An Erdős—Ko—Rado type theorem II

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1. Introduction and results

Let R denote the interval $[1, r]$ of the first r positive integers. Let k be an integer with $0 \leqq k \leqq r$. The set of all k-element subsets of R will be denoted by $\binom{R}{k}$. The aim of this paper is to present the

Theorem 1. *Let* $\mu \geqq 4$ *and* $\nu \geqq 4$ *be integers. If* $F \subseteqq \binom{R}{k}$,

$$\frac{r-1}{\nu}+1 \leqq k \leqq \frac{\mu-1}{\mu}(r-1), \tag{1}$$

and F satisfies

$$X_1 \cap X_2 \cap \ldots \cap X_\mu \neq \emptyset \quad \text{for all} \quad X_1, X_2, \ldots, X_\mu \in F, \tag{2}$$

as well as

$$X_1 \cup X_2 \cup \ldots \cup X_\nu \neq R \quad \text{for all} \quad X_1, X_2, \ldots, X_\nu \in F, \tag{3}$$

then

$$|F| \leqq \binom{r-2}{k-1}.$$

This is best possible. The families $F_{x,y} = \left\{X \in \binom{R}{k} : x \in X, y \notin X\right\}$, *where x and y are different fixed elements of R, are maximal.*

This theorem was proved, for $\mu \geqq 6$ and $\nu \geqq 6$ and for some partial cases of k if $\mu = 4,5$ or $\nu = 4,5$, in *Gronau* [2]. Our proof here uses the same method but in a refined version.

Condition (1) is natural. For all other k's one of the conditions (2) or (3) is satisfied automatically, and the problem reduces to the generalized *Erdős—Ko—Rado* theorem by *Frankl* [1]. For another simple proof, see *Gronau* [3].

Theorem 2. *(generalized Erdős—Ko—Rado theorem)* *Let* $\mu \geqq 2$ *be an integer. If* $F \subseteqq \binom{R}{k}$, $0 \leqq k \leqq \frac{\mu-1}{\mu} r$, *and F satisfies* (2), *then*

$$|F| \leqq \binom{r-1}{k-1}.$$

Turning to the complements we obtain a dual version.

Theorem 2'. *Let* $\nu \geqq 2$ *be an integer. If* $F \subseteqq \binom{R}{k}$, $\frac{r}{\nu} \leqq k \leqq r$, *and F satisfies* (3), *then*

$$|F| \leqq \binom{r-1}{k}.$$

2. Some reductions

Let $\mu, \nu \geqq 4$, k and $F \subseteqq \binom{R}{k}$ be given such that (1), (2) and (3) hold. If

$$\bigcap_{X \in F} X \neq \emptyset \quad \text{or} \quad \bigcup_{X \in F} X \neq R \quad \text{then} \quad |F| \leqq \binom{r-2}{k-1}$$

follows by Theorem 2 or 2′ immediately. Since the described families $F_{x,y}$ have cardinality $\binom{r-2}{k-1}$ and satisfy (2) as well as (3), the proof of Theorem 1 will be completed by proving

Theorem 3. *Let* $\mu \geqq 4$ *and* $\nu \geqq 4$ *be integers. If* $F \subseteqq \binom{R}{k}$ *and F satisfies* (2) *and* (3) *as well as* $\bigcap_{X \in F} X = \emptyset$ *and* $\bigcup_{X \in F} X = R$, *then*

$$|F| < \binom{r-2}{k-1}.$$

Observe that here is no restriction on k. Therefore, it is sufficient to prove Theorem 3 only for $\mu = \nu = 4$. Furthermore, we may restrict ourselves to $k \leqq \frac{r}{2}$ in the proof since $k > \frac{r}{2}$ follows by duality. We make use of some results from [2].

Proposition 1. ([2, Lemma 1]).

$$|X_1 \cap X_2| \geqq 3,$$

$$|X_1 \cap X_2 \cap X_3| \geqq 2 \quad \textit{for all} \quad X_1, X_2, X_3 \in F.$$

Proposition 2. *We may suppose that for all* $X\in F$ *it holds: If* $i\notin X$, $j\in X$ *and* $i<j$, *then* $(X-\{j\})\cup\{i\}\in F$.

The last proposition is a consequence of Lemma 4 in [2], by the *Erdős—Ko—Rado* exchange operation.

