Optimization of multi valued logical functions based on evaluation graphs

A. VARGA

Dedicated to Professor K. Tandori on his 60th birthday

Abstract

In this paper we discuss simplifications of multi valued logical functions. The simplification is carried out in the following way. We associate tree graphs with the disjunctive or conjunctive normal forms of the functions. Under certain conditions some vertices of these trees can be omitted. This cancellation will correspond to reduction of terms or variables in the original function.

After all possible simplifications a normal form, which is equivalent to the function in question, is obtained.

1. Definitions, notations

Let $k(>2)$ be a natural number and ε_k the set $\{0, 1, 2, ..., k-1\}$. Any function $f\colon \varepsilon_k^n \to \varepsilon_k$ is called a k-valued logical function of n-variables where ε_k^n denotes the Cartesian product of n copies of ε_k. These functions are often given by their truth-tables and they will also be denoted by $f(\mathbf{X}^n)$ or $f(\mathbf{X}^n)=f(X_1, X_2, ..., X_n)$. The set of k-valued logical function will be denoted by P_k. Several properties valid in the theory of ordinary two-valued logic remain true in the theory of k-valued logic as well. But in the case $k \geqq 3$ certain characteristics are essentially different from those in ordinary logic.

A major problem is the definition of negation, since it can be defined in several ways.

Definition 1. Let $A_i \in \{0, 1, ..., k-1\}$, $i=1, 2, ..., n$; $n \geqq 2$. Then the operators defined by

$$A_1 \wedge A_2 \wedge ... \wedge A_n = \min(A_1, A_2, ..., A_n)$$

and

$$A_1 \vee A_2 \vee ... \vee A_n = \max(A_1, A_2, ..., A_n)$$

are called the conjunction and disjunction of the variables $A_1, A_2, \ldots, A_n$, respectively.

The following identities can easily be proved:

I. $$A \wedge B = B \wedge A,$$
$$A \vee B = B \vee A, \quad \text{for every } A, B.$$

II. $$A \wedge (B \wedge C) = (A \wedge B) \wedge C,$$
$$A \vee (B \vee C) = (A \vee B) \vee C, \quad \text{for every } A, B, C.$$

III. $$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C),$$
$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C), \quad \text{for every } A, B, C.$$

IV. $$A \vee A = A,$$
$$A \wedge A = A, \quad \text{for every } A.$$

V. $$A \wedge (k-1) = A,$$
$$A \vee \emptyset = A, \quad \text{for every } A.$$

Below we give two types of negation: one for logical constants and one for logical variables.

Definition 2. Let $A \in \varepsilon_k$. Then

$$\bar{A} = (k-1) - A.$$

Definition 3. If X is a variable then $\bar{X}$ denotes that function the actual value of which is the negation (in the sense of Definition 2) of the actual value of X. Let us introduce the following unary operator

$${}^aX^b = \begin{cases} k-1, & \text{if } a \leqq X \leqq b, \\ 0 & \text{elsewhere,} \end{cases}$$

where $a, b, X \in \varepsilon_k$ and $a \leqq b$ are fixed. It should be noticed that ${}^aX^b$ is two-valued. By Definition 3, the negation of ${}^aX^b$ is

$$\overline{{}^aX^b} = \begin{cases} 0, & \text{if } a \leqq X \leqq b, \\ k-1 & \text{elsewhere,} \end{cases}$$

where $a, b, X \in \varepsilon_k$ and $a \leqq b$ are fixed. The formulae in the theory of k-valued logic, similarly to those of two valued logic, will be given by recursive definition.

Definition 4.

(0) The elements of ε_k are k-valued logical formulae;
(1) $X_1, X_2, \ldots, X_n, {}^{a_1}X^{b_1}, {}^{a_2}X^{b_2}, \ldots, {}^{a_n}X^{b_n}$ are k-valued logical formulae;
(2) If F is a k-valued logical formula, then $\bar{F}$ is a k-valued formula;
(3) If F and G are k-valued logical formulae, then $F \vee G$, $F \wedge G$ are k-valued logical formulae;

(4) Every k-valued logical formula can be obtained by a repeated application of (0)—(3).

In what follows the letters f, g, ... will denote functions and the capital letters F, G, ... will denote formulae. By a function of n-variables we mean a k-valued logical function of n-variables ($k \geqq 3$).

Value assignement. The ordered n-tuple $(X_1, X_2, \ldots, X_i|A_i, \ldots, X_n)$ is called a value assignement of the i-th variable. If every variable has value simultaneously, then the ordered n-tuple $(\mathbf{X}^n|\mathbf{A}^n)=(X_1|A_1, X_2|A_2, \ldots, X_n|A_n)$ is simply called a value assignement.

Let $f(\mathbf{X}^n)$ be a function. Then

$$\mathrm{f}(\mathbf{X}^n|\mathbf{A}^n) = f(X_1|A_1, X_2|A_2, \ldots, X_n|A_n)$$

denotes the fact that X_i is replaced by A_i, where $A_i \in \varepsilon_k$ $i=1, 2, \ldots, n$. The value $f(X_1|A_1, X_2|A_2, \ldots, X_n|A_n)$ is called the value of $f(\mathbf{X}^n)$ under the value assignement $(X_1|A_1, X_2|A_2, \ldots, X_n|A_n)$. Below the value assignement $(X_1|A_1, X_2|A_2, \ldots, X_n|A_n)$ and the value $f(X_1|A_1, X_2|A_2, \ldots, X_n|A_n)$ will be denoted simply by $(A_1, A_2, \ldots, A_n)$ and $f(A_1, A_2, \ldots, A_n)$, respectively. One can define value assignements for formulae as well.

Definition 5. Let $f, g \in P_k$. If the value of g does not exceed that of f (in any position of the truth-table), then we say that g implies f and write $g \to f$.

Definition 6. Formulae F and G are said to be equivalent if the corresponding functions f and g are equal. In this case we write $F=G$.

An easy computation gives

Lemma 1. Let $f(\mathbf{X}^n)=f(X_1, X_2, \ldots, X_n)$, $n \geqq 2$. Then for every $i=1, 2, \ldots,$

$$f(X_1, X_2, \ldots, X_i, \ldots, X_n) = \bigvee_{j=0}^{k-1} [f(X_1, \ldots, X_{i-1}, X_i|j, X_{i+1}, \ldots, X_n) \wedge {}^jX_i^j].$$

Remark. Below the conjunction will be denoted by $\cdot$ (sometimes it will be omitted) or, in the case of several variables, by Π, and the disjunction will be denoted by $+$ or Σ. The following lemma can easily be verified.

Lemma 2. Let $f(\mathbf{X}^n)=f(X_1, X_2, \ldots, X_n)$, $n \geqq 2$. Then the relation

$$f(X_1, X_2, \ldots, X_n) = \sum_{(a_1, a_2, \ldots, a_n)} {}^{a_1}X_1^{a_1}\,{}^{a_2}X_2^{a_2} \ldots {}^{a_n}X_n^{a_n} f(a_1, a_2, \ldots, a_n)$$

holds, where Σ is taken over all the possible ordered n-tuples, and $a_i \in \varepsilon_k$, $i=1, 2, \ldots, n$.

Definition 7. By a superposition of the k-valued logical functions $f(X_1, X_2, \ldots, X_i, \ldots, X_n)$ and $g(X_1, X_2, \ldots, X_n)$ we mean the function $f(X_1, X_2, \ldots, g(X_1, X_2, \ldots, X_n), \ldots, X_n)$ which is obtained by substituting the function g for the i-th argument X_i of f.

Definition 8. The set of functions $\{f_1, f_2, \ldots, f_l\}$ is called a basis-set for P_k if every elements of P_k can be expressed by X_i $(i=1, 2, \ldots, n)$ and the functions $f_1, f_2, \ldots, f_n$ applying superpositions finitely many times. It is customary to say that the elements of a basis set form a functionally complete function system.

By virtue of Lemma 2 we get that the system $\{0, 1, \dots, k-1, {}^0X^0, {}^1X^1, \dots$ $\dots, {}^{k-1}X^{k-1}, \min(X_1, X_2), \max(X_1, X_2)\}$ is complete in P_k.

Definition 9. The expression

$$\sum_{(a_1, a_2, \dots, a_n)} {}^{a_1}X_1^{a_1}\, {}^{a_2}X_2^{a_2} \dots {}^{a_n}X_n^{a_n} f(a_1, a_2, \dots, a_n)$$

is called the full disjunctive normal form $F_\vee(X_1, X_2, \dots, X_n)$ of the function f.

Since $a_i \in \varepsilon_k$, $i=1, 2, \dots, n$, the number of all different n-tuples $(a_1, a_2, \dots, a_n)$, is k^n. Denoting the value $f(a_1^{(j)}, a_2^{(j)}, \dots, a_n^{(j)})$, concerning the j-th n-tuple (in a fixed ordering) $(a_1^{(j)}, a_2^{(j)}, \dots, a_n^{(j)})$ by α_j and the corresponding conjunction

$$^{a_1(j)}X_1^{b_1(j)}\, {}^{a_2(j)}X_2^{a_2(j)} \dots {}^{a_n(j)}X_n^{a_n(j)}$$

by E_j^n the full disjunctive normal form belonging to $f(\mathbf{X}^n)$ can be written in the form

$$F_\vee(\mathbf{X}^n) = \sum_{j=0}^{k^n-1} \alpha_j E_j^n.$$

$E_j^{(n)}$ is called a min term of n-variables. We will require some further formulae which can easily be verified.

$$^aX^c + {}^dX^b = \begin{cases} {}^aX^b & \text{if, } a \leqq d \leqq c \leqq b, \\ {}^dX^b & \text{if, } d \leqq a \leqq c \leqq b, \\ \overline{{}^{c+1}X^{d-1}} & \text{if, } a \leqq c < d \leqq b,\ a, b, c, d, X \in \varepsilon_k. \end{cases} \tag{1}$$

$$^aX^c \cdot {}^dX^b = \begin{cases} 0 & \text{if, } a \leqq c < d \leqq b, \\ {}^dX^c & \text{if, } a \leqq d \leqq c \leqq b, \\ {}^dX^b & \text{if, } a \leqq d \leqq b \leqq c,\ a, b, c, d, X \in \varepsilon_k. \end{cases} \tag{2}$$

$$\overline{^aX^b} = {}^0X^{a-1} + {}^{b+1}X^{k-1}, \tag{3}$$

where

$${}^0X^{a-1} = 0 \quad \text{if} \quad a = 0, \quad {}^{b+1}X^{k-1} = 0 \quad \text{if} \quad b = k-1,\ a, b, X \in \varepsilon_k$$

$${}^0X^{k-1} = k-1. \tag{4}$$

$${}^aX^b + \overline{^aX^b} = k-1,\ a, b, X \in \varepsilon_k. \tag{5}$$

$$\overline{\alpha\, {}^{a_1}X_1^{b_1}\, {}^{a_2}X_2^{b_2} \dots {}^{a_n}X_n^{b_n}} = \bar{\alpha} + \overline{^{a_1}X_1^{b_1}} + \overline{^{a_2}X_2^{b_2}} + \dots + \overline{^{a_n}X_n^{b_n}}. \tag{6.a}$$

$$\overline{\alpha + {}^{a_1}X_1^{b_1} + {}^{a_2}X_2^{b_2} + \dots + {}^{a_n}X_n^{b_n}} = \bar{\alpha}\, \overline{^{a_1}X_1^{b_1}}\ \overline{^{a_2}X_2^{b_2}} \dots \overline{^{a_n}X_n^{b_n}}, \tag{6.b}$$

where

$$X_i, a_i, b_i \in \varepsilon_k, \quad i = 1, 2, \dots, n.$$

Formulae (6a) and (6b) are the de Morgan's identities in the theory of multi-valued logic.

