda P_0(t) + \sum_{j=1}^{n} \mu_j P_j(t),$$

$$\vdots$$

$$P_{i_1, \dots, i_k}'(t) = \sum_{l=1}^{k} \lambda_{i_l} P_{i_1, \dots, i_{l-1}, i_{l+1}, \dots, i_k}(t) - \sigma_{i_1, \dots, i_k} P_{i_1, \dots, i_k}(t) + \tag{2}$$

$$+ \sum_{j \neq i_1, \dots i_k} \frac{\mu_j r_j}{R_{i_1, \dots, i_k, j}} P_{i_1', \dots, i_{k+1}'}(t),$$

$$\vdots$$

$$P_{1,2, \dots, n}'(t) = \sum_{l=1}^{n} \lambda_l P_{1, \dots, l-1, l+1, \dots, n}(t) - \sigma_{1, \dots, n} P_{1, \dots, n}(t),$$

where $i'_1, \ldots, i'_{k+1}$ denotes the lexicographic order of integers $i_1, \ldots, i_k, j$. Then we have:

Theorem 1. If $\lambda_i, \mu_i > 0$, $i=1, \ldots, n$ then the Markov chain $(\underline{x}(t), t \geqq 0)$ possesses unique stationary distribution

$$P_0 = \lim_{t \to \infty} P_0(t),$$

$$P_{i_1, \ldots, i_k} = \lim_{t \to \infty} P_{i_1, \ldots, i_k}(t),$$

$$(i_1, \ldots, i_k) \in C_k^n, \quad k = 1, \ldots, n,$$

which is the solution to the following system of linear equations

$$\Lambda P_0 = \sum_{j=1}^{n} \mu_j P_j,$$

$$\vdots$$

$$\sigma_{i_1, \ldots, i_k} P_{i_1, \ldots, i_k} = \sum_{l=1}^{k} \lambda_{i_l} P_{i_1, \ldots, i_{l-1}, i_{l+1}, \ldots, i_k} + \sum_{j \neq i_1, \ldots, i_k} \frac{\mu_j r_j}{R_{i_1, \ldots, i_k, j}} P_{i'_1, \ldots, i'_{k+1}}; \tag{3}$$

$$\vdots$$

$$\sigma_{1, \ldots, n} P_{1, \ldots, n} = \sum_{l=1}^{n} \lambda_l P_{1, \ldots, l-1, l+1, \ldots, n},$$

satisfying the norming condition

$$P_0 + \sum_{k=1}^{n} \sum_{C_k^n} P_{i_1, \ldots, i_k} = 1. \tag{4}$$

Proof. Since $(\underline{x}(t), t \geqq 0)$ is a continuous time finite state Markov chain with positive intensities, it is irreducible and all states are ergodic. In this case the stationary distribution exists and can be obtained as the solution to the Kolmogorov equations satisfying (4). As $t \to \infty$ from (2) we get (3).

If all $r_i = 1$, $i=1, \ldots, n$, the solution of (3) is

$$P_{i_1, \ldots, i_k} = P_0 K! \prod_{j=1}^{k} \lambda_{ij}/\mu_{ij},$$

(cf. Csige—Tomko [4], Asztalos [1]).

In the following we give an algorithm for calculating the stationary distribution $(P_0, P_{i_1}, \ldots, i_k, (i_1, \ldots, i_k) \in C_k^n, k=1, \ldots, n)$.

