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On compositions of root-to-frontier tree transformations

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0. Introduction

It is well known that the family of (nondeterministic) root-to-frontier tree transformations is not closed with respect to the composition, see [2]. In this paper we introduce the notion of k-synchronized root-to-frontier tree transducer. Transducers of this type are capable of inducing all the relations which are compositions of k root-to-frontier tree transformations. Conversely, we shall show that any relation induced by a k-synchronized tree transducer is a composition of k root-to-frontier tree transformations. We mention that similar results are obtained by M. Dauchet in his dissertation [1] using the theory of magmoids.

1. Preliminaires

In this chapter we shall review the basic notions and notations used in the paper and give a reformalized notation of root-to-frontier tree transducers.

Definition 1.1. An operator domain is a set G together with a mapping $\nu\colon G\to\{0, 1, 2, \ldots\}$ that assigns to every $g\in G$ an arity, or rank, $\nu(g)$. For any $m\geqq 0$,

$$G^m = \{g\in G \mid \nu(g) = m\}$$

is the set of m-ary operators.

From now on, by an operator domain we mean a finite one, that means G is a finite set. The letters F and G always denote operator domains.

Definition 1.2. Let Y be a set disjoint from the operator domain G. The set $T_G(Y)$ of G-trees over Y is defined as follows:

(1) $G^0\cup Y\subseteq T_G(Y)$,

(2) $g(p_1, \ldots, p_m)\in T_G(Y)$ whenever $m\geqq 1$, $g\in G^m$ and $p_1, \ldots, p_m\in T_G(Y)$, and

(3) every G-tree over Y can be obtained by applying the rules (1) and (2) a finite number of times.

The set $T \subseteq T_G(Y)$ is called a G-forest over Y.

Definition 1.3. Let $p \in T_G(Y)$ be a G-tree over Y. The set sub (p) of subtrees of p is defined by the following rules:

(1) $\operatorname{sub}(p) = \{p\}$ if $p \in G^0 \cup Y$,

(2) $\operatorname{sub}(p) = \{p\} \cup \bigcup_{i=1}^{m} \operatorname{sub}(p_i)$ if

$p = g(p_1, \ldots, p_m)$, $g \in G^m$ and

$p_1, \ldots, p_m \in T_G(Y)$.

Definition 1.4. Let $p \in T_G(Y)$ be a tree. The root root (p) and height $h(p)$ are defined as follows:

(1) If $p \in G^0 \cup Y$, then root $(p) = p$, $h(p) = 0$.

(2) If $p = g(p_1, \ldots, p_m)$ $(m > 0)$, then root $(p) = g$ and $h(p) = \max(h(p_i) | i = 1, \ldots, m) + 1$.

Definition 1.5. Let $u \in N^*$ be a word over the set of natural numbers. The word u induces a partial function u: $T_G(Y) \to T_G(Y)$ in the following way:

(1) If $u = e$ then $u(p) = p$ for every $p \in T_G(Y)$, where e denotes the empty word.

(2) If $u = iv$, $i \in N$, $v \in N^*$ and $p \in T_G(Y)$, then

$$u(p) = \begin{cases} v(p_i) & \text{if } p = g(p_1, \ldots, p_m),\ g \in G^m,\ 1 \leqq i \leqq m \\ \text{else undefined.} \end{cases}$$

The elements of $T_G(Y)$ may be visualized as tree like directed ordered labelled graphs. In this case every path from the root to a given node in the graph is determined by a word over N. For every word $u \in N^*$, if there exists a node r such that u is the path from the root of p to r, then $u(p)$ denotes the subtree (subgraph) with root r.

Definition 1.6. Let Y be a set disjoint from G. We may assume without loss of generality that $N^* \cap T_G(Y) = \emptyset$ and $G \cap N^* = \emptyset$ hold in the rest of the paper. The set $P_G(Y)$ of quasi G-trees over Y is defined by the following rule:

$$P_G(Y) = \{p \in T_G(Y \cup N^*) | \forall u \in N^* \text{ if } u(p) \in N^* \text{ then } u(p) = u\}.$$

Definition 1.7. The mapping S: $P_G(Y) \to 2^{N^*}$ assigns a subset $S(p)$ of N^* to every quasi tree p which is defined by

$$S(p) = \{u(p) | u \in N^*\} \cap N^*.$$

It is clear that $S(p)$ is a finite set for every $p \in P_G(Y)$. The set $S(p)$ is also denoted by S_p. Members of S_p are called arguments of p.

Definition 1.8. Let Z be an arbitrary set and let φ: $S_p \to Z$ be a given function for a given quasi tree $p \in P_G(Y)$. Replacing every element u of S_p by $\varphi(u)$ in the tree p we obtain a G-tree over $Y \cup Z$, which is denoted by $p[S_p, \varphi]$.

Example. Let $G = \{g_1, g_2\}$ be an operator domain with $\nu(g_1) = 1$, $\nu(g_2) = 2$ and let $Y = \{y_1, y_2, y_3\}$. The quasi tree $p = g_2(g_1(11), g_2(21, y_1))$ may be visualized by the graph on Fig. 1.

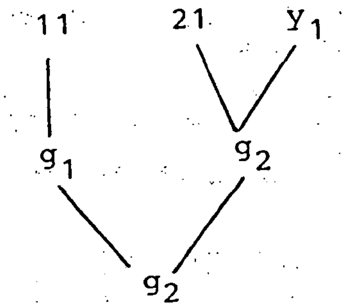

Figure 1

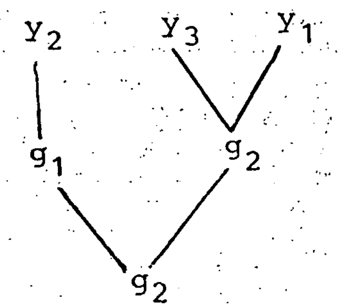

Figure 2

Let us define the mapping φ: $\{11, 21\} \to \{y_2, y_3\}$ as follows:

$$\varphi(11) = y_2, \varphi(21) = y_3.$$

The quasi tree $p[S_p, \varphi]$ may be visualized by the graph on Fig. 2.

Binary relations $\tau \subseteq T_F(X) \times T_G(Y)$ are called tree transformations. The composition $\tau_1 \circ \tau_2$ of the tree transformations $\tau_1 (\subseteq T_F(X) \times T_G(Y))$ and $\tau_2 (\subseteq T_G(Y) \times \times T_H(Z))$ is defined by

$$\tau_1 \circ \tau_2 = \{(p, q) | (p, r) \in \tau_1,\ (r, q) \in \tau_2 \text{ for some } r\}.$$

The composition $\tau_1 \circ \tau_2 \circ \ldots \circ \tau_l$ $|l \geqq 3)$ of the tree transformations $\tau_1, \tau_2, \ldots, \tau_l$ is defined by

$$\tau_1 \circ \tau_2 \circ \ldots \circ \tau_l = (\tau_1 \circ \ldots \circ \tau_{l-1}) \circ \tau_l.$$

Definition 1.9. A state set A is an operator domain consisting of unary operators only. If A is a state set and D is an arbitrary set then AD will denote the forest

$$AD = \{a(d) | a \in A,\ d \in D\}.$$

Moreover, if $a \in A$ and $d \in D$ then we generally write ad for $a(d)$.

If $A_1, \ldots, A_j$ are state sets ($j \in N$) then $A_j \ldots A_1$ denotes the state set $A_j \times \ldots \times A_1$ which is the Cartesian product of the sets A_i $(1 \leqq i \leqq j)$.

Elements of $A_j \ldots A_1$ are denoted by sequences $a_j \ldots a_1$, where $a_i \in A_i$, $i = 1, \ldots, j$.

For every non-negative integer l, $\{1, \ldots, l\}$ denotes the set $\{i | 1 \leqq i \leqq l\}$.

Definition 1.10. A root-to-frontier tree transducer (R-transducer) is a system $\mathfrak{A} = (F, X, A, G, Y, A', \Sigma)$, where

(1) F and G are operator domains.

(2) A is an operator domain consisting of unary operators, the state set of $\mathfrak{A}$. (It will be assumed that $A \cap T_F(X) = \emptyset$ and that $A \cap T_G(Y) = \emptyset$.)

(3) X and Y are finite sets.

(4) $A' \subseteq A$ is the set of initial states.

(5) Σ is a finite set of productions (rewriting rules) of the following two types:

(i) $ax \to q(a \in A, x \in X, q \in T_G(Y))$,

(ii) $af \to q[S_q, \varphi](q \in P_G(Y), f \in F^m, \varphi: S_q \to A\{1, \ldots, m\})$.

$\mathfrak{A}$ is said to be a deterministic R-transducer if A' is a singleton and there are no distinct productions in Σ with the same left-hand side.

Definition 1.11. Let $\mathfrak{A}=(F, X, A, G, Y, A', \Sigma)$ be an R-transducer and let $p_1, p_2 \in T_G(Y \cup N^* \cup AT_F(X \cup N^*))$ be trees. We say that p_1 directly derives p_2 in $\mathfrak{A}$, in symbols $p_1 \Rightarrow_{\mathfrak{A}} p_2$, if p_2 can be obtained from p_1 by

(i) replacing an occurrence of a subtree $ax(\in AX)$ in p_1 by the right side q of a production $ax \to q$ in Σ, or by

(ii) replacing an occurrence of a subtree $af(1, ..., m)[\{1, ..., m\}, \alpha](f \in F^m, \ \alpha\colon \{1, ..., m\} \to T_F(X \cup N^*))$ in p_1 by $q[S_q, \beta]$, where $af \to q[S_q, \varphi]$ is in Σ and β is a mapping $\beta\colon S_q \to AT_F(X \cup N^*)$ such that for each $s \in S_q$ if $\varphi(s)=ct(c \in A, t \in \{1, ..., m\})$ then $\beta(s)=c\alpha(t)$.

Each application of steps (i) and (ii) is called a direct derivation in $\mathfrak{A}$.

The reflexive-transitive closure of $\Rightarrow_{\mathfrak{A}}$ is denoted by $\Rightarrow_{\mathfrak{A}}^*$.

Using the notation $\Rightarrow_{\mathfrak{A}}^*$ the transformation $\tau_{\mathfrak{A}}$ induced by a root-to-frontier tree transducer $\mathfrak{A}=(F, X, A, G, Y, A', \Sigma)$ is defined by:

$$\tau_{\mathfrak{A}} = \{(p, q) \mid p \in T_F(X), \ q \in T_G(Y), \ ap \Rightarrow_{\mathfrak{A}}^* q \ \text{ for some } \ a \in A'\}.$$

The range of a mapping $\varphi\colon A \to B$ is denoted by $\mathrm{rg}(\varphi)$. Let $U_0, U_1, ..., U_l$ be sets, and let V be a subset of the set $(U_0 \times U_1 \times ... \times U_l) \cup (U_0 \times U_1 \times ... \times U_{l-1}) \cup ... ... \cup (U_0 \times U_1) \cup U_0$, where $U_0 \times U_1 \times ... \times U_l$ the Cartesian product of the sets U_i $(0 \leqq i \leqq l)$. Then for an index j, $(0 \leqq j \leqq l)$ $[V]_j$ denotes the set

$$\{u_j \mid \exists (u_0, ..., u_j, ..., u_n) \in V, \ \ 0 \leqq n \leqq l, \ \ 0 \leqq j \leqq n)\}.$$

Definition 1.12. Let u be an element of N^*. The mapping $\omega_u\colon T_G(Y \cup N^*) \to \to T_G(Y \cup N^*)$ is defined as follows:

(1) $\omega_u(p) = p$ if $p = y(\in Y)$ or $p = f(\in G^0)$,

(2) $\omega_u(p) = up$ if $p \in N^*$,

(3) $\omega_u(p) = f(\omega_u(p_1), ..., \omega_u(p_l))$ if
$p = f(p_1, ..., p_l)$, $f \in G^l$, $l \geqq 1$, $p_i \in T_G(Y \cup N^*)$, $i = 1, ..., l$.

2. Derivation sequences

In this chapter we shall deal with the description of derivations according to root-to-frontier tree transducers.

In the rest of the paper k denotes a natural number, not less than two, moreover let $\mathfrak{A}_i=(G_{i-1}, Y_{i-1}, A_i, G_i, Y_i, A_i', \Sigma_{\mathfrak{A}_i})$ be R-transducers, $1 \leqq i \leqq k$.

Now we give a procedure P. The input of P is a derivation in the form

(1) $a_j p_{j-1} \Rightarrow_{\mathfrak{A}}^* p_j \ (a_j \in A_j, p_{j-1} \in T_{G_{j-1}}(Y_{j-1}), \ p_j \in T_{G_j}(Y_j))$

for some $j \in \{1, ..., k\}$ and a decomposition

$$p_{j-1} = r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}] \ (r_{j-1} \in P_{G_{j-1}}(Y_{j-1}), \varphi_{j-1}\colon S_{r_{j-1}} \to T_{G_{j-1}}(Y_{j-1})).$$

The procedure P produces two derivations denoted by (2) and (3) which are defined by induction on the height of $r_{j-1}(\in T_{G_{j-1}}(Y_{j-1}\cup N^*))$. The derivations (2) and (3) will have the following forms:

$$(2)\quad a_j r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}] \Rightarrow^*_{\mathfrak{A}_j} r_j[S_{r_j}, \psi_j] \Rightarrow^*_{\mathfrak{A}_j} r_j[S_{r_j}, \varphi_j] = p_j,$$

$$(\psi_j\colon S_{r_j} \to A_j\,\mathrm{rg}\,(\varphi_{j-1}),\ \varphi_j\colon S_{r_j} \to T_{G_j}(Y_j)),$$

for each

$$s_j\in S_{r_j},\ \psi_j(s_j) \Rightarrow^*_{\mathfrak{A}_j} \varphi_j(s_j)$$

holds,

$$(3)\quad a_j r_{j-1} \Rightarrow^*_{\mathfrak{A}_j} r_j[S_{r_j}, \bar\psi_j], (\bar\psi_j\colon S_{r_j} \to A_j S_{r_{j-1}}),$$

and for each $s_j\in S_{r_j}$ if $\bar\psi_j(s_j)=a_j s_{j-1}$ then $\psi_j(s_j)=a_j\varphi_{j-1}(s_{j-1})$ holds.