The full conjunctive normal form can be defined in a similar manner.

Definition 10. By the full conjunctive normal form of a k-valued function $f(\mathbf{X}^n)$ we mean the formula

$$F_\wedge(\mathbf{X}^n) = \prod_{j=0}^{k^n-1} (\alpha_j + \bar{E}_j^n),$$

where $\bar{E}_j^n$ can be obtained from E_j^n by the de Morgan's identities and denotes the so called max terms. Using the usual rules of the theory of two-valued logic the full normal forms can immediately be found from the truth-table. Every conjunction term and disjunction term of the full conjunctive and disjunctive normal forms contains the expression ${}^{a_1}X_1^{b_1}$, ${}^{a_2}X_2^{b_2}$, ..., ${}^{a_n}X_n^{b_n}$ of the variables $X_1, X_2, ..., X_n$.

The full disjunctive and conjunctive normal forms can be written in the following ways

$$F_\vee(\mathbf{X}^n) = F_1 + F_2 + ... + F_{k-1} = \sum_{i=1}^{k-1} F_i,$$

and

$$F_\wedge(\mathbf{X}^n) = F_1' F_2' ... F_{k-1}' = \prod_{j=1}^{k-1} F_j',$$

where F_i (F_j') is the sub-formula consisting only of min terms (max terms) which determine the i-th (j-th) value of the function.

Definition 11. Let F be a disjunctive normal form of $f \in P_k$, and let G be a conjunction term of F. We say that G is an implicant of f if $G \to f$. G is called prime implicant if, for every G' obtained by omitting any variable of G, $G' \nrightarrow f$ holds.

Remark. The above defined min and max operations are mutually distributive (see identity III). Using this fact and the duality of the two operations we can treat the disjunction terms in a conjunctive normal form in the same way as we treat the conjunction terms in a disjunctive normal form. A normal form is called irredundant if the following properties hold:

(1) each of its terms is a primimplicant, and
(2) no expression obtained by omitting any term in the normal form implies the original function.

A normal form is called redundant if it is not irredundant.

2. Representation of formulae of functions.

The tree-construction procedure

We will work with a fixed order of our variables, which will be denoted by S. We agree that if we write $f(\mathbf{X}^n) = f(X_1, X_2, ..., X_n)$ then $S \equiv (X_1, X_2, ..., X_n)$. The simplification procedure we are going to discuss depends on S, therefore to some of the objects in the procedure we will affix S. By successive evaluation we mean successive evalution determined by S(i.e. we change first the first variable for logical values then the second one etc.)

Let $f(\mathbf{X}^n)$ and $S\equiv(X_1, X_2, ..., X_n)$ be given, and let

$$
\begin{aligned}
f(\mathbf{X}^n) = f(X_1, X_2, ..., X_n) &= f_{0,1}\\
f(0, X_2, ... X_n) &= f_{1,1}\\
f(1, X_2, ... X_n) &= f_{1,2}\\
\vdots \qquad\qquad &\quad \vdots\\
f(i, X_2, ..., X_n) &= f_{1,i+1}\\
\vdots \qquad\qquad &\quad \vdots\\
f(k-1, X_2, ..., X_n) &= f_{1,k}\\
f(0, 0, X_3, ..., X_n) &= f_{2,1}\\
\vdots \qquad\qquad &\quad \vdots\\
f(0, k-1, X_3, ..., X_n) &= f_{2,k}\\
f(1, 0, X_3, ..., X_n) &= f_{2,k+1}\\
\vdots \qquad\qquad &\quad \vdots\\
f(k-1, k-1, X_3, ..., X_n) &= f_{2,k^2}\\
\vdots \qquad\qquad &\quad \vdots\\
f(k-1, k-1, ..., k-1, X_n) &= f_{n-1,k^{n-2}}\\
f(k-1, k-1, ..., k-1, 0) &= f_{n,1}\\
\vdots \qquad\qquad &\quad \vdots\\
f(k-1, k-1, ..., k-1, k-1) &= f_{n,k^n}
\end{aligned}
$$

Using the results of Lemma 1, the following arrangement can be given (Fig. 1).

The functions $f_{0,1}, f_{1,1}, ..., f_{n,k^n}$ are called level-functions. Every function $f_{m,j}$ $(0\leqq m<n, 1\leqq j\leqq k^m)$ determines k new functions on the $(m+1)$-th level in the following way:

$$f_{m+1,jk-(i-(k-1))}(..., X_j, ...) = f_{m,j}(..., i, ...).$$

So there are k^{m+1} level-functions on the $(m+1)$-th level. The ${}^iX_j^i s$ $(i=0, 1, ..., k-1, j=1, 2, ..., n)$ appearing at the edges of the tree above indicate that the variable X_j is replaced by the constant i. The functions $f_{n,1}$, being on the n-th level, are logical values.

This way we can associate a k-ary tree with every function $f(\mathbf{X}^n)$.

The tree which has just been obtained will be denoted by Φ_S (notice that the construction depends on the fixed order S of the variables). Since Φ_S contains all the possible level functions, Φ_s will be called complete.

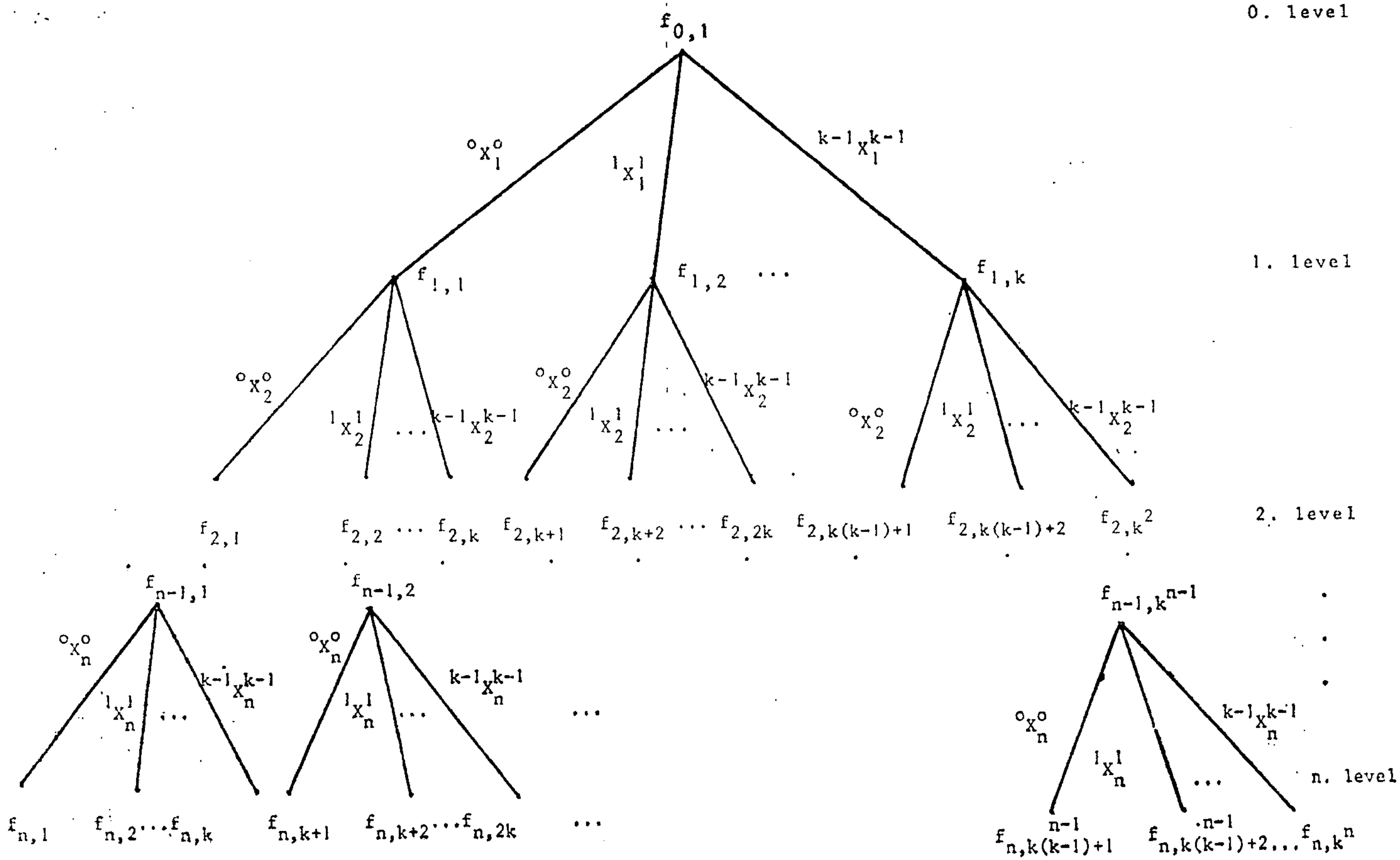


Fig. 1

Notion of endpoint and path

By endpoints we mean the "leaves" of the tree (the vertices on the lowest, n-th level). Any sequence of edges joining the root with some endpoint will be called a path. Some more notations:

Let Φ_S be a tree belonging to $f(\mathbf{X}^n)$, and let $S \equiv (X_1, X_2, \ldots, X_n)$ be fixed. Suppose that the edge denoted by ${}^aX_{i+1}^a$ connects the level functions $f_{i,j}$ and $f_{i+1,1}$.