Let $\underline{Y}_k$ be the vector

$$\begin{pmatrix} P_{1, \ldots, k} \\ \vdots \\ P_{i_1, \ldots, i_k} \\ \vdots \\ P_{n-k+1, \ldots, n} \end{pmatrix}$$

of dimension $\binom{n}{k}$. The components $P_{i_1}, \ldots, i_k$ are listed in the lexicographic order of their indexes, $k=1, \ldots, n$. Notice that eq. (3) can be written in the following neat form

$$\underline{Y}_0 = \mathbf{B}_0\underline{Y}_1,$$

$$\underline{Y}_1 = \mathbf{A}_1\underline{Y}_0+\mathbf{B}_1\underline{Y}_2,$$

$$\vdots$$

$$\underline{Y}_k = \mathbf{A}_k\underline{Y}_{k-1}+\mathbf{B}_k\underline{Y}_{k+1},$$

$$\vdots$$

$$\underline{Y}_n = \mathbf{A}_n\underline{Y}_{n-1},$$

where matrix $\mathbf{A}_k$ is of order $\binom{n}{k}\times\binom{n}{k-1}$ $k=1, \ldots, n$, $\mathbf{B}_k$ is of order $\binom{n}{k}\times\binom{n}{k+1}$ $k=0, 1, \ldots, n-1$, $Y_0=P_0$. The elements of $\mathbf{A}_k$, $\mathbf{B}_k$ can be determined by the help of eq. (3). The solution to (5) can be obtained by an iterative manner $\underline{Y}_k=\mathbf{F}_k\underline{Y}_{k-1}$, $k=1, \ldots, n$. To verify this let $\mathbf{F}_n=\mathbf{A}_n$, furthermore assume that $\underline{Y}_{k+1}=\mathbf{F}_k\underline{Y}_k$. Let us consider equation $\underline{Y}_k=\mathbf{A}_k\underline{Y}_{k-1}+\mathbf{B}_k\underline{Y}_{k+1}$. After substituting we get $(1-\mathbf{B}_k\mathbf{F}_{k+1})\underline{Y}_k=\mathbf{A}_k\underline{Y}_{k-1}$ then

$$\underline{Y}_k = (1-\mathbf{B}_k\mathbf{F}_{k+1})^{-1}\mathbf{A}_k\underline{Y}_{k-1}.$$

Let

$$\mathbf{F}_k = (1-\mathbf{B}_k\mathbf{F}_{k+1})^{-1}\mathbf{A}_k,$$

so $\underline{Y}_k=\mathbf{F}_k\underline{Y}_{k-1}$, $k=1, \ldots, n$. Starting with any Y_0 after norming the stationary distribution is given.

3. Performance measures

In the following $(\underline{x}(t), t\geqq 0)$ is supposed to be in equilibrium.

(i) *CPU utilization.* Using renewal-theoretic arguments it is well-known that

$$P_0 = (1/\Lambda)(1/\Lambda+M\delta)^{-1},$$

where $M\delta$ denotes the mean CPU busy period length and $1/\Lambda$ is its average idle period length. If the CPU utilization, which is the long-run fraction of time the CPU is busy, is denoted by U we have

$$U = 1-P_0 = (M\delta)(1/\Lambda+M\delta)^{-1}.$$

Consequently,

$$M\delta = (1-P_0)/\Lambda P_0.$$

(ii) *Mean response times.* During the execution a job is served by the CPU and takes I/0 operations. If these periods are considered as cycles, then in steady state these cycles lengths are identically distributed but not independent random variables.

Let $P^{(i)}$ denote the stationary probability that job i is in compute period and let the average response time be designated by R_i. Furthermore, let H_i be the event that the program i is under service. Introduce the function

$$Z_{H_i}(t) = \begin{cases} 1 & \text{if } \underline{x}(t)\in H_i, \\ 0 & \text{otherwise.} \end{cases}$$

By the virtue of Theorem 1 and 3 in Tomkó [9] we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T Z_{H_i}(t)\,dt = P^{(i)} = \frac{R_i}{1/\lambda_i + R_i}.$$