Let $h(r_{j-1})=0$.

Case 1. $r_{j-1}=f$, $f\in G^0_{j-1}$. In this case $S_{r_{j-1}}=\emptyset$, $\varphi_{j-1}=\emptyset$ and $r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}]=f$. Thus $a_j f\to p_j\in\Sigma_{\mathfrak{A}_j}$, where $p_j\in T_{G_j}(Y_j)$. Let $r_j=p_j$, thus $S_{r_j}=\emptyset$. Let $\varphi_j=\emptyset$, $\psi_j=\emptyset$, $\bar\psi_j=\emptyset$. Thus the derivation (1) takes the following forms:

$$(2)\quad a_j r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}] \Rightarrow^*_{\mathfrak{A}_j} r_j[S_{r_j}, \psi_j] \Rightarrow^*_{\mathfrak{A}_j} r_j[S_{r_j}, \varphi_j],$$

$$(3)\quad a_j r_{j-1} \Rightarrow_{\mathfrak{A}_j} r_j[S_{r_j}, \bar\psi_j].$$

Case 2. $r_{j-1}=y(\in Y_{j-1})$. This case is the same as Case 1.

Case 3. $r_{j-1}=e(\in N^*)$. In this case $\varphi_{j-1}(e)=r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}]$. Let $r_j=e$, thus $S_{r_j}=\{e\}$. Let the mappings

$$\psi_j\colon S_{r_j} \to A_j T_{G_{j-1}}(Y_{j-1}),\ \bar\psi_j\colon S_{r_j} \to A_j S_{r_{j-1}}$$

and

$$\varphi_j\colon S_{r_j} \to T_{G_j}(Y_j)$$

be defined as follows:

$$\psi_j(e) = a_j r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}],\ \bar\psi_j(e) = a_j e,\ \varphi_j(e) = p_j.$$

Thus

$$r_j[S_{r_j}, \psi_j] = a_j r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}],\ r_j[S_{r_j}, \bar\psi_j] = a_j e.$$

Thus we have obtained the desired derivations (2) and (3), and

$$\bar\psi_j(e) = a_j e,\ \psi_j(e) = a_j\varphi_{j-1}(e),\quad \text{where}\quad S_{r_j} = \{e\}.$$

We have proved the basic step of the induction. Let

$$r_{j-1} = f(\bar p_1, \dots, \bar p_l) = f(\omega_1(p_1), \dots, \omega_l(p_l))$$

$$(p_1, \dots, p_l\in P_{G_{j-1}}(Y_{j-1}),\ S_{r_{j-1}} = 1\cdot S_{p_1}\cup 2\cdot S_{p_2}\cup\dots\cup l\cdot S_{p_l},$$

where

$$i\cdot S_{p_i} = \{is \mid s\in S_{p_i}\},\ i=1, \dots, l).$$

$$r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}] = f(p_1[S_{p_1}, \mu_1], \dots, p_l[S_{p_l}, \mu_l]),$$

where for each $i\in\{1, \dots, l\}$, and $s\in S_{p_i}$ $\mu_i(s)=\varphi_{j-1}(is)$ holds. The production applied in the first step in derivation (1) must be of the form $a_j f\to q[S_q, \varepsilon]$, where

$q \in P_{G_j}(Y_j)$, $f \in G^l_{j-1}$ for some l, $\varepsilon\colon S_q \to A_j\{1, \ldots, l\}$. Consequently derivation (1) can be written in the following form:

$$a_j r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}] \Rightarrow^*_{\mathfrak{A}_j} q[S_q, \varrho] \Rightarrow^*_{\mathfrak{A}_j} q[S_q, \tau]$$
$$(\varrho\colon S_q \to A_j \operatorname{rg}(\varphi_{j-1}), \tau\colon S_q \to T_{G_j}(Y_j)),$$

where the mapping ϱ satisfies the following formula: for every $s \in S_q$ if $\varepsilon(s) = b_j t$ $(1 \leqq t \leqq l, b_j \in A_j)$ then $\varrho(s) = b_j p_t[S_{p_t}, \mu_t]$. This implies that $\varrho(s) = b_j p_t(S_{p_t}, \mu_t] \Rightarrow^*_{\mathfrak{A}_j} \Rightarrow^*_{\mathfrak{A}_j} \tau(s)$ holds. The desired derivations will take the following forms:

$$(2) \quad a_j f(p_1[S_{p_1}, \mu_1], \ldots, p_l[S_{p_l}, \mu_l]) \Rightarrow_{\mathfrak{A}_j} q[S_q, \varrho] \Rightarrow^*_{\mathfrak{A}_j} q[S_q, \varkappa] \Rightarrow^*_{\mathfrak{A}_j} q[S_q, \tau],$$

where

$$\varkappa\colon S_q \to T_{G_j}(Y_j \cup A_j T_{G_{j-1}}(Y_{j-1})),$$
$$\tau\colon S_q \to T_{G_j}(Y_j).$$

$$(3) \quad a_j f(\omega_1(p_1), \ldots, \omega_l(p_l)) \Rightarrow_{\mathfrak{A}_j} q[S_q, \bar{\varrho}] \Rightarrow^*_{\mathfrak{A}_j} q[S_q, \bar{\varkappa}],$$

where

$$\bar{\varrho}\colon S_q \to A_j T_{G_{j-1}}(Y_{j-1} \cup N^*), \quad \bar{\varkappa}\colon S_q \to T_{G_j}(Y_j \cup A_j S_{q_{j-1}}).$$

We shall define the mappings $\varkappa$, $\bar{\varrho}$, $\bar{\varkappa}$. For each $s \in S_q$ let us consider the derivation

$$(4) \quad \varrho(s) = b_j p_t[S_{p_t}, \mu_t] \Rightarrow^*_{\mathfrak{A}_j} \tau(s),$$

where

$$\varepsilon(s) = b_j t \quad \text{holds.}$$

Since $h(p_t) < h(r_{j-1})$ we may apply the induction hypothesis to derivation (4) and decomposition $p_t[S_{p_t}, \mu_t]$. The derivations (5) and (6) are obtained by applying procedure P to (4) and decomposition $p_t[S_{p_t}, \mu_t]$.

$$(5) \quad b_j p_t[S_{p_t}, \mu_t] \Rightarrow^*_{\mathfrak{A}_j} q_s[S_{q_s}, \eta_s] \Rightarrow^*_{\mathfrak{A}_j} q_s[S_{q_s}, \xi_s] = \tau(s),$$
$$(q_s \in P_{G_j}(Y_j), \eta_s\colon S_{q_s} \to A_j \operatorname{rg}(\mu_t), \xi_s\colon S_{q_s} \to T_{G_j}(Y_j)),$$

such that for every $v \in S_{q_s}$, $\eta_s(v) \Rightarrow^*_{\mathfrak{A}_j} \xi_s(v)$ holds.

$$(6) \quad b_j p_t \Rightarrow^*_{\mathfrak{A}_j} q_s[S_{q_s}, \bar{\eta}_s] \, (\bar{\eta}_s\colon S_{q_s} \to A_j S_{p_t}),$$

and for every $v \in S_{q_s}$ if $\bar{\eta}_s(v) = b_j z$ for some $b_j \in A_j$ and $z \in S_{p_t}$ then $\eta_s(v) = b_j \mu_t(z)$.

In this case $\varkappa$, $\bar{\varrho}$, $\bar{\varkappa}$ are defined by

$$\varkappa(s) = q_s[S_{q_s}, \eta_s], \quad \bar{\varrho}(s) = \omega_t(b_j p_t),$$
$$\bar{\varkappa}(s) = \omega_t(q_s[S_{q_s}, \bar{\eta}_s]).$$

The derivation $\varrho(s) \Rightarrow^*_{\mathfrak{A}_j} \varkappa(s) \Rightarrow^*_{\mathfrak{A}_j} \tau(s)$ is the same as derivation (5).

The derivation $\bar{\varrho}(s) \Rightarrow^*_{\mathfrak{A}_j} \bar{\varkappa}(s)$ is obtained from derivation (6) by applying the mapping ω_t to each step of derivation (6).

We give a procedure S. The input of S is a derivation sequence $D=D_1, \ldots, D_k$ given in the following form:

$$D_1: a_1 p_0 \Rightarrow^*_{\mathfrak{A}_1} p_1, \ (p_0 \in T_{G_0}(Y_0), a_1 \in A_1, p_1 \in T_{G_1}(Y_1)),$$

$$D_2: a_2 p_1 \Rightarrow^*_{\mathfrak{A}_2} p_2, \ (a_2 \in A_2, p_2 \in T_{G_2}(Y_2)),$$

$$\vdots$$

$$D_k: a_k p_{k-1} \Rightarrow^*_{\mathfrak{A}_k} p_k (a_k \in A_k, p_k \in T_{G_k}(Y_k))$$

and a decomposition $p_0=r_0[S_{r_0}, \varphi_0]$. S produces two derivation sequences denoted by $D^{r_0}=D_1^{r_0}, D_2^{r_0}, \ldots, D_k^{r_0}$ and $\bar{D}^{r_0}=\bar{D}_1^{r_0}, \bar{D}_2^{r_0}, \ldots, \bar{D}_k^{r_0}$. The derivation sequences D^{r_0} and $\bar{D}^{r_0}$ will have the following forms:

$$D_1^{r_0}: a_1 r_0[S_{r_0}, \varphi_0] \Rightarrow^*_{\mathfrak{A}_1} r_1[S_{r_1}, \psi_1] \Rightarrow^*_{\mathfrak{A}_1} r_1[S_{r_1}, \varphi_1] =$$

$$= p_1 \ (r_1 \in P_{G_1}(Y_1), \psi_1 \colon S_{r_1} \to A_1 \operatorname{rg}(\varphi_0), \varphi_1 \colon S_{r_1} \to T_{G_1}(Y_1))$$

and for each $s_1 \in S_{r_1}$ the derivation $\psi_1(s_1) \Rightarrow^*_{\mathfrak{A}_1} \varphi_1(s_1)$ is valid,

$$D_2^{r_0}: a_2 r_1[S_{r_1}, \varphi_1] \Rightarrow^*_{\mathfrak{A}_2} r_2[S_{r_2}, \psi_2] \Rightarrow^*_{\mathfrak{A}_2} r_2[S_{r_2}, \varphi_2] =$$

$$= p_2 \ (r_2 \in P_{G_2}(Y_2), \psi_2 \colon S_{r_2} \to A_2 \operatorname{rg}(\varphi_1), \varphi_2 \colon S_{r_2} \to T_{G_2}(Y_2))$$

and for each $s_2 \in S_{r_2}$ the derivation $\psi_2(s_2) \Rightarrow^*_{\mathfrak{A}_2} \varphi_2(s_2)$ is valid,

$$\vdots$$

$$D_k^{r_0}: a_k r_{k-1}[S_{r_{k-1}}, \varphi_{k-1}] \Rightarrow^*_{\mathfrak{A}_k} r_k[S_{r_k}, \psi_k] \Rightarrow^*_{\mathfrak{A}_k} r_k[S_{r_k}, \varphi_k] =$$

$$= p_k (r_k \in P_{G_k}(Y_k), \ \ \psi_k \colon S_{r_k} \to A_k \operatorname{rg}(\varphi_{k-1}), \varphi_k \colon S_{r_k} \to T_{G_k}(Y_k))$$

and for each $s_k \in S_{r_k}$ the derivation $\psi_k(s_k) \Rightarrow^*_{\mathfrak{A}_k} \varphi_k(s_k)$ is valid.

$$\bar{D}_1^{r_0}: a_1 r_0 \Rightarrow^*_{\mathfrak{A}_1} r_1[S_{r_1}, \bar{\psi}_1](\bar{\psi}_1 \colon S_{r_1} \to A_1 S_{r_0}),$$

$$\bar{D}_2^{r_0}: a_2 r_1 \Rightarrow^*_{\mathfrak{A}_2} r_2[S_{r_2}, \bar{\psi}_2] (\bar{\psi}_2 \colon S_{r_2} \to A_2 S_{r_1}),$$

$$\vdots$$

$$\bar{D}_k^{r_0}: a_k r_{k-1} \Rightarrow^*_{\mathfrak{A}_k} r_k[S_{r_k}, \bar{\psi}_k](\bar{\psi}_k \colon S_{r_k} \to A_k S_{r_k}).$$

For every $j \in \{1, \ldots, k\}$ and $s_j \in S_{r_j}$ if $\bar{\psi}_j(s_j)=b_j s_{j-1}$ for some $b_j \in A_j$ and $s_{j-1} \in S_{r_{j-1}}$ then $\psi_j(s_j)=b_j \varphi_{j-1}(s_{j-1})$. Applying the procedure P to the derivation D_1 and the decomposition $p_0=r_0[S_{r_0}, \varphi_0]$ we obtain the derivations $D_1^{r_0}, \bar{D}_1^{r_0}$.

Assume that the derivations $D_{j-1}^{r_0}, \bar{D}_{j-1}^{r_0}$ are constructed for an index j $(2 \leqq j \leqq k)$. Then the derivations $D_j^{r_0}, \bar{D}_j^{r_0}$ are obtained by applying the procedure P to D_j and decomposition $p_{j-1}=r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}]$, where the decomposition $r_{j-1}[S_{r_{j-1}}, \varphi_{j-1}]$ of p_{j-1} is given in the derivation $D_{j-1}^{r_0}$.