Below $f_{i,j}$ and $f_{i+1,j}$ will be called the start-point and the endpoint of the edge ${}^aX_{i+1}^a$, respectively. Obviously there is a one-to one correspondence between the evaluations of a function $f(\mathbf{X}^n)$ and the paths of the corresponding tree Φ_S. If we know the tree Φ_s corresponding to a function $f(\mathbf{X}^n)$ then it is easy to determine the $F_\vee(\mathbf{X}^n)$ full disjunctive and $F_\wedge(\mathbf{X}^n)$ full conjunctive normal forms of $f(\mathbf{X}^n)$. To obtain $F_\vee(\mathbf{X}^n)$ we have to take the conjunction of the variables ${}^jX_i^j$ along paths together with the logical value of the endpoint of the path and take the disjunction of all these expressions for every possible paths. If we interchange here "disjunction" and "conjunction" and "variable" for "negation of variable" we obtain $F(\mathbf{X}^n)$.

This method shows that the tree Φ_S is a representation of the formulae $F_\vee$ and $F_\wedge$. It can also be seen that Φ_S is equivalent to the truth-table of the function, the difference between them is that Φ_S can be obtained by successive evaluation while the truth-table is given by simultaneous evaluation.

Theorem 1. Let $f(\mathbf{X}^n) \in P_k$, $S \equiv (X_1, X_2, \ldots, X_n)$. Then the tree-construction procedure associates a uniquely determined k-ary tree to f.

Proof. The level function $f_{m+1, jk-(i-(k-1))}$ has fewer variables than $f_{m,j}$. Since $f_{0,1}$ contains a finite number of variables, the procedure must necessarily stop after the construction of a finite number of levels, which gives the existence of the tree. The unicity can be obtained from the equivalence of simultaneous and successive evaluations.

Definition 12. Any function f with domain $D(f) \subset \varepsilon_k^n$ is called a partially defined function. Those places where f is not defined will be marked by $(*)$ in the truth-table and at the "leafs" of the tree.

In the process of simplification we can assign any value to these places, which, in certain cases, yields a simpler representation.

3. The simplification procedure

Let $f(\mathbf{X}^n) \in P_k$ and let S be fixed. In order to construct an irredundant equivalent of $f(\mathbf{X}^n)$ first we construct the tree Φ_S and choose that subtrees Φ_s^t $(t=0, 1, \ldots, k-1)$ of Φ_S which consists of those paths of Φ_S that have t at their end.

Definition 13. Those points of Φ_s^t $(t=0, 1, \ldots, k-1)$ from which exactly k edges start will be called complete branching points, and the k edges starting from such a point will be called a complete edge-system. A complete branching point of a subtree is called m-multiple complete branching point if the subtree has altogether m total branching points with the same complete edge-system as the given point (more precisely the variables attached to the complete edge systems must be the same).

Let p be an arbitrary path of Φ_S^t $(t=0, 1, \ldots, k-1)$ and let n be the number of its edges.

Let

$$
{}^*X_{n_j}^* = \begin{cases} {}^0X_{n_j}^0 & \text{if } {}^0X_{n_j}^0 \text{ belongs to } p, \\ {}^1X_{n_j}^1 & \text{if } {}^1X_{n_j}^1 \text{ belongs to } p, \\ \vdots & \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \\ {}^{k-1}X_{n_j}^{k-1} & \text{if } {}^{k-1}X_{n_j}^{k-1} \text{ belongs to } p, \quad (1 \leqq n_{j-1} < n_j \leqq n,\ 2 \leqq j \leqq m) \end{cases}
$$

and

$$\{{}^*X_{n_1}^*, {}^*X_{n_2}^*, \ldots, {}^*X_{n_m}^*\} \equiv \bar{X} = \bar{X}\{p, n_j, m\}$$

some edges of p.

Those edges of p (if there is any) which do not belong to $\bar{X}$ will be called connecting sequences of $\bar{X}$ (relative to p) and will be denoted $\varkappa = \{\varkappa_1, \varkappa_2, \ldots, \varkappa_S\}$.

Let Φ_S^t be given, and let p be a path of Φ_S^t. A subtree $\Phi_S'^t$ of Φ_s^t will be called maximally simplifiable subtree of order m (below briefly MSST) if

(1) $\Phi_S'^t$ contains p,

(2) there exists such an edge set $\bar{X}=\bar{X}\{p, n_j, m\}$ $(1 \leqq n_{j-1} < n_j \leqq n,\ 2 \leqq j \leqq m)$ of p taken in the fixed order determined by S that the edges marked by ${}^*X_{n_{i+1}}^*$ $(i=0, 1, \ldots, m-1)$ belong to k^i-multiple total edge systems of $\Phi_S'^t$, and if p' is any other path of $\Phi_S'^t$ then the connecting sequences of $\bar{X}=\bar{X}\{p, n_j, m\}$ and $\bar{X}'=\bar{X}'\{p', n_j, m\}$ relative to p and p' are the same (more precisely, are marked in order with the same variables ${}^uX_v^u$).

(3) There exist no subtree $\Phi_S''^t$ of $\Phi_S'^t$ that has properties (1) and (2) and which has more than m total branching points.

The structure of an MSST of order m is shown on Fig. 2.

Remark. $\varkappa_i$ is the sequence of edges between ${}^*X_{n-1}^*$ and ${}^*X_{n_i}^*$ in the order determined by S. If $n_i = n_{i-1}+1$ then $\varkappa_i$ is empty. If $m=0$ then $\Phi_S'^t = p$. It is obvious that if a tree Φ_S^t and its path p are given then there exists at least one MSST containing p.

Theorem 2. Let $f(\mathbf{X}^n) \in P_k$, Φ_S^t $(t=0, 1, \ldots, k-1)$ a tree belonging to a fixed S, p a path of Φ_S^t and $\mathfrak{M}$ an MSST of p. Let the n-term conjunction of variables along the paths of $\mathfrak{M}$ be: $p_1, p_2, \ldots, p_{k^l}$ $(1 \leqq l \leqq n)$, and the (variables at the) connecting sequence $\varkappa_1, \varkappa_2, \ldots, \varkappa_m$. Then

$$\sum_{j=1}^{k^1} p_j = \prod_{i=1} \varkappa_i$$

holds.

Proof. $\mathfrak{M}$ contains k^1 paths, so there are $k^0, k^1, \ldots, k^{m-1}$ total branchings on the different levels. In other words the formula F_t corresponding to Φ_S^t does not depend on the variables appearing in the total branchings because it takes the value t independently of these variables, so they can be omitted.

This theorem shows that every $\mathfrak{M}$ yields one term. The term which is obtained by the method above is called the simplified formula of $\mathfrak{M}$. The disjunction of such simplifications of MSST's is the simplified formula of the function.

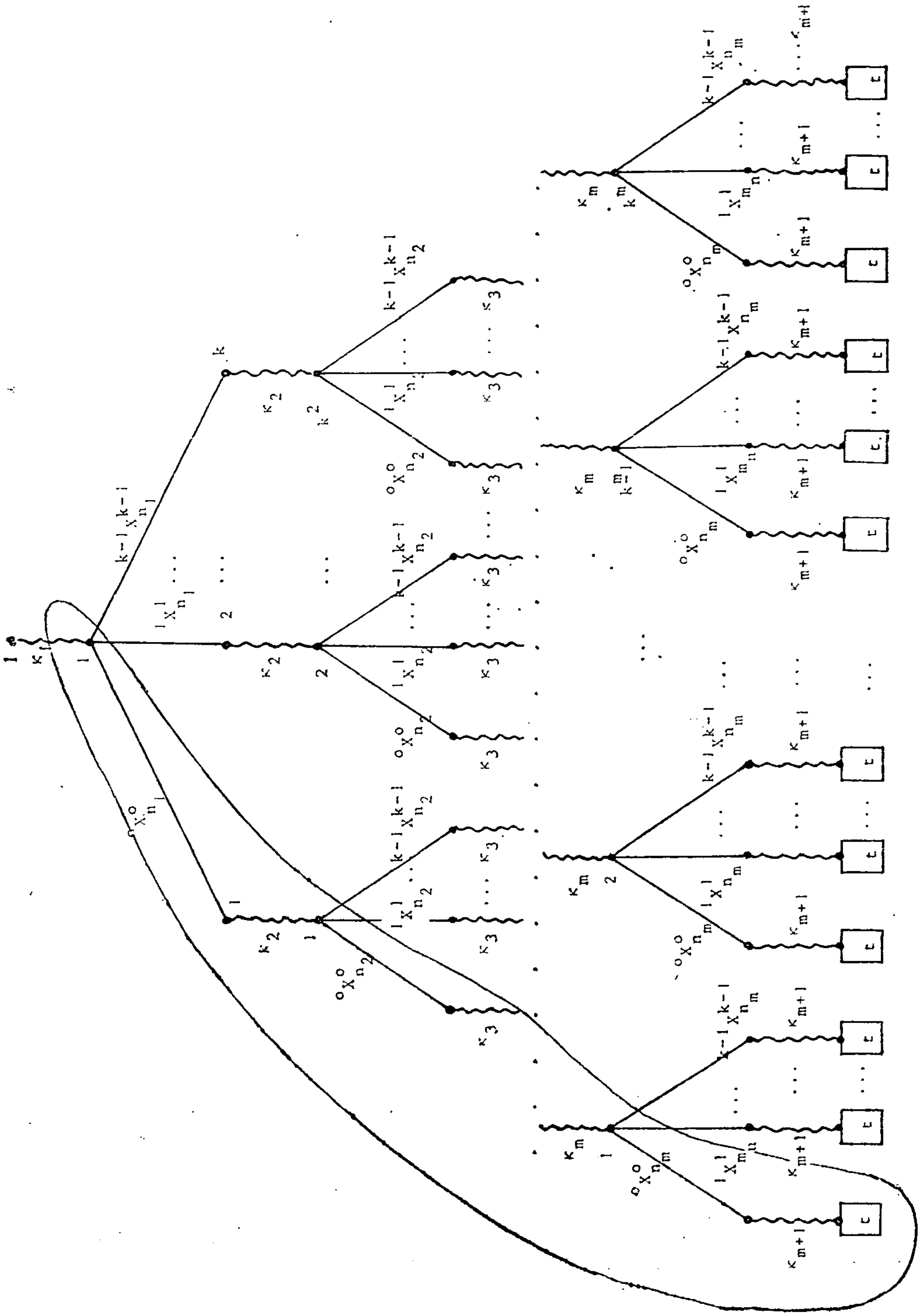


Fig. 2

4. Irredundant coverings

Definition 14. A set of MSST-s of a tree Φ_S^t $(t=0, 1, ..., k-1)$ is called a covering if each path of the tree belongs to at least one of the MSST-s of the set.