Since $P^{(i)}$ can be easily evaluated as

$$P^{(i)} = \sum_{k=1}^{n} \sum_{i\in(i_1,\dots,i_k)\in C_k^n} P_{i_1,\dots,i_k},$$

for the expected response time of job i we obtain

$$R_i = P^{(i)}/\lambda_i(1-P^{(i)}).$$

It is clear that $\Sigma Z_{H_i}(t)$ gives the number of jobs statying at the CPU at time t. Thus in equilibrium the mean number of programs processed by the CPU is $\bar{n} = \Sigma P^{(i)}$. In addition, the Little's formula is valid

$$\Sigma\lambda_i(1-P^{(i)})R_i = \Sigma P^{(i)} = \bar{n}.$$

(iii) *Waiting ratio.* Defining the waiting ratio for job i by $\hat{W}_i = \mu_i(R_i - 1/\mu_i)$ the system waiting ratio can be obtained as

$$\hat{W} = \Sigma\hat{W}_i = \Sigma\mu_i R_i - n.$$

(iv) *Throughput.* Denote by T the throughput of the system, which is the mean number of jobs serviced in unit time. If T_i denotes the throughput for job i, we have

$$T_i = \lambda_i(1-P^{(i)}).$$

Thus, we get

$$T = \Sigma T_i = \Sigma\lambda_i(1-P^{(i)}).$$

4. L—S transform of the CPU busy period and response times

Let us denote by η_α a random variable distributed exponentially with rate α. If η_α and η_β are independent, then $\min(\eta_\alpha, \eta_\beta) = \eta_{\alpha+\beta}$ which is a well-known fact. Furthermore, let the notation $(\eta_{\alpha_i} = \eta_{\alpha_1+\dots+\alpha_s})$ mean the event that $\min(\eta_{\alpha_1}, \dots$ $\dots, \eta_{\alpha_s}) = \eta_{\alpha_i}$, probability of which is $\alpha_i / \sum_{j=1}^{s} \alpha_j$, where $\alpha_i > 0$, $i = 1, \dots, s$.

Let $\chi(A)$ denote the characteristic function of event A, i.e.

$$\chi(A) = \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

(i) *CPU busy period length.* Let the random variable $\delta_{i1}, \dots, i_k$ denote a busy time interval of the CPU initiated by state $(i_1, \dots, i_k)$, $(i_1, \dots, i_k) \in C_k^n$, $k = 1, \dots, n$.

Similarly to Csige—Tomkó [4] the following recursive relations hold

$$\delta_i \doteq \eta\sigma_i + \sum_{l\neq i} \delta_{i',l'}\chi(\eta_{\lambda_l} = \eta\sigma_i),$$
$$\vdots$$
$$\delta_{i_1,\dots,i_k} \doteq \eta\sigma_{i_1,\dots,i_k} + \sum_{j=1}^{k} \delta_{i_1,\dots,i_{j-1},i_{j+1},\dots i_k}\chi\left(\eta\frac{\mu_{ij}r_{ij}}{R_{i_1,\dots,i_k}} = \eta\sigma_{i_1,\dots,i_k}\right) + \tag{6}$$
$$+ \sum_{l\neq i_1,\dots,i_k} \delta_{i'_1,\dots,i'_{k+1}}\chi(\eta_{\lambda_l} = \eta\sigma_{i_1,\dots,i_k}),$$
$$\vdots$$
$$\delta_{1,\dots,n} \doteq \eta\sigma_{1,\dots,n} + \sum_{j=1}^{n} \delta_{1,\dots,j-1,j+1,\dots,n}\chi\left(\eta\frac{\mu_j r_j}{R_{1,\dots,n}} = \eta\sigma_{1,\dots,n}\right)$$

where $\doteq$ denotes that the equality is meant in distribution. Introducing the L—S transform

$$g_{i_1,\dots,i_k}(s) = Me^{-s\delta_{i_1,\dots,i_k}},$$

from (6) we get

$$g_i(s) = \frac{1}{s+\sigma_i}\left[\mu_i + \sum_{l\neq i}\lambda_l g_{i',l'}(s)\right],$$
$$\vdots$$
$$g_{i_1,\dots,i_k}(s) = \frac{1}{s+\sigma_{i_1,\dots,i_k}}\left[\sum_{j=1}^{k}\frac{\mu_{ij}r_{ij}}{R_{i_1,\dots,i_k}} g_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_k}(s) + \tag{7}\right.$$
$$\left. + \sum_{l\neq i_1,\dots,i_k}\lambda_l g_{i'_1,\dots,i'_{k+1}}(s)\right],$$
$$g_{1,\dots,n}(s) = \frac{1}{s+\sigma_{1,\dots,n}}\sum_{j=1}^{n}\frac{\mu_j r_j}{R_{1,\dots,n}} g_{1,\dots,j-1,j+1,\dots,n}(s).$$