Let $\mathfrak{A}_i=(G_{i-1}, Y_{i-1}, A_i, G_i, Y_i, A_i', \Sigma_{\mathfrak{A}_i})$ $(i=1, \ldots, k)$ be R-transducers. Let us denote the arity function of the operator domain G_0 by ν. We fix these notations for this chapter. Let $D=D_1, \ldots, D_k$ be the following derivation sequence:

$$D_1\colon a_1 p_0 \Rightarrow^*_{\mathfrak{A}_1} p_1 \ (p_0 \in T_{G_0}(Y_0),\ p_1 \in T_{G_1}(Y_1),\ a_1 \in A_0'),$$

$$D_2\colon a_2 p_1 \Rightarrow^*_{\mathfrak{A}_2} p_2 \ (p_2 \in T_{G_2}(Y_2),\ a_2 \in A_2'),$$

$$\vdots$$

$$D_k\colon a_k p_{k-1} \Rightarrow^*_{\mathfrak{A}_k} p_k \ (p_k \in T_{G_k}(Y_k),\ a_k \in A_k'),$$

moreover, we assume that $p_0=q_0[S_{q_0}, \gamma_0]$ holds for some

$$q_0 \in P_{G_0}(Y_0),\ \gamma_0\colon S_{q_0} \to T_{G_0}(Y_0).$$

Applying the procedure S to the derivation sequence D and decomposition $p_0=q_0[S_{q_0}, \varphi_0]$ we obtain derivation sequences D^{q_0} and $\bar{D}^{q_0}$.

$$D_1^{q_0}\colon a_1 q_0[S_{q_0}, \gamma_0] \Rightarrow^*_{\mathfrak{A}_1} q_1[S_{q_1}, \alpha_1] \Rightarrow^*_{\mathfrak{A}_1} q_1[S_{q_1}, \gamma_1] = p_1,$$

$$(q_1 \in P_{G_1}(Y_1),\ \alpha_1\colon S_{q_1} \to A_1 \operatorname{rg}(\gamma_0),\ \gamma_1\colon S_{q_1} \to T_{G_1}(Y_1)),$$

and for every $s_1 \in S_{q_1}$, $\alpha_1(s_1) \Rightarrow^*_{\mathfrak{A}_1} \gamma_1(s_1)$ holds.

$$D_2^{q_0}\colon a_2 q_1[S_{q_1}, \gamma_1] \Rightarrow^*_{\mathfrak{A}_2} q_2[S_{q_2}, \alpha_2] \Rightarrow^*_{\mathfrak{A}_2} q_2[S_{q_2}, \gamma_2] = p_2,$$

$$(q_2 \in P_{G_2}(Y_2),\ \alpha_2\colon S_{q_2} \to A_2 \operatorname{rg}(\gamma_1),\ \gamma_2\colon S_{q_2} \to T_{G_2}(Y_2)),$$

and for every $s_2 \in S_{q_2}$, $\alpha_2(s_2) \Rightarrow^*_{\mathfrak{A}_2} \gamma_2(s_2)$ holds.

$$\vdots$$

$$D_k^{q_0}\colon a_k q_{k-1}[S_{q_{k-1}}, \gamma_{k-1}] \Rightarrow^*_{\mathfrak{A}_k} q_k[S_{q_k}, \alpha_k] \Rightarrow^*_{\mathfrak{A}_k} q_k[S_{q_k}, \gamma_k] = p_k,$$

$$(q_k \in P_{G_k}(Y_k),\ \alpha_k\colon S_{q_k} \to A_k \operatorname{rg}(\gamma_{k-1}),\ \gamma_k\colon S_{q_k} \to T_{G_k}(Y_k)),$$

and for every $s_k \in S_{q_k}$, $\alpha_k(s_k) \Rightarrow^*_{\mathfrak{A}_k} \gamma_k(s_k)$ holds.

$$\bar{D}_1^{q_0}\colon a_1 q_0 \Rightarrow^*_{\mathfrak{A}_1} q_1[S_{q_1}, \bar{\alpha}_1],\ (\bar{\alpha}_1\colon S_{q_1} \to A_1 S_{q_0}),$$

$$\bar{D}_2^{q_0}\colon a_2 q_1 \Rightarrow^*_{\mathfrak{A}_2} q_2[S_{q_2}, \bar{\alpha}_2],\ (\bar{\alpha}_2\colon S_{q_2} \to A_2 S_{q_1}),$$

$$\vdots$$

$$\bar{D}_k^{q_0}\colon a_k q_{k-1} \Rightarrow^*_{\mathfrak{A}_k} q_k[S_{q_k}, \bar{\alpha}_k],\ (\bar{\alpha}_k\colon S_{q_k} \to A_k S_{q_{k-1}}),$$

and for every $j \in \{1, \ldots, k\}$ and $s_j \in S_{q_j}$ if

$$\bar{\alpha}_j(s_j) = b_j s_{j-1} \quad \text{for some} \quad b_j \in A_j,\ s_{j-1} \in S_{q_{j-1}},$$

then

$$\alpha_j(s_j) = b_j \gamma_{j-1}(s_{j-1}).$$

We shall define a set $Z_{(D, q_0)}$ and mappings

$$\Omega_{(D,q_0)}\colon Z_{(D,q_0)} \to (A_1 \cup A_2 A_1 \cup \ldots \cup A_k \ldots A_1) \operatorname{rg}(\gamma_0),$$

$$\theta_{(D,q_0)}\colon S_{q_k} \to Z_{(D,q_0)}$$

and

$$\psi_{(D,q_0)}\colon S_{q_k} \to A_k \ldots A_1 T_{G_0}(Y_0)$$

in the following way:

$$Z_{(D,q_0)} = \{(s_0, s_1, \ldots, s_j) | s_0 \in S_{q_0}, s_1 \in S_{q_1}, \ldots, s_j \in S_{q_j},$$
$$1 \leqq j \leqq k \text{ and } (j=k \text{ or } (j<k \text{ and there are no } s_{j+1} \in S_{q_{j+1}} \text{ and } b_{j+1} \in A_{j+1} \text{ such that } \bar{\alpha}_{j+1}(s_{j+1}) = b_{j+1}s_j))$$
$$\text{and } \bar{\alpha}_i(s_i) = b_i s_{i-1} \ (b_i \in A_i) \text{ for } i=1, \ldots, j\}.$$

For every $(s_0, s_1, \ldots, s_j) \in Z_{(D,q_0)}$

$$\Omega_{(D,q_0)}((s_0, s_1, \ldots, s_j)) = b_j \ldots b_1 \gamma_0(s_0)$$

iff

$$\bar{\alpha}_i(s_i) = b_i s_{i-1} \text{ for } i = 1, \ldots, j.$$

For every $s_k \in S_{q_k}$, $\theta_{(D,q_0)}(s_k) = (s_0, s_1, \ldots, s_k)$ iff

$$\bar{\alpha}_i(s_i) = b_i s_{i-1} (b_i \in A_i) \text{ for } i = 1, \ldots, k.$$

For each $s_k \in S_{q_k}$, $\psi_{(D,q_0)}(s_k) = b_k \ldots b_1 \gamma_0(s_0)$ iff

$$\theta_{(D,q_0)}(s_k) = (s_0, s_1, \ldots, s_k)$$

and

$$\Omega_{(D,q_0)}((s_0, s_1, \ldots, s_k)) = b_k \ldots b_1 \gamma_0(s_0).$$

One can see the equality $\psi_{(D,q_0)} = \theta_{(D,q_0)} \circ \Omega_{(D,q_0)}$ holds.

For the derivation sequence D and a decomposition

$$p_0 = q_0[S_{q_0}, \gamma_0] \ (q_0 \in P_{G_0}(Y_0), \gamma_0 \colon S_{q_0} \to T_{G_0}(Y_0))$$

we can determine the configuration

$$K_{(D,q_0)} \colon (q_k[S_{q_k}, \psi_{(D,q_0)}], \theta_{(D,q_0)}, Z_{(D,q_0)}, \Omega_{(D,q_0)}).$$

For the sake of a unified formalism, in the sequel we use the following convention. Let G be an operator domain with arity function $\bar{v}$, and let Y be a set disjoint with G. If $u \in G^0 \cup Y$ then $u(1, \ldots, \bar{v}(u))$ means the G-tree u over Y, moreover, $u(1, \ldots, \bar{v}(u))[\{1, \ldots, \bar{v}(u)\}, \vartheta]$ means u for arbitrary ϑ.

We continue the analysis of derivation sequence D. For each $s_0 \in S_{q_0}$ the tree $\gamma_0(s_0)$ can be written in the following form:

$$\gamma_0(s_0) = u_0(1, \ldots, v(u_0))[\{1, \ldots, v(u_0)\} \vartheta_0],$$

where $u_0 \in G_0 \cup Y_0$ and $\vartheta_0 \colon \{1, \ldots, v(u_0)\} \to T_{G_0}(Y_0)$. There are two cases.

1. *Case* $Z_{(D,q_0)} = \emptyset$. Take the quasi tree $r_0 \in P_{G_0}(Y_0)$ defined by $r_0 = q_0[S_{q_0}, \xi_0]$, where the mapping $\xi_0 \colon S_{q_0} \to T_{G_0}(Y_0 \cup N^*)$ is determined by the following formula: for each

$$s_0 \in S_{q_0} \ \xi_0(s_0) = \omega_{s_0}(u_0(1, \ldots, v(u_0))) \text{ if } \gamma_0(s_0) = u_0(1, \ldots, v(u_0))[\{1, \ldots, v(u_0)\}, \vartheta_0]$$
$$(u_0 \in G_0 \cup Y_0, \vartheta_0 \colon \{1, \ldots, v(u_0)\} \to T_{G_0}(Y_0)).$$

One can see $K_{(D,q_0)} = K_{(D,r_0)}$ holds.

2. *Case* $Z_{(D,q_0)} \neq \emptyset$. Using these decompositions of the trees $\gamma_0(s_0)$ we obtain the derivation sequence $E = E_1, \ldots, E_k$ from D. For every $i \in \{1, \ldots, k\}$ the derivation E_i

is the same as D_i disregarding the order of direct derivations in D_i. We shall introduce the derivation sequence $\bar{E}=\bar{E}_1, \ldots, \bar{E}_k$ too.

$$E_1\colon a_1 q_0[S_{q_0}, \gamma_0] \Rightarrow^*_{\mathfrak{A}_1} q_1[S_{q_1}, \alpha_1] \Rightarrow^*_{\mathfrak{A}_1} q_1[S_{q_1}, \beta_1] \Rightarrow^*_{\mathfrak{A}_1}$$
$$\Rightarrow^*_{\mathfrak{A}_1} q_1[S_{q_1}, \gamma_1] \; (q_1 \in P_{G_1}(Y_1), \alpha_1\colon S_{q_1} \to A_1 \operatorname{rg}(\gamma_0),$$
$$\beta_1\colon S_{q_1} \to T_{G_1}(Y_1 \cup A_1 T_{G_0}(Y_0)), \gamma_1\colon S_{q_1} \to T_{G_1}(Y_1)).$$
$$\bar{E}_1\colon a_1 q_0[S_{q_0}, \xi_0] \Rightarrow^*_{\mathfrak{A}_1} q_1[S_{q_1}, \bar{\beta}_1]$$
$$(\xi_0\colon S_{q_0} \to T_{G_0}(Y_0 \cup N^*), \bar{\beta}_1\colon S_{q_1} \to T_{G_1}(Y_1 \cup A_1 N^*)).$$

ξ_0 is defined by the following formula: for each

$$s_0 \in S_{q_0} \quad \text{if} \quad \gamma_0(s_0) = u_0(1, \ldots, \nu(u_0))[\{1, \ldots, \nu(u_0)\}, \vartheta_0]$$

for some $u_0 \in G_0 \cup Y_0$ and mapping

$$\vartheta_0\colon \{1, \ldots, \nu(u_0)\} \to T_{G_0}(Y_0) \quad \text{then} \quad \xi_0(s_0) = \omega_{s_0}(u_0(1, \ldots, \nu(u_0))).$$

We shall define the mappings β_1 and $\bar{\beta}_1$. For every $s_1 \in S_{q_1}$ let us consider the subderivation

(1) $\alpha_1(s_1) \Rightarrow^*_{\mathfrak{A}_1} \gamma_1(s_1)$ of D.

Let us assume that $\bar{\alpha}_1(s_1) = b_1 s_0$ and

$$\alpha_1(s_1) = b_1 \gamma_0(s_0) = b_1 u_0(1, \ldots, \nu(u_0))[\{1, \ldots, \nu(u_0)\}, \vartheta_0],$$

where

$$s_0 \in S_{q_0}, b_1 \in A_1, u_0 \in G_0 \cup Y_0, \vartheta_0\colon \{1, \ldots, \nu(u_0)\} \to T_{G_0}(Y_0).$$

Applying procedure P to derivation (1) and decomposition

$$\gamma_0(s_0) = u_0(1, \ldots, \nu(u_0))[\{1, \ldots, \nu(u_0)\}, \vartheta_0]$$

we obtain derivations (2), (3).

(2) $b_1 u_0(1, \ldots, \nu(u_0))[\{1, \ldots, \nu(u_0)\}, \vartheta_0] \Rightarrow_{\mathfrak{A}_1} u_1[S_{u_1}, \delta_1] \Rightarrow^*_{\mathfrak{A}_1}$
$\Rightarrow^*_{\mathfrak{A}_1} u_1[S_{u_1}, \vartheta_1] = \gamma_1(s_1)$, where $u_1 \in P_{G_1}(Y_1)$,
$\delta_1\colon S_{u_1} \to A_1 T_{G_0}(Y_0), \vartheta_1\colon S_{u_1} \to T_{G_1}(Y_1)$,

and for each

$v_1 \in S_{u_1}, \delta_1(v_1) \Rightarrow^*_{\mathfrak{A}_1} \vartheta_1(v_1)$ holds.