A covering is called irredundant if any MSST in it contains at least one path belonging only to this MSST.

Theorem 3. Let Φ_S^t represent the disjunctive normal form of an $f(X^n)\in P_s$, $(t=0, 1, ..., k-1)$, together with one of its irredundant coverings. Let $\tilde{F}_\vee$ denote the disjunction of simplified formulae obtained from the elements of the set of MSST-s giving the irredundant covering in question. Then $\tilde{F}_\vee$ is irredundant.

Proof. Suppose that $\tilde{F}_\vee$ is redundant. Then there exist two cases.

(1) some disjunction term of $\tilde{F}_\vee$ can be omitted;

(2) at least one variable can be omitted from some conjunction term of $\tilde{F}_\vee$.

First suppose that a term $F^{(1)}$ of $\tilde{F}_\vee$ can be omitted. Since every MSST gives only one conjunction term, omitting this is equivalent to omitting the MSST from the covering, but taking into account the irredundancy, this is impossible.

Secondly we note that, if an $F^{(i)}$ can be replaced by an $F'^{(i)}$ obtained from $F^{(i)}$ by omitting some variables, then the MSST giving $F'^{(i)}$ contains the MSST which gave $F^{(i)}$, but this contradict the definition of MSST.

Remark. Theorem 3 is formulated for full disjunctive normal forms, but because of the principle of duality it is true for full conjunctive normal forms as well.

5. Simplifiable paths, simplification algorithm

Definition 15. Let Φ_S^t $(t=0, 1, ..., k-1)$ be given, and take a path p of Φ_S^t.

— p is called singular if the MSST coincides with p.

— p is called simply covered if p is covered by one and only one MSST.

— p is multiply covered if it is covered by at least two MSST-s.

Theorem 4. Let $f(X^n)\in P_k$ be given by either its disjunctive or conjunctive full normal forms. If f is given by its full disjunctive normal form $F_\vee$ and some max term E_j^n is simultaneously represented by formulae $F_{l-m}, F_{l-m+1}, ..., F_{l-m+i}$, then

$$\min(F_{l-m}, F_{l-m+1}, ..., F_{l-m+i}) = F_{l-m}. \tag{1}$$

If f is given by the full conjunctive normal form $F_\wedge$ and some max term $\bar{E}_j^n$ is simultaneously represented by formulae $F_{l-m}, F_{l-m+1}, ..., F_{l-m+i}$, then

$$\max(F_{l-m}, F_{l-m+1}, ..., F_{l-m+i}) = F_{l-m+i}. \tag{2}$$

The statement can easily be proved taking into account the definitions of the min and max operators.

Formula (1) means that the simplification procedure of a function f (or tree Φ_S which is representing the function and is written from the disjunctive normal form) value (for example in case $\varepsilon_k=\{0, 1, ..., k-1\}$ with Φ_S^{k-1}). After the first step of the simplification the endpoints marked with $(k-1)$ can be considered of $(*)$-value, that is undefined, in the tree Φ_S. Let us introduce the following notations:

(i) $$\Phi_S^{k-2,*} = \Phi_S^{k-1} \cup \Phi_S^{k-2}$$

and

(ii) $$\Phi_S^{k-i,*} = \Phi_S^{k-(i-1),*} \cup \Phi_S^{k-i} \quad (i > 2)$$

where $l=k-j$ and $(*)$ is written on the places $j<i$. By virtue of Theorem 4, there are subtrees which may give more favourable conditions for simplifiction.

On the other hand relation (2) shows in case of tree of functions given by full conjunctive normal form that simplification has to be started with the simplification of that subtree determined by the path with smallest logical value and we have to apply the method above. Below the procedure will be shown only for functions given by their full disjunctive normal forms. The case of full conjunctive normal forms can be treated in a similar way.

Now we can give the simplification procedure.

6. Simplification algorithm for representations of irredundant formulae

(1) Let $i=1$. Mark the paths with endpoint $t=k-1$ in the tree Φ_S (that is we start from the subtree Φ_S^{k-1}). If in the tree Φ_S originally there are endpoints marked with $(*)$, then we begin with $\Phi_S^{k-1} \cup \Phi_S^*$.

We choose a path and an MSST containing it. We take a record of the simplified formulae corresponding to this MSST and mark the paths in it.

(2) We choose an unmarked path and determine an MSST covering it, preferably with unmarked endpoints (this will speed up the algorithm). This way such an MSST is chosen which is necessary for an irredundant covering. The simplified formula belonging to the MSST we have just obtained will be taken record of and the so far unmarked paths of the MSST will be marked.

Repeat step 3 until we can find unmarked paths in Φ_S^{k-1}.

If there is no unmarked path, then let $i=i+1$. If $i<k$, then consider the subtree $\Phi_S^{k-1,*}$ and carry out the above steps (1), (2), (3). If $i=k$ the algorithm is over.

Finally the simplified formula of the function $f(X^n)$ can be determined as follows:

Let

$$F_{s,1}^{k-1}, F_{s,2}^{k-1}, \ldots, F_{s,i_1}^{k-1}$$

$$F_{s,1}^{k-2}, F_{s,2}^{k-2}, \ldots, F_{s,i_2}^{k-2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$F_{s,1}^{1}, F_{s,2}^{1}, \ldots, F_{s,i_{k-1}}^{1},$$

denote the simplified formulae obtained from the subtrees $\Phi_S^{k-1}, \Phi_S^{k-2,*}, \ldots, \Phi_S^{1,*}$;

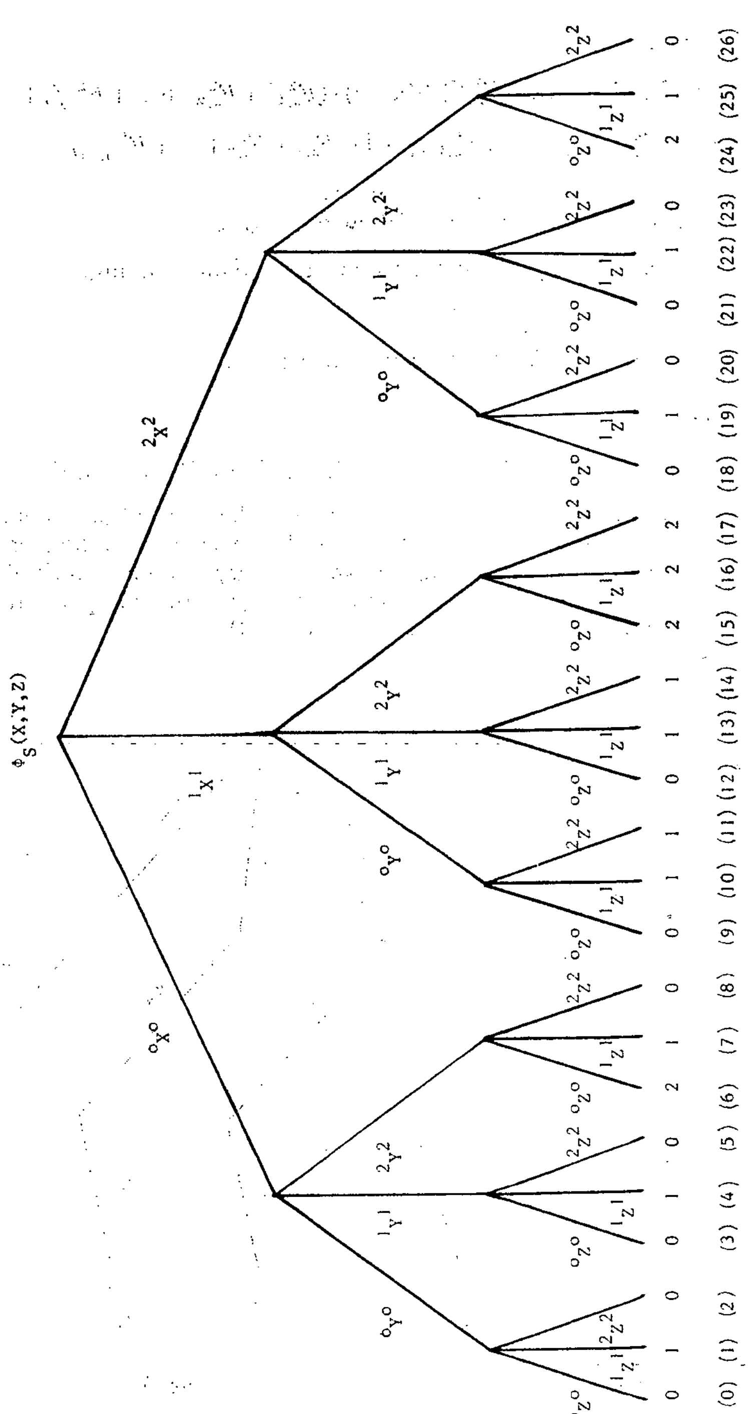

$\Phi_S(X,Y,Z)$
${}^0X^0$
${}^1X^1$
${}^2X^2$
${}^0Y^0$
${}^1Y^1$
${}^2Y^2$
${}^0Z^0$
${}^1Z^1$
${}^2Z^2$
0 1 0 0 1 0 2 1 0 0 1 1 0 1 1 2 2 2 0 1 0 0 1 0 2 1 0
(0) (1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) (13) (14) (15) (16) (17) (18) (19) (20) (21) (22) (23) (24) (25) (26)

Fig. 3

respectively. Then the formula:

$$(k-1)\cdot(F^{k-1}_{s,1}+F^{k-1}_{s,2}+\dots+F^{k-1}_{s,i_1})+(k-2)\cdot(F^{k-2}_{s,1}+F^{k-2}_{s,2}+\dots+F^{k-2}_{s,i_2})+\dots$$
$$+2\cdot(F^{2}_{s,1}+F^{2}_{s,2}+\dots+F^{2}_{s,k-2})+1\cdot(F^{1}_{s,1}+F^{1}_{s,2}+\dots+F^{1}_{s,i_{k-1}})$$

corresponds to an irredundant covering of Φ_S.