Finally,

$$Me^{-s\delta} = \sum_{i=1}^{n}\frac{\lambda_i}{\Lambda} g_i(s).$$

Let $\underline{G}_k(s)$ be the vector

$$\begin{vmatrix} g_{1,\dots,k}(s) \\ \vdots \\ g_{i_1,\dots,i_k}(s) \\ \vdots \\ g_{n-k+1,\dots,n}(s) \end{vmatrix}$$

of dimension $\binom{n}{k}$. The components $g_{i_1}, \dots, i_k(s)$ are listed in the lexicographic order of their indexes $(i_1, \dots, i_k)$. Thus (7) can be expressed as

$$\underline{G}_1(s) = \mathbf{A}_1(s)\underline{G}_0 + \mathbf{B}_1(s)\underline{G}_2(s),$$
$$\vdots$$
$$\underline{G}_k(s) = \mathbf{A}_k(s)\underline{G}_{k-1}(s) + \mathbf{B}_k\underline{G}_{k+1}(s), \tag{8}$$
$$\vdots$$
$$\underline{G}_n(s) = \mathbf{A}_n(s)\underline{G}_{n-1}(s),$$

where $\underline{G}_0=(\mu_1, \dots, \mu_n)^*$, the matrix $\mathbf{A}_k(s)$ is of order $\binom{n}{k}\times\binom{n}{k-1}$, $k=1, \dots, n$ $\mathbf{B}_k(s)$ is of order $\binom{n}{k}\times\binom{n}{k+1}$, $k=0, 1, \dots, n-1$. The elements of $\mathbf{A}_k(s)$, $\mathbf{B}_k(s)$ can be readily determined with the aid of eq. (7). It is easy to see that the solution to (8) can be evaluated in the same way as in (5), that is

$$\underline{G}_k(s) = \mathbf{F}_k(s)\underline{G}_{k-1}(s), \tag{9}$$

where

$$\mathbf{F}_n(s)= \mathbf{A}_n(s),\ \mathbf{F}_k(s) = \big(1-\mathbf{B}_k(s)\mathbf{F}_{k+1}(s)\big)^{-1}\mathbf{A}_k(s), \quad k = 1, \dots, n-1.$$

Finally,

$$Me^{-s\delta} = \sum \frac{\lambda_i}{\Lambda} g_i(s).$$

The moments of busy period δ can be obtained from eq. (8) by differentiating and setting $s=0$. If $\mathbf{A}^{(i)}(s)$ denotes the i-th derivate of matrix $\mathbf{A}(s)$, which is meant by elements, then it is easy to see that

$$(\mathbf{A}(s)\mathbf{B}(s))^{(i)} = \sum_{l=0}^{i} \binom{i}{l} \mathbf{A}^{(l)}(s)\mathbf{B}^{(i-l)}(s).$$

If we define $M_{i_1,\dots,i_k}^{(i)}$ and $\underline{M}_k^{(i)}$ by

$$M_{i_1,\dots,i_k}^{(i)} = (-1)^i \frac{d^i g_{i_1,\dots,i_k}(s)}{ds^i}\bigg|_{s=0},$$

and

$$\underline{M}_k^{(i)} = (-1)^i \underline{G}_k^{(i)}(0),$$

then

$$\underline{G}_k^{(i)}(0) = \sum_{l=0}^{i}\binom{i}{l}\mathbf{A}_k^{(l)}(0)\underline{G}_{k-1}^{(i-l)}(0)+\sum_{l=0}^{i}\binom{i}{l}\mathbf{B}_k^{(l)}(0)\underline{G}_{k+1}^{(i-l)}(0),$$

which yields

$$\underline{M}_k^{(i)} = \sum_{l=0}^{i}(-1)^l\binom{i}{l}\mathbf{A}_k^{(l)}(0)\underline{M}_{k-1}^{(i-l)}+\sum_{l=0}^{i}(-1)^l\binom{i}{l}\mathbf{B}_k^{(l)}(0)\underline{M}_{k+1}^{(i-l)}. \tag{10}$$