(3) $b_1 u_0(1, \ldots, \nu(u_0)) \Rightarrow_{\mathfrak{A}_1} u_1[S_{u_1}, \bar{\delta}_1]$, where
$\bar{\delta}_1\colon S_{u_1} \to A_1\{1, \ldots, \nu(u_0)\}$ and for each $v_1 \in S_{u_1}$ if
$\bar{\delta}_1(v_1) = c_1 t_0$ $(c_1 \in A_1, t_0 \in \{1, \ldots, \nu(u_0)\})$ then $\delta_1(v_1) = c_1 \vartheta_0(t_0)$.

In this case β_1 and $\bar{\beta}_1$ are defined by

$$\beta_1(s_1) = u_1[S_{u_1}, \delta_1], \; \bar{\beta}_1(s_1) = \omega_{s_0}(u_1[S_{u_1}, \bar{\delta}_1]).$$

Let l be an index of the transducers in consideration such that $2 \leq l \leq k$. The derivations E_l and $\bar{E}_l$ are the following:

$$E_l: a_l q_{l-1}[S_{q_{l-1}}, \gamma_{l-1}] \Rightarrow^*_{\mathfrak{A}_l} q_l[S_{q_l}, \alpha_l] \Rightarrow^*_{\mathfrak{A}_l} q_l[S_{q_l}, \beta_l] \Rightarrow^*_{\mathfrak{A}_l} q_l[S_{q_l}, \gamma_l]$$

$$(q_l \in P_{G_l}(Y_l), \alpha_l: S_{q_l} \to A_l \operatorname{rg}(\gamma_{l-1}),$$

$$\beta_l: S_{q_l} \to T_{G_l}(Y_l \cup A_l T_{G_{l-1}}(Y_{l-1})), \gamma_l: S_{q_l} \to T_{G_l}(Y_l)).$$

$$\bar{E}_l: a_l q_{l-1}[S_{q_{l-1}}, \xi_{l-1}] \Rightarrow^*_{\mathfrak{A}_l} q_l[S_q, \bar{\beta}_l]$$

$$(\xi_{l-1}: S_{q_{l-1}} \to T_{G_{l-1}}(Y_{l-1} \cup N^*),$$

$$\bar{\beta}_l: S_{q_l} \to T_{G_l}(Y_l \cup A_l N^*)).$$

ξ_{l-1} is defined by the following formula: for every $s_{l-1} \in S_{q_{l-1}}$ if $\beta_{l-1}(s_{l-1}) = u_{l-1}[S_{u_{l-1}}, \delta_{l-1}]$ then $\xi_{l-1}(s_{l-1}) = \omega_{s_{l-1}}(u_{l-1})$. We shall define the mappings β_l and $\bar{\beta}_l$. For every $s_l \in S_{q_l}$ let us consider the subderivation

(1) $\alpha_l(s_l) \Rightarrow^*_{\mathfrak{A}_l} \gamma_l(s_l)$ of D_l.

Let us assume that

$$\bar{\alpha}_l(s_l) = b_l s_{l-1} \quad \text{and} \quad \alpha_l(s_l) = b_l \gamma_{l-1}(s_{l-1}) = b_l u_{l-1}[S_{u_{l-1}}, \vartheta_{l-1}],$$

where

$$s_{l-1} \in S_{q_{l-1}}, b_l \in A_l, u_{l-1} \in P_{G_{l-1}}(Y_{l-1}),$$

and the decomposition $\gamma_{l-1}(s_{l-1}) = u_{l-1}[S_{u_{l-1}}, \vartheta_{l-1}]$ of $\gamma_{l-1}(s_{l-1})$ is the same as in E_{l-1}. Applying the procedure P to derivation (1) and decomposition $\gamma_{l-1}(s_{l-1}) = u_{l-1}[S_{u_{l-1}}, \vartheta_{l-1}]$ we obtain derivations (2), (3).

(2) $b_l u_{l-1}[S_{u_{l-1}}, \vartheta_{l-1}] \Rightarrow^*_{\mathfrak{A}_l} u_l[S_{u_l}, \delta_l] \Rightarrow^*_{\mathfrak{A}_l} u_l[S_{u_l}, \vartheta_l] = \gamma_l(s_l)$,

where

$$u_l \in P_{G_l}(Y_l), \delta_l: S_{u_l} \to A_l T_{G_{l-1}}(Y_{l-1}), \vartheta_l: S_{u_l} \to T_{G_l}(Y_l),$$

and for every $v_l \in S_{u_l}$ the derivation $\delta_l(v_l) \Rightarrow^*_{\mathfrak{A}_l} \vartheta_l(v_l)$ is valid.

(3) $b_l u_{l-1} \Rightarrow^*_{\mathfrak{A}_l} u_l[S_{u_l}, \bar{\delta}_l]$, where $\bar{\delta}_l: S_{u_l} \to A_l S_{u_{l-1}}$

and for each $v_l \in S_{u_l}$ if

$\bar{\delta}_l(v_l) = c_l t_{l-1}(c_l \in A_l, t_{l-1} \in S_{u_{l-1}})$

then

$\delta_l(v_l) = c_l \vartheta_{l-1}(t_{l-1})$.

In this case β_l and $\bar{\beta}_l$ are defined by

$$\beta_l(s_l) = u_l[S_{u_l}, \delta_l], \bar{\beta}_l(s_l) = \omega_{s_{l-1}}(u_l[S_u, \bar{\delta}_l]).$$

Take the quasi-tree $r_0 \in P_{G_0}(Y_0)$ defined by $r_0 = q_0[S_{q_0}, \xi_0]$, where the mapping $\xi_0\colon S_{q_0} \to T_{G_0}(Y_0 \cup N^*)$ as in $\bar{E}_1$. Let $\lambda_0\colon S_{r_0} \to T_{G_0}(Y_0)$ be the mapping such that

$$\lambda_0(s_0 i) = \vartheta_0(i) \quad \text{if} \quad \gamma_0(s_0) = u_0(1, \ldots, \nu(u_0))[\{1, \ldots, \nu(u_0)\}, \vartheta_0],$$

where

$$s_0 \in S_{q_0},\ u_0 \in G_0 \cup Y_0,\ \vartheta_0\colon \{1, \ldots, \nu(u_0)\} \to T_{G_0}(Y_0), \quad i \in \{1, \ldots, \nu(u_0)\}.$$

For these r_0 and λ_0 we have that $q_0[S_{q_0}, \gamma_0] = r_0[S_{r_0}, \lambda_0]$ holds. We take the quasi tree $r_1 \in P_{G_1}(Y_1)$ which is defined by $r_1 = q_1[S_{q_1}, \xi_1]$, $\xi_1\colon S_{q_1} \to T_{G_1}(Y_1 \cup N^*)$, for every $s_1 \in S_{q_1}$, $\xi_1(s_1) = \omega_{s_1}(u_1)$ if $\beta_1(s_1) = u_1[S_{u_1}, \delta_1]$. It can be seen that

$$S_{r_1} = \{s_1 t_1 | s_1 \in S_{q_1}, \xi_1(s_1) = \omega_{s_1}(u_1), t_1 \in S_{u_1}\}$$

holds. Let us define the mappings $\eta_1\colon S_{r_1} \to A_1 T_{G_0}(Y_0)$, $\bar{\eta}_1\colon S_{r_1} \to A_1 S_{r_0}$ and $\lambda_1\colon S_{r_1} \to T_{G_1}(Y_1)$ as follows: for each $\bar{s}_1 \in S_{r_1}$ let us consider its unique decomposition $\bar{s}_1 = s_1 t_1$, where $s_1 \in S_{q_1}$, $\bar{\alpha}_1(s_1) = b_1 s_0$ for some $b_1 \in A_1$ and

$$s_0 \in S_{q_0}, \xi_1(s_1) = \omega_{s_1}(u_1), t_1 \in S_{u_1}, \beta_1(s_1) = u_1[S_{u_1}, \delta_1],$$

$$\bar{\beta}_1(s_1) = \omega_{s_0}(u_1[S_{u_1}, \bar{\delta}_1]), \gamma_1(s_1) = u_1[S_{u_1}, \vartheta_1]$$

and $\alpha_1, \beta_1, \bar{\beta}_1, \gamma_1, \delta_1, \bar{\delta}_1, \vartheta_1$ as in $E_1, \bar{E}_1$.

Let $\eta_1(s_1 t_1) = \delta_1(t_1)$, $\bar{\eta}_1(s_1 t_1) = \omega_{s_0}(\bar{\delta}_1(t_1))$, $\lambda_1(s_1 t_1) = \vartheta_1(t_1)$. The derivation $\delta_1(t_1) \Rightarrow^*_{\mathfrak{A}_1} \vartheta_1(t_1)$ holds, which implies that the derivation $\eta_1(s_1 t_1) \Rightarrow^*_{\mathfrak{A}_1} \lambda_1(s_1 t_1)$ is valid. Thus we obtain the derivations E_1' and $\bar{E}_1'$ from E_1 and $\bar{E}_1$, respectively.

$$E_1'\colon a_1 r_0[S_{r_0}, \lambda_0] \Rightarrow^*_{\mathfrak{A}_1} r_1[S_{r_1}, \eta_1] \Rightarrow^*_{\mathfrak{A}_1} r_1[S_{r_1}, \lambda_1],$$

$$\bar{E}_1'\colon a_1 r_0 \Rightarrow^*_{\mathfrak{A}_1} r_1[S_{r_1}, \bar{\eta}_1],$$

and for each $v_1 \in S_{r_1}$ if $\bar{\eta}_1(v_1) = c_1 v_0$ for some $c_1 \in A_1$, $v_0 \in S_{r_0}$, then $\eta_1(v_1) = c_1 \lambda_0(v_0)$. For each $2 \leq l \leq k$ we take the quasi trees $r_l \in P_{G_l}(Y_l)$ which is defined by

$$r_l = q_l[S_{q_l}, \xi_l], \xi_l\colon S_{q_l} \to T_{G_l}(Y_l \cup N^*),$$

for every

$$s_l \in S_{q_l}, \xi_l(s_l) = \omega_{s_l}(u_l) \quad \text{if} \quad \beta_l(s_l) = u_l[S_{u_l}, \delta_l].$$

It can be seen that

$$S_{r_l} = \{s_l t_l | \xi_l(s_l) = \omega_{s_l}(u_l), t_l \in S_{ui}\}$$

holds. Let us define the mappings $\eta_l\colon S_{r_l} \to A_l T_{G_{l-1}}(Y_{l-1})$, $\bar{\eta}_l\colon S_{r_l} \to A_l S_{r_{l-1}}$ and $\lambda_l\colon S_{r_l} \to T_{G_l}(Y_l)$ as follows: for each $\bar{s}_l \in S_{r_l}$ let us consider its unique decomposition

$$\bar{s}_l = s_l t_l, \quad \text{where} \quad s_l \in S_{q_l}, \xi_l(s_l) = \omega_{s_l}(u_l), t_l \in S_{u_l},$$

$$\beta_l(s_l) = u_l[S_{u_l}, \delta_l], \bar{\beta}_l(s_l) = u_l[S_{u_l}, \bar{\delta}_l], \gamma_l(s_l) = u_l[S_{u_l}, \vartheta_l],$$

($\bar{\alpha}_l, \beta_l, \bar{\beta}_l, \gamma_l, \delta_l, \bar{\delta}_l, \vartheta_l$ as in $E_l, \bar{E}_l$) and $\bar{\alpha}_l(s_l) = b_l s_{l-1}$ for some $b_l \in A_l$ and $s_{l-1} \in S_{q_{l-1}}$. In this case $\eta_l, \bar{\eta}_l$ and λ_l are defined by $\eta_l(s_l t_l) = \delta_l(t_l)$, $\bar{\eta}_l(s_l t_l) = \omega_{s_{l-1}}(\bar{\delta}_l(t_l))$, $\lambda_l(s_l t_l) = \vartheta_l(t_l)$. The derivation $\delta_l(t_l) \Rightarrow^*_{\mathfrak{A}_l} \vartheta_l(t_l)$ holds, which implies that the derivation $\eta_l(s_l t_l) \Rightarrow^*_{\mathfrak{A}_l} \lambda_l(s_l t_l)$ is valid. Thus we obtain the derivations

E_l' and $\bar{E}_l'$ from E_l and $\bar{E}_l$, respectively.

$$E_l': a_l r_{l-1}[S_{r_{l-1}}, \lambda_{l-1}] \Rightarrow^*_{\mathfrak{A}_l} r_l[S_{r_l}, \eta_l] \Rightarrow^*_{\mathfrak{A}_l} r_l[S_{r_l}, \lambda_l],$$

$$\bar{E}_l': a_l r_{l-1} \Rightarrow^*_{\mathfrak{A}_l} r_l[S_{r_l}, \bar{\eta}_l],$$

and for each $v_l \in S_{r_l}$ if $\bar{\eta}_l(v_l) = c_l v_{l-1}$ for some $c_l \in A_i$, $v_{l-1} \in S_{r_{l-1}}$, then $\eta_l(s_l) = c_l \lambda_{l-1}(v_{l-1})$.

For the sequence of root-to-frontier tree transducers $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ we shall define the sets $\Sigma(l)$ and V_l, $(0 \leqq l \leqq k)$ in the following way:

$\Sigma(0) = \{u_0 | u_0 \in G_0 \cup Y_0\}$,

$V_0 = \Sigma(0)$;

$\Sigma(1) = \{(b_1, u_0, u_1[S_{u_1}, \varphi_1], \varrho_1, W_1, \tau_1) | b_1 u_0 \to u_1[S_{u_1}, \varphi_1] \in \Sigma_{\mathfrak{A}_1}$,

$u_1 \in P_{G_1}(Y_1)$, $\varphi_1\colon S_{u_1} \to A_1 S_{u_0}$,

$W_1 = \{(t_0, t_1) | \varphi_1(t_1) = c_1 t_0, c_1 \in A_1\}$,

$\varrho_1\colon S_{u_1} \to W_1$; $\varrho_1(t_1) = (t_0, t_1)$ if $\varphi_1(t_1) = c_1 t_0$,

$\tau_1\colon W_1 \to A_1 S_{u_0}$; $\tau_1((t_0, t_1)) = c_1 t_0$ if $\varphi_1(t_1) = c_1 t_0\}$.