Finally we prove Theorem 3 for small k, similarly to [2], by a short argument.

Lemma 1. *Theorem* 3 *is true for* $k\leqq\frac{r}{4}+\frac{3}{2}$.

Proof. By Theorem 6 in [2], $|F|\leqq\binom{r}{k-3}$. Hence,

$$\frac{|F|}{\binom{r-2}{k-1}}\leqq\frac{\binom{r}{k-3}}{\binom{r-2}{k-1}}=\frac{r(r-1)(k-1)(k-2)}{(r-k+3)(r-k+2)(r-k+1)(r-k)}\leqq$$

$$\leqq\frac{r(r-1)\left(\frac{r}{4}+\frac{1}{2}\right)\left(\frac{r}{4}-\frac{1}{2}\right)}{\left(\frac{3}{4}r+\frac{3}{2}\right)\left(\frac{3}{4}r+\frac{1}{2}\right)\left(\frac{3}{4}r-\frac{1}{2}\right)\left(\frac{3}{4}r-\frac{3}{2}\right)}=\frac{16}{27}\frac{r}{r+\frac{2}{3}}\frac{r-1}{r-\frac{2}{3}}\frac{r+2}{r+2}\frac{r-2}{r-2}<1.\ \square$$

3. An upper bound for |F|

Suppose that F satisfies the suppositions of Theorem 3, and $\frac{r}{4}+\frac{3}{2}<k\leqq\frac{r}{2}$. We decompose F into F_1, F_2, and F_3 according to

$$F_1=\{X\in F:\ \{1,2\}\subseteqq X\},$$

$$F_2=\{X\in F:\ 1\in X,2\notin X\},$$

$$F_3=\{X\in F:\ 1\notin X\}.$$

i) Let $F_1'=\{X:\ X\cup\{1,2\}\in F_1,\ \{1,2\}\cap X=\emptyset\}$. Then F_1' is a family of $(k-2)$-element subsets of the $(r-2)$-element set $\{3,4,\dots,r\}$ satisfying (3) for $v=4$. Since $k-2>\left(\frac{r}{4}+\frac{3}{2}\right)-2=\frac{r-2}{4}$, we may apply Theorem 2′ and obtain

$$|F_1|=|F_1'|\leqq\binom{r-3}{k-2}. \tag{4}$$

In order to estimate $|F_2|$ and $|F_3|$ we use the description of the families by walks in the plane. We associate with every $X\in\binom{R}{k}$ a certain walk. We start from $(0,0)$. If we are after i moves at point (a,b) then we turn to $(a,b+1)$ or $(a+1,b)$ depending on whether $i+1\in X$ or $i+1\notin X$. So every set of $\binom{R}{k}$ is associated with a walk from $(0,0)$ to $(r-k,k)$ and vice versa.

Let F_2' and F_3' denote the set of walks associated with F_2 and F_3, respectively. By the definition of F_2 and F_3, every walk of F_2' starts with (0, 0)—(0, 1)—(1, 1) whereas every walk of F_3' starts with (0, 0)—(1, 0).

ii) Every walk of F_2' meets the line $y=2x+2$, since otherwise, by Proposition 2, F_2 would contain the set $X_1=\{1, 3, 4, 6, 7, 9, 10, \ldots\}$. For the same reason, F would contain $X_2=\{1, 2, 4, 5, 7, 8, 10, \ldots\}$ and $X_3=\{1, 2, 3, 5, 6, 8, 9, \ldots\}$. But $|X_1\cap X_2\cap X_3|=|\{1\}|=1$, contradicting Proposition 1.

If a walk meets the line $y=2x+2$ the first time at $(i, 2i+2)$, $i\geqq 1$, then this walk passes through $(i, 2i-1)$, too. Hence the number of these walks is not greater than

$$\binom{3i-3}{i-1}\binom{r-3i-2}{k-2i-2}$$

since $\binom{3i-3}{i-1}$ is the total number of walks from (1, 1) to $(i, 2i-1)$, whereas $\binom{r-3i-2}{k-2i-2}$ is the total number of walks from $(i, 2i+2)$ to $(r-k, k)$. Consequently, using $\binom{0}{0}=1$, we obtain

$$|F_2|=|F_2'|\leqq\sum_{i=1}^{\left[\frac{k-2}{2}\right]}\binom{3i-3}{i-1}\binom{r-3i-2}{k-2i-2}. \tag{5}$$

iii) Every walk of F_3' meets the line $y=3x+1$. This follows by the same arguments as in the preceding case recalling (2). Thus,

$$|F_3|=|F_3'|\leqq\sum_{i=1}^{\left[\frac{k-1}{3}\right]}\binom{4i-4}{i-1}\binom{r-4i-1}{k-3i-1}. \tag{6}$$