Introducing

$$\mathbf{A}_k^{(i)} = \mathbf{A}_k^{(0)}(0),\quad \mathbf{B}_k^{(i)} = \mathbf{B}_k^{(0)}(0),\quad \underline{C}_k^{(i)} = \sum_{l=1}^{i}(-1)^l\binom{i}{l}[\mathbf{A}_k^{(l)}(0)\underline{M}_{k-1}^{(i-l)}+\mathbf{B}_k^{(l)}(0)M_{k+1}^{(i-l)}]$$

(10) can be written as

$$\underline{M}_k^{(i)} = \mathbf{A}_k^{(i)}\underline{M}_{k-1}^{(i)}+\mathbf{B}_k^{(i)}\underline{M}_{k+1}^{(i)}+\underline{C}_k^{(i)}.$$

Thus, the equations for the i-th moment of the busy period in matrix form are

$$\begin{aligned}
\underline{M}_1^{(i)} &= \mathbf{B}_1^{(i)}\underline{M}_2^{(i)}+\underline{C}_1^{(i)},\\
&\vdots\\
\underline{M}_k^{(i)} &= \mathbf{A}_k^{(i)}\underline{M}_{k-1}^{(i)}+\mathbf{B}_k^{(i)}\underline{M}_{k+1}^{(i)}+\underline{C}_k^{(i)},\\
&\vdots\\
\underline{M}_n^{(i)} &= \mathbf{A}_n^{(i)}\underline{M}_{n-1}^{(i)}+\underline{C}_n^{(i)}.
\end{aligned} \tag{11}$$

In the following we show that the solution to (11) can be obtained as

$$\underline{M}_k^{(i)} = F_k^{(i)}\underline{M}_{k-1}^{(i)}+\underline{D}_k^{(i)}.$$

To derive this, define $\mathbf{F}_n^{(i)}$ and $\underline{D}_n^{(i)}$ by

$$\mathbf{F}_n^{(i)} = \mathbf{A}_n^{(i)}, \quad \underline{D}_n^{(i)} = \underline{C}_n^{(i)}. \tag{12}$$

Assuming that $M_{k+1}^{(i)}=\mathbf{F}_{k+1}^{(i)}\underline{M}_k^{(i)}+\underline{D}_{k+1}^{(i)}$, after substituting to (11) we get

$$(1-\mathbf{B}_k^{(i)}\mathbf{F}_{k+1}^{(i)})\underline{M}_k^{(i)} = \mathbf{A}_k^{(i)}\underline{M}_{k-1}^{(i)}+\mathbf{B}_k^{(i)}\underline{D}_{k+1}^{(i)}+\underline{C}_k^{(i)},$$

which yields

$$\mathbf{F}_k^{(i)} = (1-\mathbf{B}_k^{(i)}\mathbf{F}_{k+1}^{(i)})^{-1}\mathbf{A}_k^{(i)}, \quad \underline{D}_k^{(i)} = (1-\mathbf{B}_k^{(i)}\mathbf{F}_{k+1}^{(i)})^{-1}(\mathbf{B}_k^{(i)}\underline{D}_{k+1}^{(i)}+\underline{C}_k^{(i)}). \tag{13}$$

Concluding

$$\underline{M}_1^{(i)} = \underline{D}_1^{(i)},$$
$$\vdots$$
$$\underline{M}_k^{(i)} = \mathbf{F}_k^{(i)}\underline{M}_{k-1}^{(i)}+\underline{D}_k^{(i)}, \quad k = 2, \dots, n,$$

where matrix $\mathbf{F}_k^{(i)}$ and vector $\underline{D}_k^{(i)}$ are defined by (12), (13). Finally,

$$M\delta^{(i)} = \sum \frac{\lambda_l}{\Lambda} M\delta_l^{(i)}. \tag{14}$$

In particular, if $i=1$, (14) reduces to equations found in Tomkó [10].