It can be seen that for each $t_1 \in S_{u_1}$, $\varphi_1(t_1) = \tau_1(\varrho_1(t_1))$, that is, $\varphi_1 = \varrho_1 \circ \tau_1$ holds We say that the element $(b_1, u_0, u_1[S_{u_1}, \varphi_1], \varrho_1, W_1, \tau_1)$ of $\Sigma(1)$ is generated by the production $b_1 u_0 \to u_1[S_{u_1}, \varphi_1]$.

$V_1 = \{(u_0, \sigma_1) | u_0 \in \Sigma(0), \sigma_1 \in \Sigma(1)$ and the second component of σ_1 is $u_0\}$.

Let j be an index such that $2 \leqq j \leqq k$, and assume that for each i $(1 \leqq i < j)$ the sets $\Sigma(i)$ and V_i are defined, and that for each $\sigma_i = (b_i \ldots b_1, u_0, u_i[S_{u_i}, \varphi_i], \varrho_i, W_i, \tau_i) (\in \Sigma_{\mathfrak{B}}(i))$ $\varphi_i = \varrho_i \circ \tau_i$ holds. We shall define $\Sigma(j)$ and V_j as follows:

$\Sigma(j) = \{(b_j \ldots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varrho_j, W_j, \tau_j) |$

$(b_{j-1} \ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varrho_{j-1}, W_{j-1}, \tau_{j-1}) \in \Sigma(j-1)$,

$b_j u_{j-1} \Rightarrow^*_{\mathfrak{A}_j} u_j[S_{u_j}, \varepsilon_j]$ holds, where $u_j \in P_{G_j}(Y_j)$,

$\varepsilon_j\colon S_{u_j} \to A_j S_{u_{j-1}}$,

$\varphi_j\colon S_{u_j} \to A_j A_{j-1} \ldots A_1 \{1, \ldots, v(u_0)\}$;

$\varphi_j(t_j) = c_j c_{j-1} \ldots c_1 t_0$ if $\varepsilon_j(t_j) = c_j t_{j-1}$ and

$\varphi_{j-1}(t_{j-1} = c_{j-1} \ldots c_1 t_0$,

$W_j = \{(t_0, \ldots, t_{j-1}, t_j) | \varepsilon_j(t_j) = c_j t_{j-1}, c_j \in A_j, \quad \varrho_{j-1}(t_{j-1}) = (t_0, \ldots, t_{j-1})\} \cup$

$\cup \{(t_0, \ldots, t_{j-1}) \in W_{j-1} |$ there are no t_j in S_{u_j} and $c_j \in A_j$ such that $\varepsilon_j(t_j) = c_j t_{j-1}\} \cup$

$\cup \{(t_0, \ldots, t_l) \in W_{j-1} | 1 \leqq l \leqq j-2\}$,

$\varrho_j\colon S_{u_j} \to W_j$; $\varrho_j(t_j) = (t_0, \ldots, t_{j-1}, t_j)$ if

$\varepsilon_j(t_j) = c_j t_{j-1}$ and $\varrho_{j-1}(t_{j-1}) = (t_0, \ldots, t_{j-1})$,

$\tau_j: W_j \to A_j \dots A_1\{1, \dots, \nu(u_0)\};$

$\tau_j|_{W_j \cap W_{j-1}} = \tau_{j-1}|_{W_j \cap W_{j-1}}$ and

if $(t_0, \dots, t_{j-1}, t_j) \in W_j$ and $\varepsilon_j(t_j) = c_j t_{j-1}$

then $\tau_j((t_0, \dots, t_{j-1}, t_j)) = c_j \tau_{j-1}((t_0, \dots, t_{j-1}))\}$.

We say that the element $(b_j \dots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varrho_j, W_j, \tau_j)$ of $\Sigma(j)$ is generated by the derivation $b_j u_{j-1} \Rightarrow^*_{\mathfrak{A}_j} u_j[S_{u_j}, \varepsilon_j]$ and element

$$(b_{j-1} \dots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varrho_{j-1}, W_{j-1}, \tau_{j-1}) \text{ of } \Sigma(j-1).$$

It can be seen that for each element $(b_j \dots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varrho_j, W_j, \tau_j)$ of $\Sigma(j)$, $\varphi_j = \varrho_j \circ \tau_j$ hold.

$V_j = \{(u_0, \sigma_1, \dots, \sigma_{j-1}, \sigma_j) | (u_0, \sigma_1, \dots, \sigma_{j-1}) \in V_{j-1}, \sigma_{j-1}$

has the form $(b_{j-1} \dots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varrho_{j-1}, W_{j-1}, \tau_{j-1})$, σ_j has the form $(b_j \dots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varrho_j, W_j, \tau_j)$ and σ_j is generated by the derivation $b_j u_{j-1} \Rightarrow^*_{\mathfrak{A}_j} u_j[S_{u_j}, \varepsilon_j]$ and $\sigma_{j-1}\}$.

We define mappings $\varkappa_i$: $[Z_{(D, q_0)}]_i \to \Sigma(i)$ for $0 \leqq i \leqq k$. Let $s_0 \in [Z_{(D, q_0)}]_0$, which means $s_0 \in S_{q_0}$. $\varkappa_0(s_0)$ is defined by

$$\varkappa_0(s_0) = \operatorname{root}(\gamma_0(s_0)) = u_0 \quad \text{if}$$

$$\gamma_0(s_0) = u_0(1, \dots, \nu(u_0))[\{1, \dots, \nu(u_0)\}, \vartheta_0](\vartheta_0\colon \{1, \dots, \nu(u_0)\} \to T_{G_0}(Y_0)).$$

Let $s_1 \in [Z_{(D, q_1)}]_1$, that is, $s_1 \in S_{q_1}$. Let us consider the decomposition $\alpha_1(s_1) = b_1 u_0(1, \dots, \nu(u_0))[\{1, \dots, \nu(u_0)\}, \vartheta_0]$ $(u_0 \in G_0 \cup Y_0,\ \vartheta_0\colon \{1, \dots, \nu(u_0)\} \to T_{G_0}(Y_0))$.

Applying the procedure P to the subderivation (1) $\alpha_1(s_1) \Rightarrow^*_{\mathfrak{A}_1} \gamma_1(s_1)$ of D_1 and decomposition $u_0(1, \dots, \nu(u_0))[\{1, \dots, \nu(u_0)\}, \vartheta_0]$ we obtain derivations (2) and (3).

(2) $b_1 u_0(1, \dots, \nu(u_0))[\{1, \dots, \nu(u_0)\}, \vartheta_0] \Rightarrow_{\mathfrak{A}_1} u_1[S_{u_1}, \delta_1] \Rightarrow^*_{\mathfrak{A}_1}$

$\Rightarrow^*_{\mathfrak{A}_1} u_1[S_{u_1}, \vartheta_1] = \gamma_1(s_1)$, where $u_1 \in P_{G_1}(Y_1)$,

$\delta_1\colon S_{u_1} \to A_1 T_{G_0}(Y_0)$, $\vartheta_1\colon S_{u_1} \to T_{G_1}(Y_1)$,

and for each

$v_1 \in S_{u_1}$, $\delta_1(v_1) \Rightarrow^*_{\mathfrak{A}_1} \vartheta_1(v_1)$ holds.

(3) $b_1 u_0 \Rightarrow_{\mathfrak{A}_1} u_1[S_{u_1}, \bar{\delta}_1]$, where $\bar{\delta}_1\colon S_{u_1} \to A_1\{1, \dots, \nu(u_0)\}$ and for each $v_1 \in S_{u_1}$ if $\bar{\delta}_1(v_1) = c_1 t_0$ $(c_1 \in A_1,\ t_0 \in \{1, \dots, \nu(u_0)\})$, then $\delta_1(v_1) = c_1 \vartheta_0(t_0)$.

$\beta_1(s_1) = u_1[S_{u_1}, \delta_1]$, $\bar{\beta}_1(s_1) = \omega_{s_0}(u_1[S_{u_1}, \bar{\delta}_1])$

for $\beta_1, \bar{\beta}_1$ given in derivations $E_1, \bar{E}_1$.

Let $\varkappa_1(s_1)$ be the element of $\Sigma(1)$ generated by the production $b_1 u_0 \to u_1[S_{u_1}, \bar{\delta}_1]$.

Assume that $\varkappa_i$ is defined for every $0 \leqq i \leqq j-1$. Then the mapping $\varkappa_j$ $(2 \leqq j \leqq k)$ is defined in the following way: for each $s_j (\in [Z_{(D, q_0)}]_j = S_{q_j})$, $\bar{\alpha}_j(s_j) = b_j s_{j-1}$ for some $b_j \in A_j$ and $s_{j-1} \in [Z_{(D, q_0)}]_{j-1}$. Thus $\alpha_j(s_j) = b_j \gamma_{j-1}(s_{j-1})$. $\varkappa_{j-1}(s_{j-1})$ has the form $(b_{j-1} \dots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varrho_{j-1}, W_{j-1}, \tau_{j-1}) \in \Sigma_{\mathfrak{B}}(j-1)$. Let us consider the decomposition $\gamma_{j-1}(s_{j-1}) = u_{j-1}[S_{u_{j-1}}, \vartheta_{j-1}]$ of $\gamma_{j-1}(s_{j-1})$ which is the same as in E_{j-1}.

Applying the procedure P to the subderivation
(1) $\alpha_j(s_j)\Rightarrow^*_{\mathfrak{A}_j}\gamma_j(s_j)$ of D_j and decomposition $u_{j-1}[S_{u_{j-1}}, \vartheta_{j-1}]$ we obtain derivations (2) and (3).

(2) $b_j u_{j-1}[S_{u_{j-1}}, \vartheta_{j-1}]\Rightarrow^*_{\mathfrak{A}_j} u_j[S_{u_j}, \delta_j]\Rightarrow^*_{\mathfrak{A}_j} u_j[S_{u_j}, \vartheta_j] = \gamma_j(s_j)$,

where

$u_j\in P_{G_j}(Y_j)$, $\delta_j\colon S_{u_j}\to A_j T_{G_{j-1}}(Y_{j-1})$, $\vartheta_j\colon S_{u_j}\to T_{G_j}(Y_j)$,

and for every $v_j\in S_{u_j}$ the derivation $\delta_j(v_j)\Rightarrow^*_{\mathfrak{A}_j}\vartheta_j(v_j)$ is valid.

(3) $b_j u_{j-1}\Rightarrow^*_{\mathfrak{A}_j} u_j[S_{u_j}, \bar{\delta}_j]$, where $\bar{\delta}_j\colon S_{u_j}\to A_j S_{u_{j-1}}$ and for each $v_j\in S_{u_j}$ if $\bar{\delta}_j(v_j)=c_j t_{j-1}$ $(c_j\in A_j,\ t_{j-1}\in S_{u_{j-1}})$ then $\delta_j(v_j)=c_j\vartheta_{j-1}(t_{j-1})$.

Let $\varkappa_j(s_j)$ be the element of $\Sigma(j)$ generated by derivation (3) and $\varkappa_{j-1}(s_{j-1})$ $(\in\Sigma(j-1))$.

We associate the configuration

$$K_{(D,r_0)} = \big(r_k[S_{r_k}, \psi_{(D,r_0)}], \Theta_{(D,r_0)}, Z_{(D,r_0)}, \Omega_{(D,r_0)}\big)$$

with the derivation sequence D and decomposition $p_0=r_0[S_{r_0}, \lambda_0]$.

Using the derivation sequences E_l' and $\bar{E}_l'$ we shall show the connection between the configurations $K_{(D,q_0)}$ and $K_{(D,r_0)}$.

(1) $r_k=q_k[S_{q_k}, \xi_k]$, which was established in E_k', moreover we know that for each $s_k\in S_{q_k}$, $\xi_k(s_k)=\omega_{s_k}(u_k)$, where

$\varkappa_k(s_k) = (b_k \dots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$.

(2) $Z_{(D,r_0)} = \{(s_0 t_0, s_1 t_1, \dots, s_j t_j) \mid (s_0, s_1, \dots, s_l)\in Z_{(D,q_0)}$

for some l $(1\leq j\leq l\leq k)$ and

$\varkappa_l(s_l) = (b_l \dots b_1, u_0, u_l[S_{u_l}, \varphi_l], \varrho_l, W_l, \tau_l)$ and $(t_0, t_1, \dots, t_j)\in W_l\}$.

(3) For every $\bar{s}_k\in S_{r_k}$ let us consider its unique decomposition $\bar{s}_k=s_k t_k$, where $s_k\in S_{q_k}$, $\varkappa_k(s_k)$ has the form

$\varkappa_k(s_k) = (b_k \dots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$,

$\xi_k(s_k) = \omega_{s_k}(u_k)$ and $t_k\in S_{u_k}$.

If $\Theta_{(D,q_0)}(s_k) = (s_0, s_1, \dots, s_k)$ and $\varrho_k(t_k) = (t_0, t_1, \dots, t_k)$ then

$\Theta_{(D,r_0)}(\bar{s}_k) = (s_0 t_0, s_1 t_1, \dots, s_k t_k)$.