All these can be summarised in the following theorem.

Theorem 5. Every tree Φ_S has at least one irredundant covering.

7. Some demonstrative examples

I. Consider the function

$$f^3(X,Y,Z) = 1\Sigma(1,4,7,10,11,13,14,19,22,25)+2\Sigma(6,15,16,17,24)$$

given by its full disjuntive normal form (here we use the conventional notation of binary logic; only the numbers in brackets should be considered as numbers in the number system with base k instead of 2). We will simplify the function $f^3(X, Y, Z)$. Let $S\equiv(X, Y, Z)$ be the order of evaluation. Fig. 3 gives the complete tree of f^3 With $k=2$ pick the tree Φ^2_S and let us analyse it (Fig. 4).

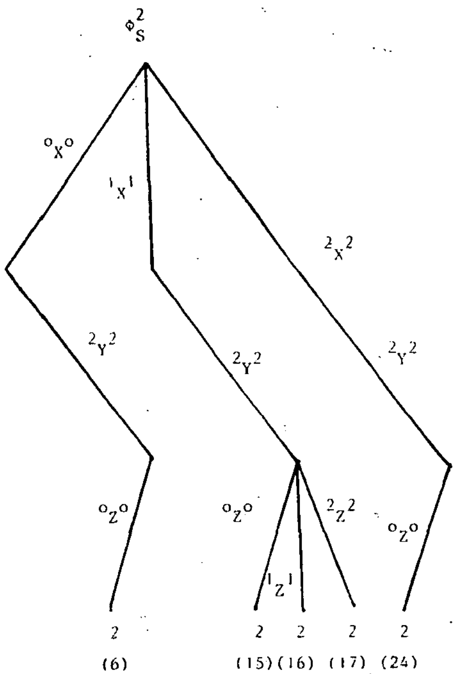


Fig. 4

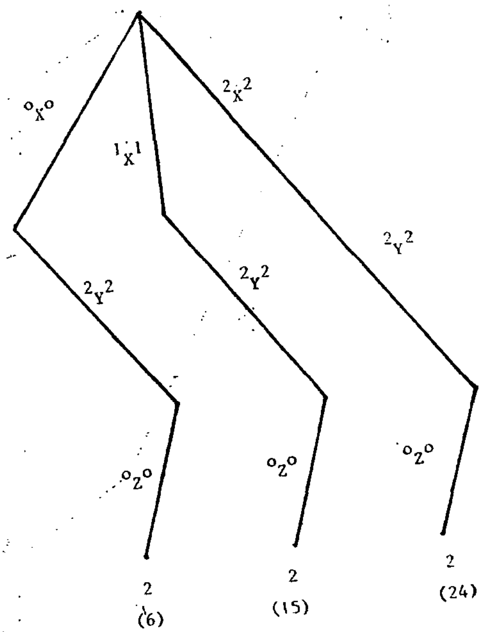


Fig. 5

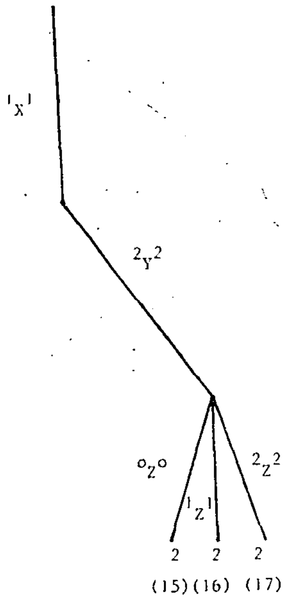


Fig. 6

Let us investigate the paths of this tree moving from the left to the right

1. There is no singular path.

2. Simply covered paths: (6), (16), (17). The MSST belonging to (6) is (6—15—24) (Fig. 5). The next path is (16) and the corresponding MSST is (15—16—17) (Fig. 6).

The simplified formulae

$$^2Y^2 \cdot {}^0Z^0$$

$$^1X^1\, {}^2Y^2$$

3. There is no more unmarked path.

We write $(*)$ instead of 2 and consider $\Phi_S^{1,*}$ with $k=1$. (Fig. 7)

1. There is no singular path.

2. Simply covered paths are:

(1) and the corresponding MSST is (1—4—7—10—13—16—19—22—25) (Fig. 8), (11) and the MSST is (11—14—17) (Fig. 9).

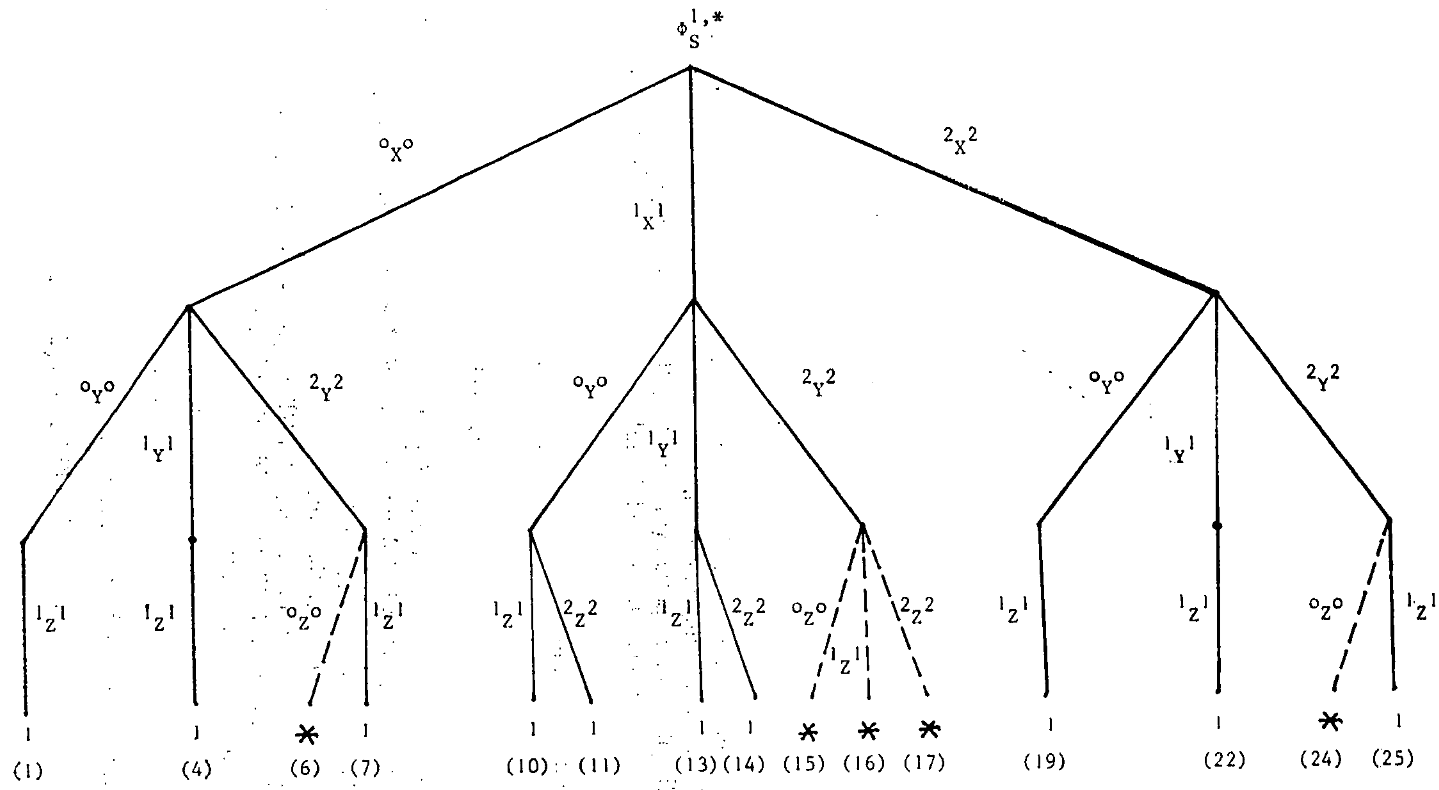
The simplified formulae are:

$$^1Z^1$$

$$^1X^1 \cdot {}^2Z^2$$

The simplified irredundant formula is:

$$2({}^1X^1\, {}^2Y^2 + {}^2Y^2\, {}^0Z^0) + 1({}^1Z^1 + {}^1X^1\, {}^2Z^2).$$



$\Phi_S^{1,*}$
$^0X^0$
$^1X^1$
$^2X^2$
$^0Y^0$
$^1Y^1$
$^2Y^2$
$^0Z^0$
$^1Z^1$
$^2Z^2$
(1) (4) (6) (7) (10) (11) (13) (14) (15) (16) (17) (19) (22) (24) (25)