(ii) *Response times.* Let $\{\tau_k,\ k\geqq 0\}$ denote the instants of consecutive changes of states in Markov chain $(\underline{x}(t),\ t\geqq 0)$. Let us consider the following imbedded Markov chain $(\underline{Y}_m,\ m\geqq 0)$ defined by $\underline{Y}_m=\underline{X}(\tau_m+0)$. If we define by

$$(q_0,\ q_{i_1,\dots,i_k};\ (i_1,\dots,i_k)\in c_k^n,\ k = 1, \dots, n) \tag{15}$$

the stationary distribution of $(\underline{Y}_m,\ m\geqq 0)$ then it is clear that

$$P_0 = \frac{q_0}{\Lambda}\Bigg/\left(q_0/\Lambda+\sum_{l=1}^{n}\sum_{c_l^n}\frac{q_{i_1,\dots,i_l}}{\sigma_{i_1,\dots,i_l}}\right),$$

$$P_{i_1,\dots,i_k} = \frac{q_{i_1,\dots,i_k}}{\sigma_{i_1,\dots,i_k}}\Bigg/\left(q_0/\Lambda+\sum_{l=1}^{n}\sum_{c_l^n}\frac{q_{i_1,\dots,i_l}}{\sigma_{i_1,\dots,i_l}}\right), \tag{16}$$

$k=1, \dots, n$, (cf. Tomkó [9]).

From (16), for (15) we get the following system of linear equations

$$q_0\left(\frac{P_0-1}{\Lambda}\right)+\sum_l\sum_{c_l^n}\frac{P_0}{\sigma_{i_1,\dots,i_l}}q_{i_1,\dots,i_l} = 0,$$

$$\frac{P_{i_1,\dots,i_k}}{\Lambda}q_0+\sum_l\sum_{c_l^n,(i_1,\dots,i_l)\neq(i_1,\dots,i_k)}\frac{P_{i_1,\dots,i_k}}{\sigma_{i_1,\dots,i_l}}q_{i_1,\dots,i_l}+\frac{P_{i_1,\dots,i_k}-1}{\sigma_{i_1,\dots,i_k}}q_{i_1,\dots,i_k} = 0,$$

which can be solved by standard methods.

Let $E_{i_1,\dots,i_k}^{(i)}$ denote the event that at the arrival epoch of job i programs with indexes $(i_1, \dots, i_k)$ are processed by the CPU, $(i_1, \dots, i_k)\in C_k^n$, $i_1\neq i, \dots, i_k\neq i$, $1\leqq i\leqq n$,

$0 \leqq k \leqq n-1$. In addition, let $\gamma^{(i)}_{i_1,\dots,i_k}$ denote the response time of job i if event $E^{(i)}_{i_1,\dots,i_k}$ occurs. Denote by $q^{(i)}_{i_1,\dots,i_k}$ the steady-state probability of $E^{(i)}_{i_1,\dots,i_k}$. Then similarly to Csige—Tomkó [4], for $q^{(i)}_{i_1,\dots,i_k}$, we have

$$q_0^{(i)} = q_0 \frac{\lambda_i}{\Lambda} Q^{(i)}, \quad q^{(i)}_{i_1,\dots,i_k} = q_{i_1,\dots,i_k} \frac{\lambda_i}{\sigma_{i_1,\dots,i_k}} Q^{(i)},$$

where $Q^{(i)}$ can be determined from the norming condition by

$$Q^{(i)} = \frac{1}{\lambda_i} \left(\frac{q_0}{\Lambda} + \sum_{k=1}^{n-1} \sum_{c_k^n} \frac{q_{i_1,\dots,i_k}}{\sigma_{i_1,\dots,i_k}} \right)^{-1}.$$