(4) Let $\bar{s}_k\in S_{r_k}$ be arbitrary, and consider its unique decomposition $\bar{s}_k=s_k t_k$, where $s_k\in S_{q_k}$, $\varkappa_k(s_k)$ has the form

$\varkappa_k(s_k) = (b_k \dots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$, $\xi_k(s_k) = \omega_{s_k}(u_k)$

and $t_k\in S_{u_k}$. Then if $\varphi_k(t_k=c_k \dots c_1 t_0$ and

$\psi_{(D,q_0)}(s_k) = u_0(1, \dots, \nu(u_0))[\{1, \dots, \nu(u_0)\}, \vartheta_0]$

$(u_0\in G_0\cup Y_0$, $\vartheta_0\colon \{1, \dots, \nu(u_0)\}\to T_{G_0}(Y_0))$, then

$\psi_{(D,r_0)}(s_k t_k) = c_k \dots c_1 \vartheta_0(t_0)$.

(5) From the definition of $Z_{(D, r_0)}$ it follows that for every $(\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_j) \in Z_{(D, r_0)}$ there is a vector $(s_0, s_1, \ldots, s_l) \in Z_{(D, q_0)}$ for some $l(\geqq j)$ such that

$$\varkappa_l(s_l) = (b_l \ldots b_1, u_0, u_l[S_{u_l}, \varphi_l], \varrho_l, W_l, \tau_l),$$

and

$$\bar{s}_0 = s_0 t_0,\ \bar{s}_1 = s_1 t_1, \ldots, \bar{s}_j = s_j t_j$$

hold for some $(t_0, t_1, \ldots, t_j) \in W_l$.

If $\tau_l((t_0, t_1, \ldots, t_j)) = c_j \ldots c_1 t_0$ and

$$\Omega_{(D, q_0)}((s_0, s_1, \ldots, s_l)) = b_l \ldots b_1 u_0(1, \ldots, \nu(u_0))[\{1, \ldots, \nu(u_0)\}, \vartheta_0]$$

for some $u_0 \in G_0 \cup Y_0$ and $\vartheta_0\colon \{1, \ldots, \nu(u_0)\} \to T_{G_0}(Y_0)$, then

$$\Omega_{(D, r_0)}((\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_j)) = c_j \ldots c_1 \vartheta_0(t_0).$$

3. k-synchronized R-transducers

In this chapter we shall introduce the notion of a k-synchronized R-transducer and prove that the relations induced by this type of transducers are exactly those relations which can be obtained by compositions of k relations induced by root-to-frontier tree transducers.

Definition 3.1. A k-synchronized R-transducer is a system

$$\mathfrak{B} = (G_0, G_1, \ldots, G_k, Y_0, Y_1, \ldots, Y_k, A_1, \ldots, A_k, A_1', \ldots, A_k', \Sigma_{\mathfrak{B}}, V),$$

where

(1) $k \geqq 2$,

(2) $G_0, G_1, \ldots, G_k$ are operator domains,

(3) $A_1, \ldots, A_k$ are state sets, for $i = 1, \ldots, k$,

$$A_i \ldots A_1 \cap T_{G_0}(Y_0) = \emptyset, \quad \text{and} \quad A_k \ldots A_1 \cap T_{G_k}(Y_k) = \emptyset.$$

(4) $A_1' \subseteq A_1, \ldots, A_k' \subseteq A_k$ are the sets of initial states,

(5) $\Sigma_{\mathfrak{B}}$ is a finite set of productions, which is a disjoint union

$$\Sigma_{\mathfrak{B}} = \Sigma_{\mathfrak{B}}(0) \cup \Sigma_{\mathfrak{B}}(1) \cup \ldots \cup \Sigma_{\mathfrak{B}}(k),$$

$V = V_0 \cup V_1 \cup \ldots \cup V_k$, where $V_0 = \Sigma_{\mathfrak{B}}(0)$, and for $i = 1, \ldots, k$, $V_i \subseteq \Sigma_{\mathfrak{B}}(0) \times \Sigma_{\mathfrak{B}}(1) \times \ldots \times \Sigma_{\mathfrak{B}}(i)$ and $[V]_i = \Sigma_{\mathfrak{B}}(i)$.
$\Sigma_{\mathfrak{B}}(0) = \{u_0 | u_0 \in G_0 \cup Y_0\}$ and the members σ_j of the production sets $\Sigma_{\mathfrak{B}}(j)$ $(j \geqq 1)$ have the form:

$$\sigma_j = (b_j \ldots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varrho_j, W_j, \tau_j), \quad \text{where}$$

$$b_i \in A_i \text{ for } i = 1, \ldots, j,\ u_0 \in G_0 \cup Y_0,$$

$$u_j \in P_{G_j}(Y_j),\ \varphi_j\colon S_{u_j} \to A_j \ldots A_1\{1, \ldots, \nu(u_0)\},$$

W_j is a finite subset of $(N^*)^2\cup\ldots\cup(N^*)^{j+1}$, where $(N^*)^1=N^*$ and for each $l\geqq 1$, $(N^*)^{l+1}=(N^*)^l\times N^*$.

$\varrho_j\colon S_{u_j}\to W_j$, $\tau_j\colon W_j\to(A_j\ldots A_2A_1\cup\ldots\cup A_2A_1\cup A_1)[W_j]_0$,

and the following requirements are satisfied:

a) $j=1$

i) $W_1=\{(t_0, t_1)|t_1\in S_{u_1}, \varphi_1(t_1)=c_1t_0$ for some $c_1\in A_1\}$,

ii) for every $t_1\in S_{u_1}$ if $\varphi_1(t_1)=c_1t_0$ then $\varrho_1(t_1)=(t_0, t_1)$,

iii) $\varphi_1=\varrho_1\circ\tau_1$,

iv) $V_1=\{(u_0, \sigma_1)|u_0\in G_0\cup Y_0$, and the second component of $\sigma_1(\in\Sigma_{\mathfrak{B}}(1))$ is $u_0\}$.

b) $j>1$ if $(u_0, \sigma_1, \ldots, \sigma_{j-1}, \sigma_j)\in V_j$ and σ_{j-1} has the form $(b_{j-1}\ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varrho_{j-1}, W_{j-1}, \tau_{j-1})$ then $(u_0, \sigma_1, \ldots, \sigma_{j-1})\in V_{j-1}$ and there is a mapping $\varepsilon_j\colon S_{u_j}\to A_j[W_{j-1}]_{j-1}$ such that i)—iv) hold:

i) $W_j=\{(t_0, \ldots, t_{j-1}, t_j)|\varepsilon_j(t_j)=c_jt_{j-1}, c_j\in A_j, t_j\in S_{u_j}, t_{j-1}\in S_{u_{j-1}},$

$\varrho_{j-1}(t_{j-1})=(t_0, \ldots, t_{j-1}, t_j)\}\cup$

$\cup\{(t_0, \ldots, t_l)\in W_{j-1}|1\leqq l\leqq j-2\}\cup$

$\cup\{(t_0, \ldots, t_{j-1})\in W_{j-1}|$ there are no $t_j\in S_{u_j}$ and $c_j\in A_j$

such that $\varepsilon_j(t_j)=c_jt_{j-1}\}$.

ii) For

τ_j, $\tau_j|_{W_j\cap W_{j-1}}=\tau_{j-1}|_{W_j\cap W_{j-1}}$ and

if $(t_0, \ldots, t_{j-1}, t_j)\in W_j$, $\varepsilon_j(t_j)=c_jt_{j-1}$ $(c_j\in A_j, t_{j-1}\in[W_{j-1}]_{j-1})$ and

$\tau_{j-1}((t_0, \ldots, t_{j-1}))=c_{j-1}\ldots c_1t_0$ then $\tau_j((t_0, \ldots, t_{j-1}, t_j))=c_j\ldots c_1t_0$.

iii) For each $t_j\in S_{u_j}$ if

$\varepsilon_j(t_j)=c_jt_{j-1}(c_j\in A_j, t_{j-1}\in[W_{j-1}]_{j-1})$ and

$\varrho_{j-1}(t_{j-1})=(t_0, \ldots, t_{j-1})$ then $\varrho_j(t_j)=(t_0, \ldots, t_{j-1}, t_j)$.

iv) $\varphi_j=\varrho_j\circ\tau_j$.

(One can see that for each $t_j\in S_{u_j}$, $\varepsilon_j(t_j)=c_jt_{j-1}$ $(c_j\in A_j, t_{j-1}\in S_{u_{j-1}})$ iff $\varrho_j(t_j)=(t_0, \ldots, t_{j-1}, t_j)$ and $\tau_j((t_0, \ldots, t_{j-1}, t_j))=c_j\ldots c_1t_0$.)

In the rest of the paper we shall denote the arity function of G_0 by v.

Definition 3.2. Let $\mathfrak{B}$ be a k-synchronized R-transducer as in Definition 3.1. A configuration of $\mathfrak{B}$ is a system $(q[S_q, \psi], \Theta, Z, \Omega)$, where $q\in P_{G_k}(Y_k)$, $\psi\colon S_q\to A_k\ldots A_1T_{G_0}(Y_0)$, $\Theta\colon S_q\to Z$; for each $s_k\in S_q$, $\Theta(s_k)=(s_0, \ldots, s_{k-1}, s_k)$ for some $s_0, \ldots, s_{k-1}\in N^*$.

Z is a finite subset of $(N^*)^2\cup\ldots\cup(N^*)^k\cup(N^*)^{k+1}$ such that the following two conditions hold:

i) for $j=0, \ldots, k$ and arbitrary $s_j, \bar{s}_j\in[Z]_j$ if $s_j=\bar{s}_j\bar{\bar{s}}_j$, then $s_j=\bar{s}_j$ and $\bar{\bar{s}}_j=e$,

ii) for each $s_k \in S_q$ $\Theta(s_k)$ is the only element of Z which has the form $(s_0, \ldots, \ldots, s_{k-1}, s_k)$ for some $s_0, \ldots, s_{k-1} \in N^*$.

Ω: $Z \to (A_k A_{k-1} \ldots A_1 \cup A_{k-1} \ldots A_1 \cup \ldots \cup A_1) T_{G_0}(Y_0)$ is a mapping such that $\psi = \Theta \circ \Omega$ holds, that is, the diagram in Figure 3 is commutative.

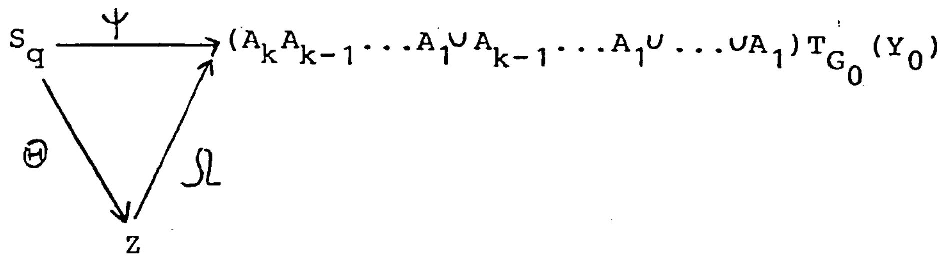

Figure 3

A configuration $(q[S_q, \psi], \Theta, Z, \Omega)$ is said to be a starting configuration, if q is the quasi tree $e \in N^*$ (empty word) and $\psi(e) \in A'_k \ldots A'_1 T_{G_0}(Y_0)$, moreover $Z = \{\underbrace{(e, \ldots, e)}_{k \text{ times}}\}$.

Definition 3.3. Let $K_1 = (q^1[S_{q^1}, \psi^1], \Theta^1, Z^1, \Omega^1)$ and $K_2 = (q^2[S_{q^2}, \psi^2], \Theta^2, Z^2, \Omega^2)$ be configurations of a k-synchronized R-transducer $\mathfrak{B} = (G_0, G_1, \ldots, G_k, Y_0, Y_1, \ldots, Y_k, A_1, \ldots, A_k, A'_1, \ldots, A'_k, \Sigma_{\mathfrak{B}}, V)$. It is said that there is a transition from K_1 to K_2 in $\mathfrak{B}$ which is denoted by $K_1 \Rightarrow_{\mathfrak{B}} K_2$ if there are mappings $\varkappa_j$: $[Z^1]_j \to \Sigma_{\mathfrak{B}}(j)$ for $j = 0, 1, \ldots, k$ such that the following requirements hold:

(1) For each $(s_0, s_1, \ldots, s_j) \in Z^1$ $(1 \leq j \leq k)$ if

$$\Omega^1((s_0, s_1, \ldots, s_j)) = b_j \ldots b_1\, u_0(1, \ldots, \nu(u_0))[\{1, \ldots, \nu(u_0)\}, \vartheta_0] \text{ for some}$$

$$u_0 \in G_0 \cup Y_0,\ b_j \ldots b_1 \in A_j \ldots A_1 \text{ and } \vartheta_0\colon \{1, \ldots, \nu(u_0)\} \to T_{G_0}(Y_0) \text{ then}$$

$$\varkappa_0(s_0) = u_0,\ \varkappa_i(s_i) = (b_i \ldots b_1, u_0, u_i[S_{u_i}, \varphi_i], \varrho_i, W_i, \tau_i)$$

for some u_i, ϱ_i, W_i and τ_i $(i = 1, 2, \ldots, j)$, and $(\varkappa_0(s_0), \varkappa_1(s_1), \ldots, \varkappa_j(s_j)) \in V_j$.

(2) $q^2 = q^1[S_{q^1}, \xi]$ for the mapping ξ: $S_{q^1} \to T_{G_k}(Y_k \cup N^*)$ which is defined by the following formula: for every $s_k \in S_{q^1}$, $\xi(s_k) = \omega_{s_k}(u_k)$ if $\varkappa_k(s_k) = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$.

(3) $Z^2 = \{(s_0 t_0, s_1 t_1, \ldots, s_j t_j) | (s_0, s_1, \ldots, s_l) \in Z^1$ for some l, $(1 \leq j \leq l \leq k)$ and $\varkappa_l(s_l) = (b_l \ldots b_1, u_0, u_l[S_{u_l}, \varphi_l], \varrho_l, W_l, \tau_l)$ and $(t_0, t_1, \ldots, t_j) \in W_l\}$.