By (4), (5), and (6) we obtain

$$|F|\leqq\binom{r-3}{k-2}+\sum_{i=1}^{\left[\frac{k-2}{2}\right]}\binom{3i-3}{i-1}\binom{r-3i-2}{k-2i-2}+\sum_{i=1}^{\left[\frac{k-1}{3}\right]}\binom{4i-4}{i-1}\binom{r-4i-1}{k-3i-1}. \tag{7}$$

4. Some lemmas

In order to estimate (7) we need the following lemmas.

Lemma 2. *For any natural numbers n and i with $n\geqq 2$,*

$$\frac{\binom{n(i+1)}{i+1}}{\binom{ni}{i}}\leqq\frac{n^n}{(n-1)^{n-1}};$$

Proof

$$\frac{\binom{n(i+1)}{i+1}}{\binom{ni}{i}} = \frac{(n(i+1))!\,i!\,((n-1)i)!}{(i+1)!\,((n-1)(i+1))!\,(ni)!} = \frac{n(i+1)}{i+1}\prod_{j=1}^{n-1}\frac{ni+j}{(n-1)i+j} \leqq$$

$$\leqq n\prod_{j=1}^{n-1}\frac{n}{n-1} = \frac{n^n}{(n-1)^{n-1}}. \quad \square$$

Lemma 3. *For integers r, k, i satisfying $k \leqq \frac{r}{2}$ and $i \geqq 1$ we have*

$$a) \quad \frac{\binom{r-3i-5}{k-2i-4}}{\binom{r-3i-2}{k-2i-2}} \leqq \frac{1}{8} \quad \text{if} \quad i \leqq \frac{k-4}{2},$$

$$b) \quad \frac{\binom{r-4i-5}{k-3i-4}}{\binom{r-4i-1}{k-3i-1}} \leqq \frac{1}{16} \quad \text{if} \quad i \leqq \frac{k-4}{3}.$$

Proof. Since, for positive α and β; $\alpha \cdot \beta \leqq \left(\frac{\alpha+\beta}{2}\right)^2$, and $k \leqq \frac{r}{2}$, we have

$$a) \quad \frac{\binom{r-3i-5}{k-2i-4}}{\binom{r-3i-2}{k-2i-2}} = \frac{(r-3i-5)!\,(k-2i-2)!\,(r-k-i)!}{(k-2i-4)!\,(r-k-i-1)!\,(r-3i-2)!} =$$

$$= \frac{(k-2i-2)(k-2i-3)(r-k-i)}{(r-3i-2)(r-3i-3)(r-3i-4)} \leqq \frac{\frac{r}{2}-2i-2}{r-3i-4}\,\frac{\left(\frac{r-3i-3}{2}\right)^2}{(r-3i-3)^2} \leqq$$

$$\leqq \frac{1}{8}\,\frac{r-4i-4}{r-3i-4} \leqq \frac{1}{8}.$$

$$b) \quad \frac{\binom{r-4i-5}{k-3i-4}}{\binom{r-4i-1}{k-3i-1}} = \frac{(r-4i-5)!\,(k-3i-1)!\,(r-k-i)!}{(k-3i-4)!\,(r-k-i-1)!\,(r-4i-1)!} =$$

$$= \frac{(k-3i-1)(k-3i-3)(k-3i-2)(r-k-i)}{(r-4i-3)(r-4i-4)(r-4i-1)(r-4i-2)} \leqq$$

$$\leqq \frac{\frac{r}{2}-3i-1}{r-4i-3}\,\frac{\frac{r}{2}-3i-3}{r-4i-4}\,\frac{\left(\frac{r-4i-2}{2}\right)^2}{(r-4i-2)^2} \leqq$$

$$\leqq \frac{1}{16}\,\frac{r-6i-2}{r-4i-3}\,\frac{r-6i-6}{r-4i-4} \leqq \frac{1}{16} \quad \text{for } i \geqq 1. \quad \square$$