Fig. 7

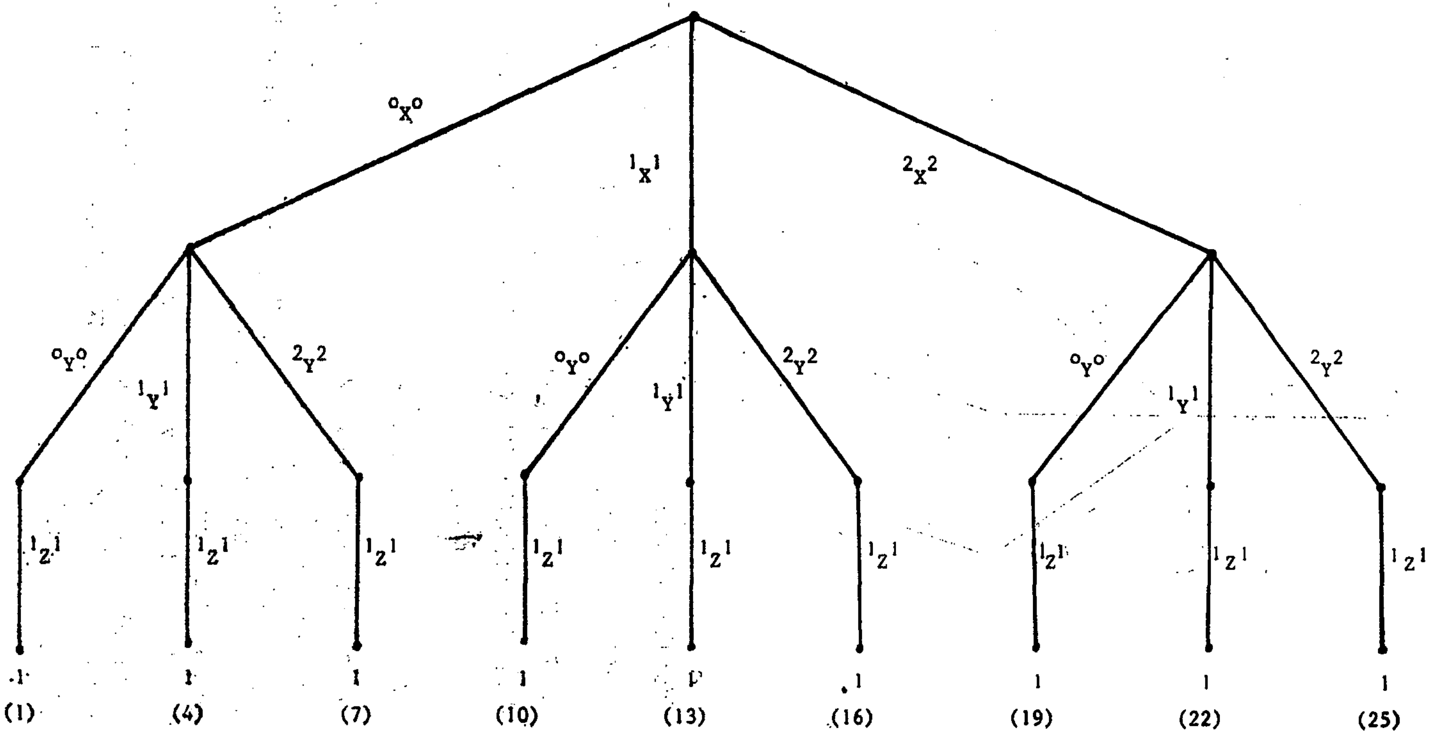


Fig. 8

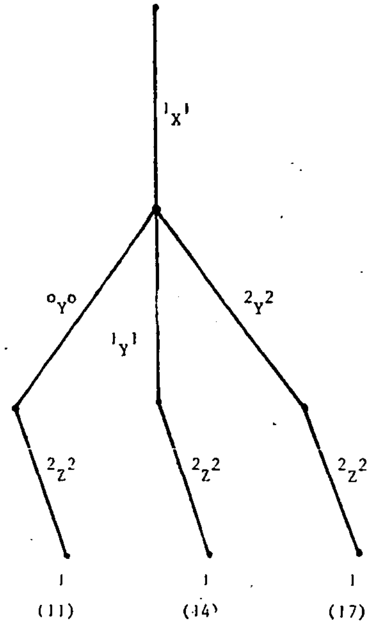


Fig. 9

II. Consider the following function

$$f^4(X, y) = 1\Sigma(3, 4, 6, 7)+2\Sigma(1, 5, 13)+3\Sigma(9, 10, 11, 14)$$

and let $S\equiv(X, Y)$. Simplify this function.

1. Singular paths are: (9), (10), (11), (14) and the formulae belonging to these are:

$$^2X^2\,{}^1Y^1,\ {}^2X^2\,{}^2Y^2,\ {}^2X^2\,{}^3Y^3,\ {}^3X^3\,{}^2Y^2.$$

2. There is no more unmarked path.

We write $(*)$ instead of 3 and let $k=2$.

1. There is no singular path.
2. Simply covered path is: (1) and the MSST is (1—5—9—13) (Fig. 13)
3. There is no more unmarked path.

The simplified formula is ${}^1Y^1$.
We write instead of 2 and 3 now $(*)$ and let $k=1$.

1. Singular path is: (3) and the corresponding formula is: ${}^0X^0\,{}^3Y^3$
2. Simply covered path is (4) and the MSST is (4—5—6—7) (Fig. 15).

The simplified formula is: ${}^1X^1$

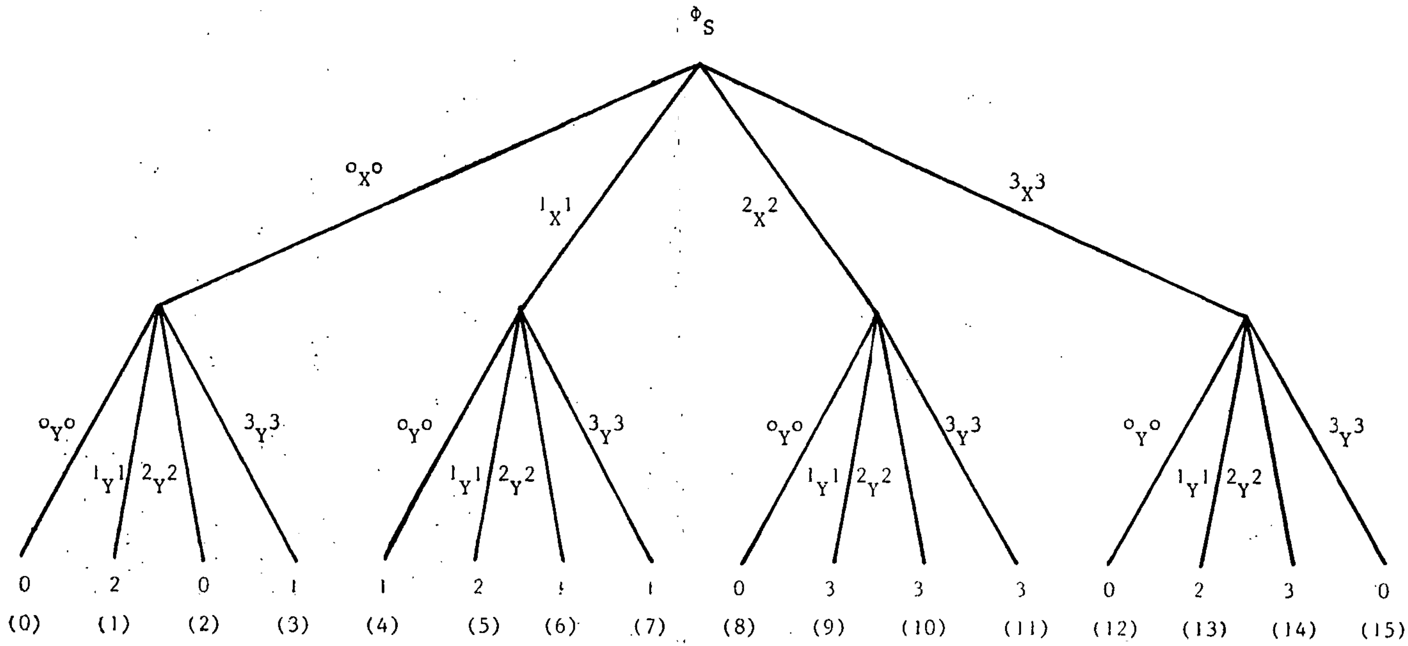

Φ_S
$^0X^0$
$^1X^1$
$^2X^2$
$^3X^3$
$^0Y^0$
$^1Y^1$
$^2Y^2$
$^3Y^3$
0 2 0 1 1 2 1 1 0 3 3 3 0 2 3 0
(0) (1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) (13) (14) (15)

Fig. 10

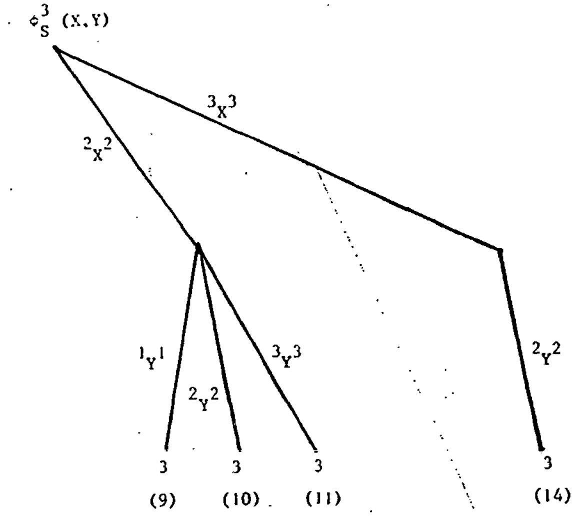


Fig. 11

3. The simplified function is:

$$3({}^2X^2\,{}^1Y^1+{}^2X^2\,{}^2Y^2+{}^2X^2\,{}^3Y^3+{}^3X^3\,{}^2Y^2)+2{}^1Y^1+1({}^0X^0\,{}^3Y^3+{}^1X^1).$$

Remark. The irredundant formula we have just obtained can be transformed by virtue of indentities treated above.

For example:

$$3({}^2X^2({}^1Y^1+{}^3Y^3)+{}^2Y^2({}^2X^2+{}^3X^3))+2{}^1Y^1+1({}^0X^0\,{}^3Y^3+{}^1X^1)=$$

$$=3({}^2X^2\,{}^1Y^3+{}^2X^3\,{}^2Y^2)+2{}^1Y^1+1({}^0X^0\,{}^3Y^3+{}^1X^1).$$

III. Let $f^3(X, Y, Z)$ be given by its truth-table (Fig. 16). Simplify this function Let $S\equiv(X, Y, Z)\cdot\Phi_S(X, Y, Z)$ is sketched in Fig. 17. For the endpoints marked with $k=2$ and $*$ we have:

1. There is no singular path.
2. Simply covered paths are:

(i) (13) and the corresponding MSST is (4—13—22) (marked with +) (Fig. 17). The simplified formula is: ${}^1Y^1\,{}^1Z^1$;

(ii) (21), the MSST is (21—22—23) (marked with o) and the simplified formula is ${}^2X^2\,{}^1Y^1$.