Using the results derived by Tomkó in [9] the following iterative relations can be written

$$\gamma_0^{(i)} \doteq \eta\sigma_i + \sum_{l \neq i} \gamma_l^{(i)} \chi(\eta_{\lambda_l} = \eta\sigma_i),$$

$$\vdots$$

$$\gamma^{(i)}_{i_1,\dots,i_k} \doteq \eta\sigma_{i_1,\dots,i_k,i} + \sum_{l \neq i_1,\dots,i_k} \gamma^{(i)}_{i'_1,\dots,i'_{k+1}} \chi(\eta_{\lambda_l} = \eta\sigma_{i_1,\dots,i_k,i}) +$$

$$+ \sum_{j=1}^{k} \gamma^{(i)}_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_k} \chi\left(\eta \frac{\mu_{i_j} r_{i_j}}{R_{i_1,\dots,i_k,i}} = \eta\sigma_{i_1,\dots,i_k,i}\right), \tag{17}$$

$$\gamma^{(i)}_{1,\dots,i-1,i+1,n} \doteq \eta\sigma_{1,\dots,n} + \sum_{j=1}^{n} \gamma^{(i)}_{1,\dots,j-1,j+1,\dots,n} \chi\left(\eta \frac{\mu_j r_j}{R_{1,\dots,n}} = \eta\sigma_{1,\dots,n}\right),$$

Introducing the L—S transform

$$v^{(i)}_{i_1,\dots,i_k}(s) = Me^{-s\gamma^{(i)}_{i_1,\dots,i_k}},$$

from (17) we get

$$v_0^{(i)}(s) = \frac{1}{s+\sigma_i} [\mu_i + \sum_{l \neq i} \lambda_l v_l^{(i)}(s)], \tag{18}$$

$$\vdots$$

$$v^{(i)}_{i_1,\dots,i_k}(s) = \frac{1}{s+\sigma_{i_1,\dots,i_k,i}} \left[\sum_{j=1}^{k} \frac{\mu_{ij} r_{ij}}{R_{i_1,\dots,i_k,i}} v^{(i)}_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_k}(s) + \sum_{l \neq i_1,\dots,i_k} v^{(i)}_{i'_1,\dots,i'_{k+1}}(s) \right],$$

$$v^{(i)}_{1,\dots,i-1,i+1,\dots,n}(s) = \frac{1}{s+\sigma_{1,\dots,n}} \sum_{j=1}^{n} \frac{u_j r_j}{R_{1,\dots,n}} v^{(i)}_{1,\dots,j-1,j+1,\dots,n}(s).$$

Finally, the L—S transform of the response time for job i can be easily obtained by

$$v^{(i)}(s) = q_0^{(i)} v_0^{(i)}(s) + \sum_{k=1}^{n-1} \sum_{c_k^n} q^{(i)}_{i_1,\dots,i_k} v^{(i)}_{i_1,\dots,i_k}(s).$$

Notice that eq. (18) can be treated in the same way as eq. (7).

5. Numerical results

The algorithm generating these performance measures were implemented in PL/1 in the Computer Centre of University of Debrecen. Some sample results for different input parameters λ_i, μ_i, n $(i=1, \dots, n)$ are shown in Tables 1—4. In Table 1 and 2 some comparisons are made with PS and PR disciplines, while in Table 3 and 4 we give the characteristics under PPS discipline.