(4) For each $\bar{s}_k = S_{q^2}$ consider its unique decomposition $\bar{s}_k = s_k t_k$, where $s_k \in S_{q^1}$, $\varkappa_k(s_k)$ has the form $\varkappa_k(s_k) = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$, $\xi_k(s_k) = \omega_{s_k}(u_k)$, and $t_k \in S_{u_k}$. If $\Theta^1(s_k) = (s_0, s_1, \ldots, s_k)$ and $\varrho_k(t_k) = (t_0, t_1, \ldots, t_k)$ then $\Theta^2(\bar{s}_k) = (s_0 t_0, s_1 t_1, \ldots, s_k t_k)$.

(5) Let $\bar{s}_k \in S_{q^2}$ be arbitrary and consider its unique decomposition $\bar{s}_k = s_k t_k$, where $s_k \in S_{q^1}$, $\varkappa_k(s_k)$ has the form $\varkappa_k(s_k) = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k,$

$\tau_k)$, $\xi_k(s_k)=\omega_{s_k}(u_k)$ and $t_k\in S_{u_k}$. If $\varphi_k(t_k)=c_k\dots c_1 t_0$ and

$$\psi^1(s_k) = u_0(1, \dots, \nu(u_0))[\{1, \dots, \nu(u_0)\}, \vartheta_0]$$

$$(u_0\in G_0\cup Y_0, \vartheta_0\colon \{1, \dots, \nu(u_0)\} \to T_{G_0}(Y_0)),$$

then $\psi^2(\bar{s}_k)=c_k\dots c_1\vartheta_0(t_0)$.

(6) For every $(\bar{s}_0, \bar{s}_1, \dots, \bar{s}_j)\in Z^2$ there is a vector $(s_0, s_1, \dots, s_l)\in Z^1$ for some l $(1\leqq j\leqq l\leqq k)$ such that $\varkappa_l(s_l)=(b_l\dots b_1, u_0, u_l[S_{u_l}, \varphi_l], \varrho_l, W_l, \tau_l)$, and $\bar{s}_0=s_0t_0$, $\bar{s}_1=s_1t_1, \dots, \bar{s}_j=s_jt_j$ hold for some $(t_0, t_1, \dots, t_j)\in W_l$. If $\tau_l((t_0, t_1, \dots, t_j))=c_j\dots c_1 t_0$ and

$$\Omega^1((s_0, s_1, \dots, s_l)) = b_l\dots b_1 u_0(1, \dots, \nu(u_0))[\{1, \dots, \nu(u_0)\}, \vartheta_0]$$

for some $u_0\in G_0\cup Y_0$ and $\vartheta_0\colon \{1, \dots, \nu(u_0)\}\to T_{G_0}(Y_0)$ then

$$\Omega^2((\bar{s}_0, \bar{s}_1, \dots, \bar{s}_j)) = c_j\dots c_1\vartheta_0(t_0).$$

Notice, that given configuration K_1 and mappings $\varkappa_i$, for $i=0, \dots, k$ satisfying condition (1), uniquely determine configuration K_2.

The reflexive and transitive closure of relation $\Rightarrow_{\mathfrak{B}}$ between configurations is denoted by $\Rightarrow^*_{\mathfrak{B}}$.

Definition 3.4. Take a k-synchronized R-transducer

$$\mathfrak{B} = (G_0, G_1, \dots, G_k, Y_0, Y_1, \dots, Y_k, A_1, \dots, A_k, A_1', \dots, A_k', \Sigma_{\mathfrak{B}}, V).$$

Then the relation

$$\tau_{\mathfrak{B}} = \{(p, q)|p\in T_{G_0}(Y_0),\ q\in T_{G_k}(Y_k),$$
$$K_0 = (e[\{e\}, \psi_0\colon e\mapsto bp], \Theta^0, Z^0, \Omega^0)\Rightarrow^*_{\mathfrak{B}}(q, \emptyset, \emptyset, \emptyset)$$
$$\text{for some starting configuration } K_0\}$$

is called the transformation induced by $\mathfrak{B}$.

Configurations of the form $(q, \emptyset, \emptyset, \emptyset)$, where $q\in T_{G_k}(Y_k)$, are said to be final.

Theorem 3.5. Let $\mathfrak{A}_i=(G_{i-1}, Y_{i-1}, A_i, G_i, Y_i, A_i', \Sigma_{\mathfrak{A}_i})$ $(i=1, \dots, k, k\geqq 2)$ be R-transducers. Then there is a k-synchronized R-transducer $\mathfrak{B}$ such that $\tau_{\mathfrak{B}}=\tau_{\mathfrak{A}_1}\circ\dots\circ\tau_{\mathfrak{A}_k}$.

Proof: We construct a k-synchronized R-transducer $\mathfrak{B}$ as follows:

$$\mathfrak{B} = (G_0, G_1, \dots, G_k, Y_0, Y_1, \dots, Y_k, A_1, \dots, A_k, A_1', \dots, A_k', \Sigma_{\mathfrak{B}}, V),$$

where $\Sigma_{\mathfrak{B}}(0)=\Sigma(0), \dots, \Sigma_{\mathfrak{B}}(k)=\Sigma(k)$ for the sets $\Sigma(i)$, which are defined in the previous chapter. $V=V_0\cup V_1\cup\dots\cup V_k$, where the sets $V_0, V_1, \dots, V_k$ are defined in the previous chapter. We may assume without loss of generality that $A_k\dots A_1\cap T_{G_k}(Y_k)=\emptyset$ and that, for $i=1, \dots, k$, $A_i\dots A_1\cap T_{G_0}(Y_0)=\emptyset$. Thus $\mathfrak{B}$ satisfies requirement (3) of Definition 3.1.

First we shall prove the inclusion

$$\tau_{\mathfrak{A}_1}\circ\dots\circ\tau_{\mathfrak{A}_k}\subseteq\tau_{\mathfrak{B}}.$$

Assume that $(p_0, p_k) \in \tau_{\mathfrak{A}_1} \circ \tau_{\mathfrak{A}_2} \circ \dots \circ \tau_{\mathfrak{A}_k}$. Then there are initial states $a_1 \in A'_1, \dots, a_k \in A'_k$ and there is a derivation sequence D: $a_1 p_0 \Rightarrow^*_{\mathfrak{A}_1} p_1$, $a_2 p_1 \Rightarrow^*_{\mathfrak{A}_2} p_2, \dots$ $\dots, a_k p_{k-1} \Rightarrow^*_{\mathfrak{A}_k} p_k$, where $p_i \in T_{G_i}(Y_i)$ for $i=0, \dots, k$.

Take an arbitrary decomposition $p_0 = q_0[S_{q_0}, \gamma_0]$ of the tree p_0, where $q_0 \in P_{G_0}(Y_0)$ and γ_0: $S_{q^0} \to T_{G_0}(Y_0)$. We have constructed a configuration

$$K_{(D,q_0)} = \big(q_k[S_{q_k}, \psi_{(D,q_0)}], \Theta_{(D,q_0)}, Z_{(D,q_0)}, \Omega_{(D,q_0)}\big)$$

for D and q_0 in Chapter 2.

One can see that $K_{(D,q_0)}$ is a configuration of the k-synchronized R-transducer $\mathfrak{B}$. Let $r_0 = q_0[S_{q^0}, \xi_0]$ for the mapping ξ_0: $S_{q_0} \to T_{G_0}(Y_0 \cup N^*)$ which is defined by $\xi_0(s_0) = \omega_{s_0}(u_0(1, \dots, v(u_0)))$ for each $s_0 \in S_{q_0}$, where

$$\gamma_0(s_0) = u_0(1, \dots, v(u_0))[\{1, \dots, v(u_0)\}, \vartheta_0],$$

$$(u_0 \in G_0 \cup Y_0, \vartheta_0\colon \{1, \dots, v(u_0)\} \to T_{G_0}(Y_0)).$$

$K_{(D,r_0)}$ is again a configuration of $\mathfrak{B}$.

It follows from the definition of the relation $\Rightarrow_{\mathfrak{B}}$ that

$$K_{(D,q_0)} = K_{(D,r_0)} \quad \text{or} \quad K_{(D,q_0)} \Rightarrow_{\mathfrak{B}} K_{(D,r_0)}$$

holds.

Let $p^0, p^1, \dots, p^l \in P_{G_0}(Y_0)$ be quasi trees for $l = h(p)+1$ such that for every i

$$(0 \leqq i \leqq l), \quad p_0 = p^i[S_{p^i}, \gamma^i] \quad (\gamma^i\colon S_{p^i} \to T_{G_0}(Y_0)),$$

where

i) $p^0 = e$, $\gamma^0(e) = p_0$, and

ii) $p^{i+1} = p^i[S_{p^i}, \xi^{i+1}]$ for the mapping ξ^{i+1}: $S_{p^i} \to T_{G_0}(Y_0 \cup N^*)$ such that for every $s^i \in S_{p^i}$

$$\xi^{i+1}(s^i) = \omega_{s^i}(u_0(1, \dots, v(u_0))),$$

where

$$\gamma^i(s^i) = u_0(1, \dots, v(u_0))[\{1, \dots, v(u_0)\}, \vartheta_0]$$

for some $u_0 \in G_0 \cup Y_0$ and ϑ_0: $\{1, \dots, v(u_0)\} \to T_{G_0}(Y_0)$. In this case every $s^{i+1} \in S_{p^{i+1}}$ has a unique decomposition $s^{i+1} = s^i t^i$,

$$s^i \in S_{p^i}, \gamma^i(s^i) = u_0(1, \dots, v(u_0))[\{1, \dots, v(u_0)\}, \vartheta_0]$$

for some

$$u_0 \in G_0 \cup Y_0, \vartheta_0\colon \{1, \dots, v(u_0)\} \to T_{G_0}(Y_0) \text{ and } t^i \in \{1, \dots, v(u_0)\}.$$

Then $\gamma^{i+1}(s^{i+1}) = \vartheta_0(t^i)$.

We know that $K_{(D,p^i)} = K_{(D,p^{i+1})}$ or $K_{(D,p^i)} \Rightarrow_{\mathfrak{B}} K_{(D,p^{i+1})}$ holds for $i=0, \dots, l-1$. It has remained to prove that $K_{(D,p^0)}$ is a starting configuration and $K_{(D,p^l)}$ is a final configuration of $\mathfrak{B}$. The first part of the statement trivially holds. Since $p^l = p_0$ and $p_0 \in T_{G_0}(Y_0)$, S_{p^l} must be the empty set, thus $K_{(D,p^l)} = (p_k, \emptyset, \emptyset, \emptyset)$. We have proved that $(p_0, p_k) \in \tau_{\mathfrak{B}}$.

We shall prove the reverse inclusion:

$$\tau_{\mathfrak{B}} \subseteq \tau_{\mathfrak{A}_1} \circ \tau_{\mathfrak{A}_2} \circ \dots \circ \tau_{\mathfrak{A}_k}.$$

Let $K_0 \Rightarrow_{\mathfrak{B}} K_1 \Rightarrow_{\mathfrak{B}} \ldots \Rightarrow_{\mathfrak{B}} K_n$ be a sequence of transitions in $\mathfrak{B}$, where $n \geqq 1$, $K_0 = (e[\{e\}, \psi^0], \Theta^0, Z^0, \Omega^0)$ is a starting configuration and $K_i = (q^i[S_{q^i}, \psi^i], \Theta^i, Z^i, \Omega^i)$ for $i=1, \ldots, n$. Assume that $\psi^0(e) = a_k \ldots a_1 p$. Let $p^0, p^1, \ldots, p^l$ be the sequence of quasi trees constructed in the first part of the proof, where $p^l = p$. It can be seen that $n \leqq l$. Then there is a derivation sequence $D = D_1, \ldots, D_k$,

$$D_1: a_1 p^n \Rightarrow^*_{\mathfrak{A}_1} p_1[S_{p^1}, \eta_1], (p_1 \in P_{G_1}(Y_1), \eta_1: S_{p_1} \to A_1 S_{p^n}),$$

$$D_2: a_2 p_1 \Rightarrow^*_{\mathfrak{A}_2} p_2[S_{p_2}, \eta_2], (p_2 \in P_{G_2}(Y_2), \eta_2: S_{p_2} \to A_2 S_{p_1}),$$

$$\vdots$$

$$D_k: a_k p_{k-1} \Rightarrow^*_{\mathfrak{A}_k} p_k[S_{p_k}, \eta_k], (p_k \in P_{G_k}(Y_k), \eta_k: S_{p_k} \to A_k S_{p_{k-1}})$$

such that the following equalities hold:

i) $p_k = q^n$,

ii) $\psi_{(D, p^n)} = \psi^n$,

iii) $Z^n = Z_{(D, p^n)}$,

iv) $\Theta^n = \Theta_{(D, p^n)}$,

v) $\Omega^n = \Omega_{(D, p^n)}$,

where the sets $Z_{(D, p^n)}$, $\Theta_{(D, p^n)}$ and the mappings

$$\psi_{(D, p^n)}: S_{p_k} \to (A_k \ldots A_2 A_1 \cup \ldots \cup A_2 A_1 \cup A_1) S_{p^n},$$

$$\Omega_{(D, p^n)}: Z_{(D, p^n)} \to (A_k \ldots A_2 A_1 \cup \ldots \cup A_2 A_1 \cup A_1) \operatorname{rg}(\gamma^n)$$

are defined as follows:

(1) $Z_{(D, p^n)} = \{(s_0, s_1, \ldots, s_j) | s_0 \in S_{p^n}, s_1 \in S_{p_1}, \ldots, s_j \in S_{p_j}, \quad 1 \leqq j \leqq k,$

and $(j=k$ or $(j<k$ and there are no $s_{j+1} \in S_{q_{j+1}}$ and $b_{j+1} \in A_{j+1}$ such that $\eta_{j+1}(s_{j+1}) = b_{j+1} s_j))$ and $\eta_i(s_i) = b_i s_{i-1}$ $(b_i \in A_i)$ for $i=1, \ldots, j\}$.