(iii) (24), the MSST is (18—21—24) (marked with "=") and the simplified formula is ${}^2X^2\,{}^0Z^0$.

3. There is no more unmarked path with endpoint 2. Consider now the subtree with endpoints $k=1$, $2\equiv *$ and $*$

1. There is no singular path.
2. Simply covered paths are:

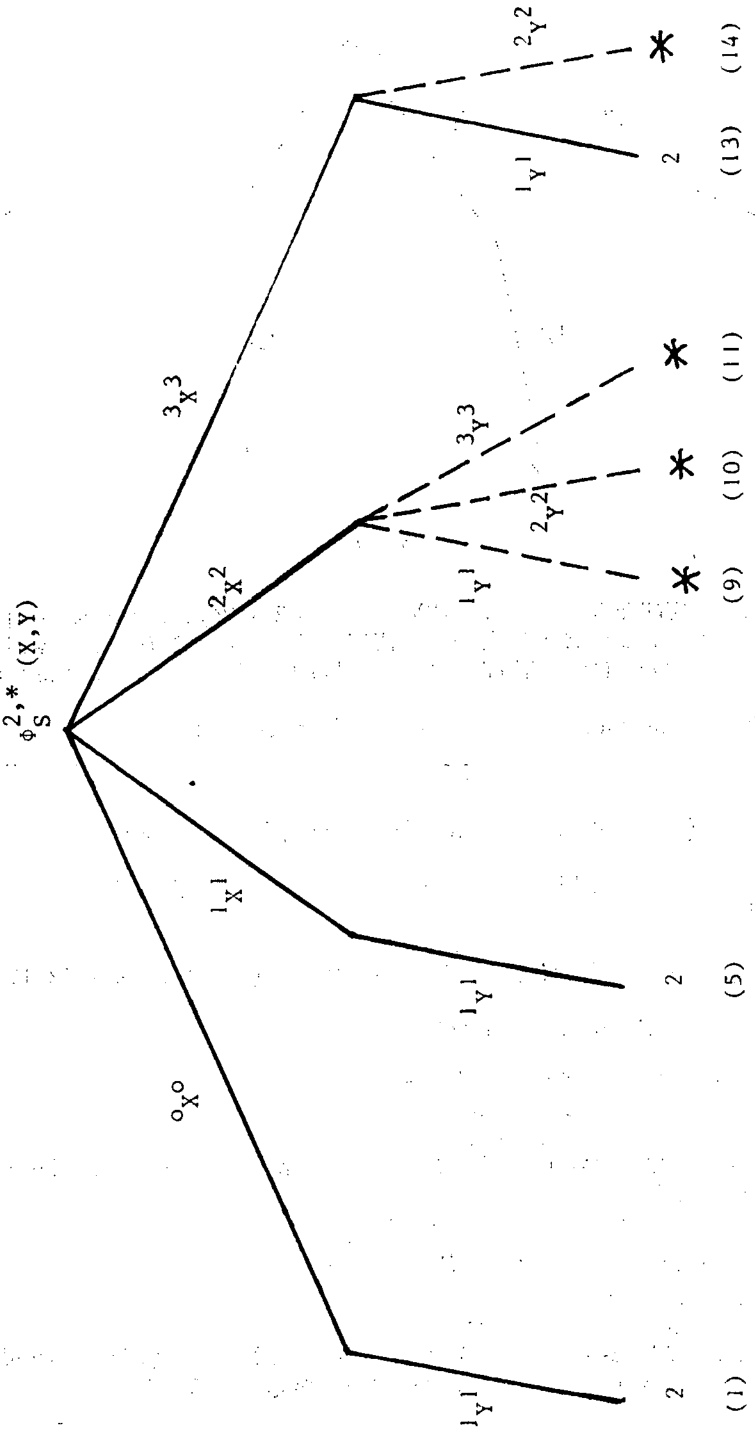


Fig. 12

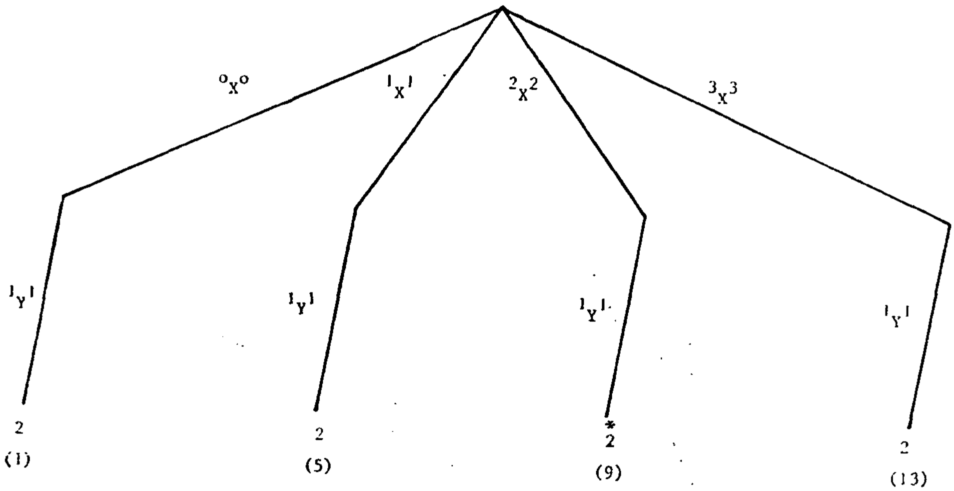


Fig. 13

(0), the MSST is (0—3—6—9—12—15—18—21—24) (marked with ⊡) and the simplified formula is: $^0Z^0$; (10), the MSST is (9—10—11) (marked with •), the formula is $^1X^1\,^0Y^0$; (16) MSST: (15—16—17) (marked with X), the formula is: $^1X^1\,^2Y^2$.

3. There is no more unmarked path with endpoint 1.

The simplified formula of the function is:

$$2(^1Y^1\,^1Z^1+{}^2X^2\,^1Y^1+{}^2X^2\,^0Z^0)+1(^0Z^0+{}^1X^1\,^0Y^0+{}^1X^1\,^2Y^2) =$$

$$=2(^1Y^1\,^1Z^1+{}^2X^2\,^1Y^1+{}^2X^2\,^0Z^0)+1(^0Z^0+{}^1X^1\,^1Y^2).$$

IV. Let

$$f^4(X,Y) = 1\Sigma(5,8,9,11)+2\Sigma(2,6,10)+3\Sigma(13,14)+*\Sigma(1,12,15)$$

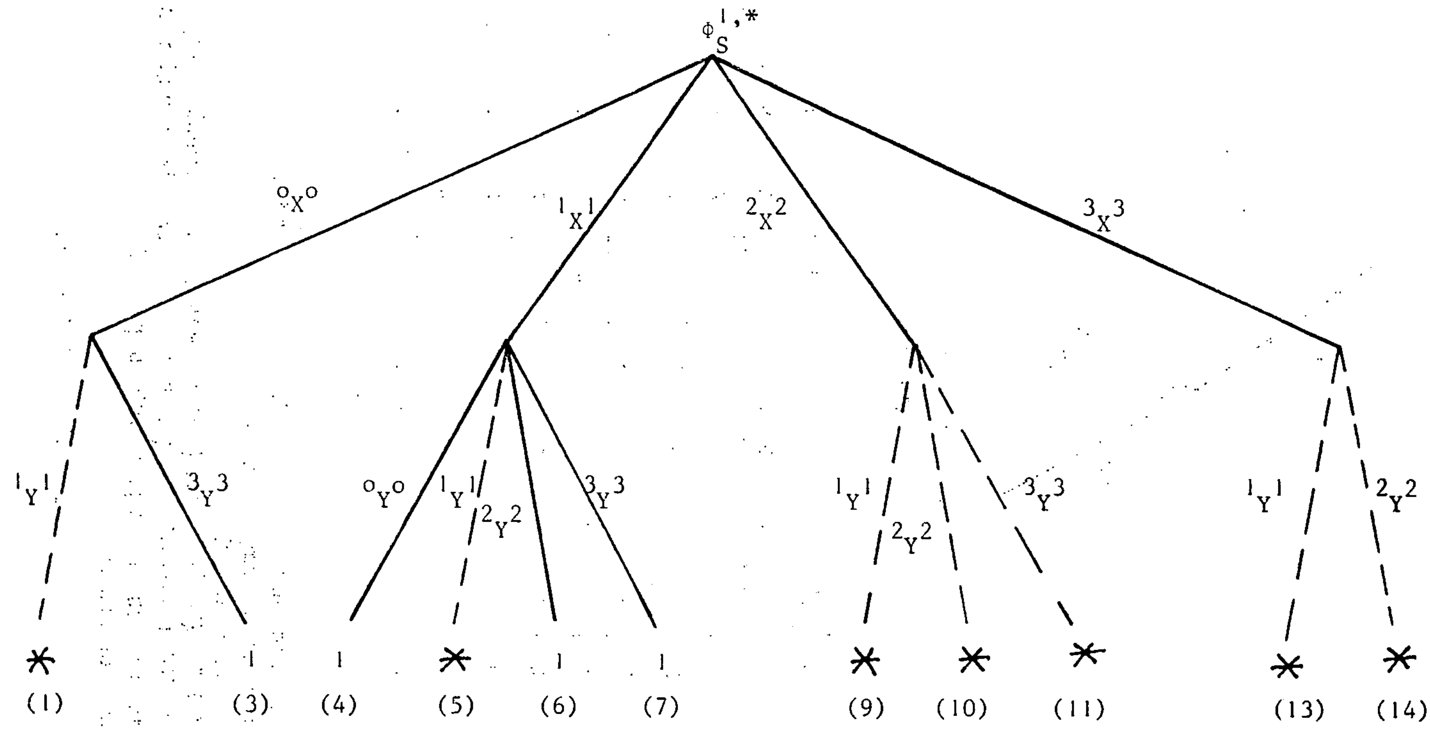
and $S\equiv(X,Y)$. Simplify this function

For the paths with endpoints $k=3$ and $*$:

1. There is no singular path,
2. Simply covered paths are:
   (13) MSST: (12—13—14—15) (marked with ∅) (Fig. 18) the simplified formula: $^3X^3$.
3. There is no more path with endpoint $k=3$.