Table. 1. Homogeneous I/0 and CPU times

Parameters: $\lambda_1=\lambda_2=\lambda_3=0.3$, $\mu_1=\mu_2=\mu_3=0.7$

	R_i	T_i	$\hat{W}_i$	U	$M\delta$	$\bar{n}$	$\hat{W}$	T
PR	1.428	0.21	0					
	2.413	0.17	0.689	0.74	3.177	1.27	2.658	0.51
	4.242	0.13	1.969					
PS	2.450	0.17	0.715					
	2.450	0.17	0.715	0.74	3.177	1.27	2.145	0.51
	2.450	0.17	0.715					
PPS weights								
125	1.467	0.21	0.026					
5	2.559	0.16	0.791	0.74	3.177	1.27	2.460	0.51
1	3.776	0.14	1.643					
1000	1.437	0.21	0.005					
10	2.485	0.17	0.739	0.74	3.177	1.27	2.512	0.51
1	3.955	0.13	1.768					
125 000	1.428	0.21	0.000					
50	2.396	0.17	0.677	0.74	3.177	1.27	2.561	0.51
1	4.120	0.13	1.884					
1 000 000	1.428	0.21	0.000					
100	2.383	0.17	0.668	0.74	3.177	1.27	2.568	0.51
1	4.143	0.13	1.900					

Table 2. Heterogeneous I/0 and CPU times

Parameters: $\lambda_1=0.5$, $\lambda_2=0.3$, $\lambda_3=0.2$
$\mu_1=0.9$, $\mu_2=0.7$, $\mu_3=0.5$

PR	1.125	0.32	0					
	2.619	0.16	0.833	0.77	3.34	1.34	3.014	0.56
	6.363	0.08	2.181					
PS	1.885	0.25	0.697					
	2.526	0.17	0.768	0.76	3.21	1.33	2.252	0.53
	3.574	0.11	0.787					
PPS								
1	2.056	0.24	0.850					
1	2.745	0.16	0.921	0.75	3.15	1.33	2.266	0.52
2	2.990	0.12	0.495					

Table 3. Homogeneous I/0 times

Parameters: $\lambda_1=\lambda_2=\lambda_3=0.2$, $\mu_1=0.4$, $\mu_2=0.6$, $\mu_3=0.8$

weights								
1	4.831	0.10	0.932					
5	2.498	0.13	0.499	0.675	3.472	1.028	1.453	0.394
125	1.277	0.15	0.021					
1	4.965	0.10	0.986					
10	2.407	0.13	0.444	0.675	3.472	1.024	1.435	0.395
100	1.256	0.15	0.004					
1	5.100	0.09	1.040					
100	2.304	0.13	0.382	0.675	3.472	1.020	1.423	0.395
1 000 000	1.250	0.15	0.000					

Table 4. Heterogeneous I/0 and CPU times

Parameters: $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=3$, $\lambda_4=4$, $\lambda_5=5$, $\lambda_6=6$
$\mu_1=6$, $\mu_2=5$, $\mu_3=4$, $\mu_4=3$, $\mu_5=2$, $\mu_6=1$

weights								
1	3.008	0.24	17.048					
2	1.834	0.42	8.170					
3	1.543	0.53	5.172	0.999	289.350	5.070	38.795	2.525
4	1.549	0.55	3.647					
5	1.853	0.48	2.706					
6	3.052	0.31	2.052					
36	0.329	0.75	0.974					
25	0.435	1.06	1.175					
16	0.659	1.00	1.636	0.999	94.777	4.145	33.764	3.775
9	1.233	0.65	2.699					
4	3.379	0.27	5.758					
1	22.522	0.04	21.522					

Conclusion

In this paper we have modelled a multiprogrammed computer system as finite-source single server queueing system with different types of customers under priority processor-shared service discipline. The system performance measures were numerically evaluated using an algorithmic approach.

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Abstract

This paper deals with a heterogeneous multiprogrammed computer system under priority processor-shared (PPS) service discipline introduced by Kleinrock. The jobs are characterized by exponentially distributed input-output (I/0) and central processing unit (CPU) times. In steady state the main performance measures, such as CPU utilization, expected CPU busy period length, mean response times, waiting ratio, throughput of the jobs and throughput of the system, are given.

In addition, a system of linear equations for Laplace—Stieltjes (L—S) transform of the response times and the CPU busy period length is obtained. Finally, by numerical examples characteristics under different service disciplines are compared with the PPS one.

Keywords: I/0 times, CPU times, utilization, mean response time, waiting ratio, throughput.

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