Immediate induction consequences of our lemmas are

$$\binom{ni}{i} \leqq \binom{n\gamma}{\gamma}\left[\frac{n^n}{(n-1)^{n-1}}\right]^{i-\gamma} \quad \text{if} \quad i \geqq \gamma,$$

$$\binom{r-3i-2}{k-2i-2} \leqq \left(\frac{1}{8}\right)^{i-1}\binom{r-5}{k-4} \quad \text{if} \quad 1 \leqq i \leqq \left\lfloor\frac{k-2}{2}\right\rfloor, \tag{9}$$

and

$$\binom{r-4i-1}{k-3i-1} \leqq \left(\frac{1}{16}\right)^{i-1}\binom{r-5}{k-4} \quad \text{if} \quad 1 \leqq i \leqq \left\lfloor\frac{k-1}{3}\right\rfloor. \tag{10}$$

Finally, by (8), with $n=3$ and 4 we have

$$\sum_{i=1}^{\infty}\binom{3i-3}{i-1}\left(\frac{1}{8}\right)^{i-1} \leqq 1+\frac{\binom{3}{1}}{8}+\frac{\binom{6}{2}}{8^2}+\frac{\binom{9}{3}}{8^3}+\frac{\binom{12}{4}}{8^4}+$$

$$+\frac{\binom{15}{5}}{8^5}\left[\sum_{i=6}^{\infty}\left(\frac{3^3}{2^2}\right)^{i-6}\left(\frac{1}{8}\right)^{i-6}\right] =$$

$$= 1+\frac{3}{8}+\frac{15}{64}+\frac{84}{512}+\frac{495}{4096}+\frac{3003}{32768}\frac{1}{1-\frac{27}{32}} < 2.481, \tag{11}$$

and

$$\sum_{i=1}^{\infty}\binom{4i-4}{i-1}\left(\frac{1}{16}\right)^{i-1} \leqq 1+\frac{\binom{4}{1}}{16}+\frac{\binom{8}{2}}{16^2}+\frac{\binom{12}{3}}{16^3}+ \tag{12}$$

$$+\frac{\binom{16}{4}}{16^4}\left[\sum_{i=5}^{\infty}\left(\frac{4^4}{3^3}\right)^{i-5}\left(\frac{1}{16}\right)^{i-5}\right] = 1+\frac{4}{16}+\frac{28}{256}+\frac{220}{4096}+\frac{1820}{65536}\frac{1}{1-\frac{16}{27}} < 1.482.$$

5. Proof of Theorem 3

Now we are able to prove the Theorem 3. Starting with (7) and using (9), (10), (11), and (12) we get

$$|F| \leqq \binom{r-3}{k-2}+\left\{\sum_{i=1}^{\infty}\binom{3i-3}{i-1}\left(\frac{1}{8}\right)^{i-1}+\sum_{i=1}^{\infty}\binom{4i-4}{i-1}\left(\frac{1}{16}\right)^{i-1}\right\}\binom{r-5}{k-4} <$$

$$< \binom{r-3}{k-2}+\{2.481+1.482\}\binom{r-5}{k-4} < \binom{r-3}{k-2}+4\binom{r-5}{k-4}.$$

Furthermore, recalling $k \leqq \frac{r}{2}$,

$$\frac{|F|}{\binom{r-2}{k-1}} < \frac{\binom{r-3}{k-2}}{\binom{r-2}{k-1}} + 4\frac{\binom{r-5}{k-4}}{\binom{r-2}{k-1}} = \frac{k-1}{r-2} + 4\frac{(k-1)(k-2)(k-3)}{(r-2)(r-3)(r-4)} \leqq$$

$$\leqq \frac{\frac{r}{2}-1}{r-2} + 4\frac{\left(\frac{r}{2}-1\right)\left(\frac{r}{2}-2\right)\left(\frac{r}{2}-3\right)}{(r-2)(r-3)(r-4)} = \frac{1}{2} + 4\frac{1}{8}\frac{(r-4)(r-6)}{(r-3)(r-4)} < 1.$$

This completes the proof. □

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