For the paths with endpoints $k=2$, $3\equiv*$ and $*$:

1. There is no singular path.
2. Simply covered paths are:
   (2) MSST: (2—6—10—14) (marked with +) the simplified formula is $^2Y^2$.
3. There is no more unmarked path with endpoint $k=2$.



$\Phi_S^{1,*}$
${}^0X^0$
${}^1X^1$
${}^2X^2$
${}^3X^3$
${}^1Y^1$
${}^3Y^3$
${}^0Y^0$
${}^1Y^1$
${}^2Y^2$
${}^3Y^3$
${}^1Y^1$
${}^2Y^2$
${}^3Y^3$
${}^1Y^1$
${}^2Y^2$
(1)
(3)
(4)
(5)
(6)
(7)
(9)
(10)
(11)
(13)
(14)

Fig. 14

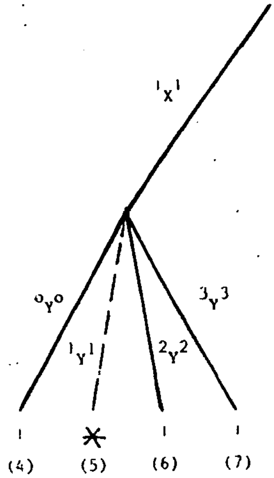


Fig. 15

X	Y	Z	f^3	X	Y	Z	f^3	X	Y	Z	f^3
0	0	0	1	1	0	0	1	2	0	0	*
0	0	1	0	1	0	1	1	2	0	1	0
0	0	2	*	1	0	2	*	2	0	2	0
0	1	0	1	1	1	0	1	2	1	0	2
0	1	1	*	1	1	1	2	2	1	1	2
0	1	2	0	1	1	2	0	2	1	2	2
0	2	0	1	1	2	0	1	2	2	0	2
0	2	1	0	1	2	1	1	2	2	1	0
0	2	2	*	1	2	2	1	2	2	2	0

Fig. 16

For the paths with endpoints $k=1$, $2\equiv *, 3\equiv *$, and $*$.

1. There exists no singular path.
2. Simply covered paths:
   (5) MSST: (1—5—9—13) (marked with □) the formula: ${}^1Y^1$;
   (8) MSST: (8—9—10—11) (marked with X) and the formula: ${}^2X^2$.
3. There is no unmarked path with endpoint $k=1$.

The simplified formula is:

$$3\,{}^3X^3+2\,{}^2Y^2+1\,({}^2X^2+{}^1Y^1).$$

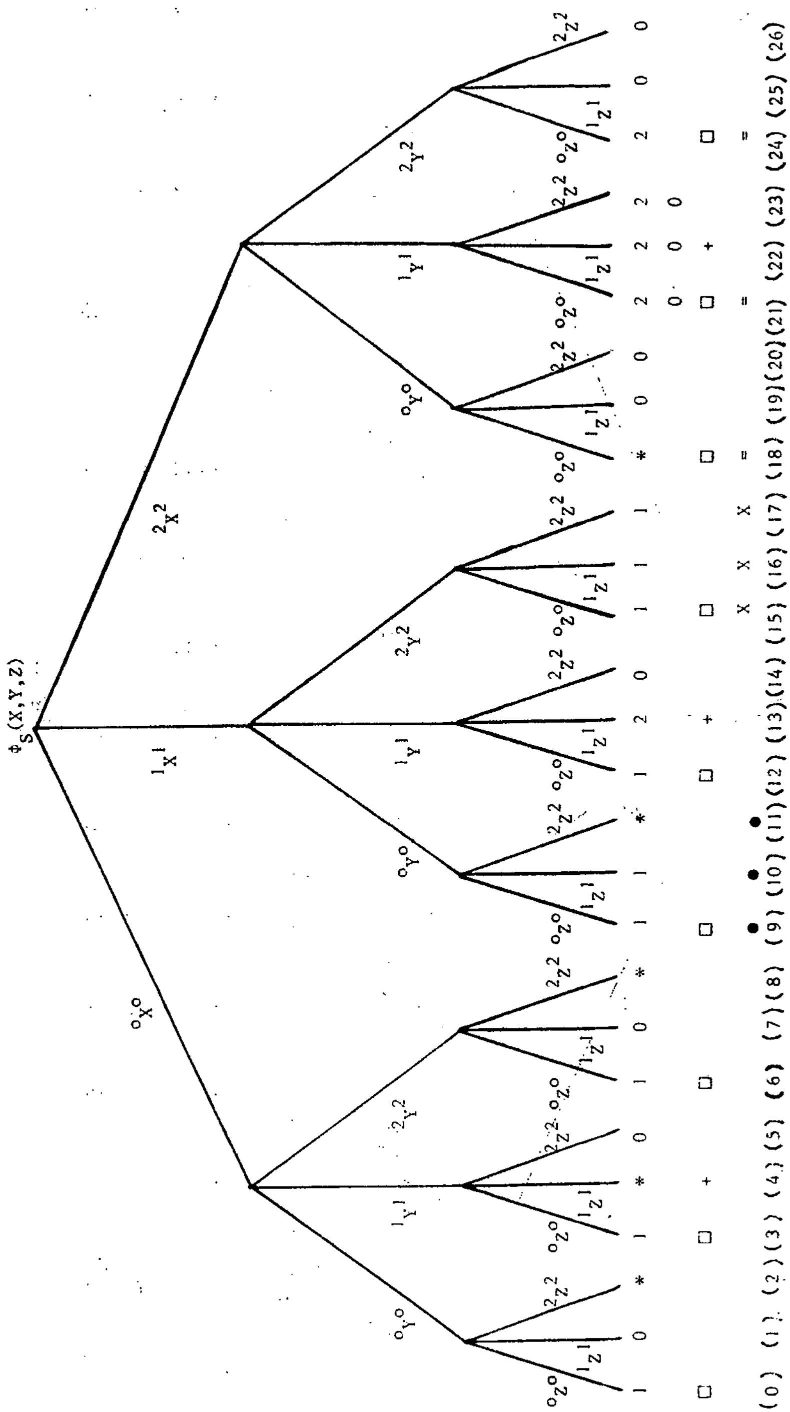
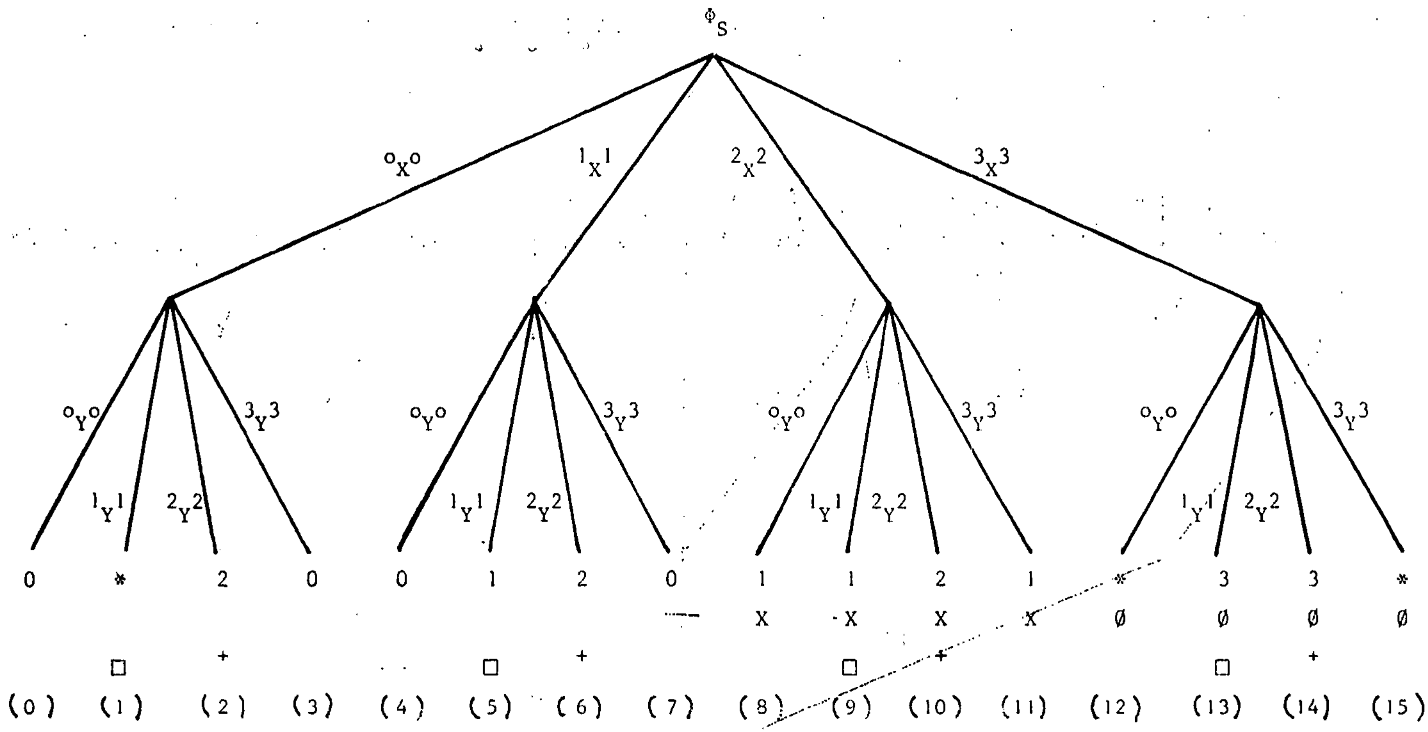

$\Phi_S(X,Y,Z)$
$^0X^0$
$^1X^1$
$^2X^2$
(0) (1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) (13) (14) (15) (16) (17) (18) (19) (20) (21) (22) (23) (24) (25) (26)

Fig. 17



ΦS
0X0
1X1
2X2
3X3
0Y0
1Y1
2Y2
3Y3
0 * 2 0 0 1 2 0 1 1 2 1 * 3 3 *
— X X X X Ø Ø Ø Ø
□ + □ + □ + □ +
(0) (1) (2) (3) (4) (5) (6) (7) (8) (9) (10) (11) (12) (13) (14) (15)

Fig. 18

JÓZSEF A. UNIVERSITY
BOLYAI INSTITUTE
SZEGED
ARADI VÉRTANÚK TERE 1.

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