(2) For every $(s_0, s_1, \ldots, s_j) \in Z_{(D, p^n)}$ $(1 \leqq j \leqq k)$, $\Omega_{(D, p^n)}((s_0, s_1, \ldots, s_j)) = b_j \ldots b_1 \gamma^n(s_0)$ iff $\eta_i(s_i) = b_i s_{i-1} (b_i \in A_i)$ for $i=1, \ldots, j$.

(3) For every $s_k \in S_{q^n}$, $\Theta_{(D, p^n)}(s_k) = (s_0, s_1, \ldots, s_k)$ iff

$$\eta_i(s_i) = b_i s_{i-1} (b_i \in A_i) \quad \text{for} \quad i = 1, \ldots, k.$$

(4) $\psi_{(D, p^n)} = \Theta_{(D, p^n)} \circ \Omega_{(D, p^n)}$.

We proceed by induction on n. Let $n=1$. In this case $p^1 = u_0(1, \ldots, v(u_0))$, $u_0 = \operatorname{root}(p)$. From the definition of the transition in $\mathfrak{B}$ it follows that there are mappings $\varkappa_i: \{e\} \to \Sigma_{\mathfrak{B}}(i)$ $(i=0, 1, \ldots, k)$ such that $\varkappa_0(e) = u_0$,

$$\varkappa_1(e) = (a_1, u_0, u_1[S_{u_1}, \varphi_1], \varrho_1, W_1, \tau_1),$$

and so on,

$$\varkappa_k(e) = (a_k \ldots a_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k),$$

and $(\varkappa_0(e), \varkappa_1(e), ..., \varkappa_k(e)) \in V_k$, and configuration K_0 and mappings $\varkappa_i$ $(i=0, ..., k)$ determine the configuration K_1.

According to the construction of the transducer $\mathfrak{B}$ and the definition of the transition in $\mathfrak{B}$, there is a derivation sequence $D=D_1, ..., D_k$,

$$D_1\colon a_1 u_0(1, ..., \nu(u_0)) \Rightarrow^*_{\mathfrak{A}_1} u_1[S_{u_1}, \varphi_1],$$

$$D_i\colon a_i u_{i-1} \Rightarrow^*_{\mathfrak{A}_i} u_i[S_{ui}, \eta_i]$$

for some

$$\eta_i\colon S_{u_i} \to A_i S_{u_{i-1}}, \quad (i = 2, ..., k)$$

such that the following equalities hold:

i) $q^1 = u_k$,

ii) $\psi_{(D, p^1)} = \psi^1$,

iii) $Z^1 = W_k = Z_{(D, p^1)}$,

iv) $\Theta^1 = \varrho_k = \Theta_{(D, p^1)}$,

v) $\Omega^1 = \tau_k = \Omega_{(D, p^1)}$.

The proof of the basic step is complete.

Assume that the statement is true for $n-1$. It means that there is a derivation sequence

$$D_1\colon a_1 p^{n-1} \Rightarrow^*_{\mathfrak{A}_1} p_1[S_{p_1}, \eta_1] \; (p_1 \in P_{G_1}(Y_1), \eta_1\colon S_{p_1} \to A_1 S_{p^{n-1}}),$$

$$D_2\colon a_2 p_1 \Rightarrow^*_{\mathfrak{A}_2} p_2[S_{p_2}, \eta_2] \; (p_2 \in P_{G_2}(Y_2), \eta_2\colon S_{p_2} \to A_2 S_{p_1}),$$

$$\vdots$$

$$D_k\colon a_k p_{k-1} \Rightarrow^*_{\mathfrak{A}_k} p_k[S_{p_k}, \eta_k] \; (p_k \in P_{G_k}(Y_k), \eta_k\colon S_{p_k} \to A_k S_{p_{k-1}})$$

such that the following equalities hold:

i) $p_k = q^{n-1}$,

ii) $\psi_{(D, p^{n-1})} = \psi^{n-1}$,

iii) $Z^{n-1} = Z_{(D, p^{n-1})}$,

iv) $\Theta^{n-1} = \Theta_{(D, p^{n-1})}$,

v) $\Omega^{n-1} = \Omega_{(D, p^{n-1})}$.

Because of the transition $K_{n-1} \Rightarrow_{\mathfrak{B}} K_n$ there are mappings $\varkappa_i\colon [Z^{n-1}]_i \to \Sigma_{\mathfrak{B}}(i)$ $(i=0, ..., k)$ which satisfy condition (1) in Definition 3.3.

Take the sequence $r_0, ..., r_k$ of quasi trees given as follows:

$$r_0 = p^n, \quad \text{for } i = 2, ..., k \quad \text{let} \quad r_i = p_i[S_{p_i}, \varepsilon_i],$$

where $\varepsilon_i\colon S_{p_i} \to T_{G_i}(Y_i \cup N^*)$ such that for every $s \in S_{p_i}$, $\varepsilon_i(s) = \omega_s(u_i)$ holds, where $u_i[S_{u_i}, \varphi_i]$ is the third component of

$$\varkappa_i(s) = (b_i \dots b_1, u_0, u_i[S_{u_i}, \varphi_i]\, \varrho_i, W_i, \tau_i).$$

Let $\xi_i\colon S_{r_i}\to A_i S_{r_{i-1}}$ $(i\in\{1, \ldots, k\})$ be the mapping satisfying the following requirements: we know that for each $\bar{s}_i\in S_{r_i}$ there is a unique decomposition $\bar{s}_i=s_i t_i$ of $\bar{s}_i$, where $s_i\in S_{p_i}$, $\varkappa_i(s_i)=(b_i\ldots b_1, u_0, u_i[S_{u_i}, \varphi_i], \varrho_i, W_i, \tau_i)$ and $t_i\in S_{u_i}$. If $\varrho_i(t_i)=(t_0, \ldots, t_{i-1}, t_i)$ $((t_0, \ldots, t_{i-1}, t_i)\in W_i)$ and $\tau_i((t_0, \ldots, t_{i-1}, t_i))=$ $=c_i\ldots c_1 t_0$ for some $c_i\in A_i, \ldots, c_1\in A_1$, and $\eta_i(s_i)=b_i s_{i-1}$ for some $s_{i-1}\in S_{p_{i-1}}$ then $\xi_i(s_i t_i)=c_i s_{i-1} t_{i-1}$. We obtain that for $i=1, \ldots, k$, $E_i\colon a_i r_{i-1}\Rightarrow^*_{\mathfrak{A}_i} r_i[S_{r_i}, \xi_i]$ holds.

From the definition of the transition in $\mathfrak{B}$ and from the definitions of r_k, $Z_{(E, p^n)}$, $\Theta_{(E, p^n)}$, $\Omega_{(E, p^n)}$, $\psi_{(E, p^n)}$ it follows that

i) $r_k = q^n$,

ii) $\psi_{(E, p^n)} = \psi^n$,

iii) $Z^n = Z_{(E, p^n)}$,

iv) $\Theta^n = \Theta_{(E, p^n)}$,

v) $\Omega^n = \Omega_{(E, p^n)}$.

Assume that $(p, q)\in\tau_{\mathfrak{B}}$. Then there are configurations $K_0, \ldots, K_n$ $(n\geq 1)$ such that K_0 is a starting configuration, $K_0=(e[\{e\}, \psi^0], \Theta^0, Z^0, \Omega^0)$ where $\psi^0(e)=$ $=a_k\ldots a_1 p$ for some $a_k\in A'_k, \ldots, a_1\in A'_1$, K_n is a final configuration, $K_n=(q, \emptyset, \emptyset, \emptyset)$, moreover, $K_{i-1}\Rightarrow_{\mathfrak{B}} K_i$ holds for $i=1, \ldots, n$.

According to the above proposition there is a derivation sequence

$$D = D_1, \ldots, D_k,$$

$$D_1\colon a_1 p^n \Rightarrow^*_{\mathfrak{A}_1} p_1[S_{p_1}, \eta_1], (p_1\in P_{G_1}(Y_1), \eta_1\colon S_{p_1}\to A_1 S_{p^n}),$$

$$D_2\colon a_2 p_1 \Rightarrow^*_{\mathfrak{A}_2} p_2[S_{p_2}, \eta_2], (p_2\in P_{G_2}(Y_2), \eta_2\colon S_{p_2}\to A_2 S_{p_1}),$$

$$\vdots$$

$$D_k\colon a_k p_{k-1} \Rightarrow^*_{\mathfrak{A}_k} p_k[S_{p_k}, \eta_k], (p_k\in P_{G_k}(Y_k), \eta_k\colon S_{p_k}\to A_k S_{p_{k-1}})$$

such that the following equalities hold:

i) $p_k = q$,

ii) $\psi_{(D, p^n)} = \psi^n = \emptyset$,

iii) $Z^n = Z_{(D, p^n)}$,

iv) $\Theta^n = \Theta_{(D, p^n)}$,

v) $\Omega^n = \Omega_{(D, p^n)}$.

Thus $Z_{(D, p^n)}=\emptyset$, $\Theta_{(D, p^n)}=\emptyset$, $\Omega_{(D, p^n)}=\emptyset$. According to the definition of $Z_{(D, p^n)}$, $S_{p^i}=[Z_{(D, p^n)}]_i$ for $i=1, \ldots, k$. Thus $p_i\in T_{G_i}(Y_i)$ for $i=1, \ldots, k$. One can see that $a_1 p^n[S_{p^n}, \gamma^n]\Rightarrow^*_{\mathfrak{A}_1} p_1$ holds. Thus $(p^n[S_{p^n}, \gamma^n], q)\in\tau_{\mathfrak{A}_1}\circ\tau_{\mathfrak{A}_2}\circ\ldots\circ\tau_{\mathfrak{A}_k}$. The proof of the theorem is complete.

Theorem 3.6. Let $\mathfrak{B}=(G_0, G_1, \ldots, G_k, Y_0, Y_1, \ldots, Y_k, A_1, \ldots, A_k, A'_1, \ldots, A'_k, \Sigma_{\mathfrak{B}}, V)$ be a k-synchronized R-transducer. Then there are R-transducers $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ such that $\tau_{\mathfrak{B}}=\tau_{\mathfrak{A}_1}\circ\ldots\circ\tau_{\mathfrak{A}_k}$.

Proof. The production sets $\Sigma_{\mathfrak{B}}(i)$ for $i=1, \ldots, k-1$ are considered to be operator domains with arity function ν^i: $\Sigma_{\mathfrak{B}}(i) \to \{0, 1, 2, \ldots\}$ as follows: for $1 \leqq i \leqq k-1$,

$$\sigma = (b_i \ldots b_1, u_0, u_i[S_{u_i}, \varphi_i], \varrho_i, W_i, \tau_i) \in \Sigma_{\mathfrak{B}}(i)$$

let $\nu^i(\sigma) = |S_{u_i}|$, where $|S_{u_i}|$ denotes the cardinality of the set S_{u_i}.

Remember, the arity function of the operator domain G_0 is denoted by ν.

Convention: Let $S \subseteq N^*$. If $S \neq \emptyset$, then ξ_0, ξ_k: $S \to S$ denote the identity function. If $S = \emptyset$ then ξ_0, ξ_k: $S \to S$ denote the empty function.

For every j $(1 \leqq j \leqq k)$ if $S \neq \emptyset$ then ξ_j: $S \to \{1, \ldots, |S|\}$ denotes the function whose value on $s \in S$ is the ordinal number of s in S with respect to the lexicographic ordering. If $S = \emptyset$ then ξ_j: $S \to S$ denotes the empty function. Thus ξ_j always denotes a bijective function which is determined by its domain.

Take the R-transducer

$$\mathfrak{A}_1 = (G_0, Y_0, A_1, \Sigma_{\mathfrak{B}}(1), \emptyset, \Sigma_{\mathfrak{A}_1}, A_1'),$$

where

$\Sigma_{\mathfrak{A}_1} = \{b_1 u_0 \to \sigma_1(1, \ldots, \nu^1(\sigma_1))[\{1, \ldots, \nu^1(\sigma_1)\}, \beta_1]|$

σ_1 has the form $(b_1, u_0, u_1[S_{u_1}, \varphi_1], \varrho_1, W_1, \tau_1) \in \Sigma_{\mathfrak{B}}(1)$,
$(\sigma_0, (b_1, u_0, u_1[S_{u_1}, \varphi_1], \varrho_1, W_1, \tau_1)) \in V_1$ and the mapping
β_1: $\{1, \ldots, \nu^1(\sigma_1)\} \to A_1\{1, \ldots, \nu(\sigma_0)\}$ is defined as follows:
Let ξ_0: $\{1, \ldots, \nu(u_0)\} \to \{1, \ldots, \nu(u_0)\}$, ξ_1: $S_{u_1} \to \{1, \ldots, |S_{u_1}|\}$.
For each $t_1 \in S_{u_1}$, $\beta_1(\xi_1(t_1)) = c_1 \xi_0(t_0)$ iff $\varrho_1(t_1) = (t_0, t_1)$ and $\tau_1((t_0, t_1)) = c_1 t_0$. (Thus for each $t_1 \in S_{u_1}$, $\beta_1(\xi_1(t_1)) = c_1 \xi_0(t_0)$ iff $\varphi_1(t_1) = c_1 t_0$.)$\}$

For $j = 2, \ldots, k-1$ consider the R-transducer $\mathfrak{A}_j = (\Sigma_{\mathfrak{B}}(j-1), \emptyset, A_j, \Sigma_{\mathfrak{B}}(j), \emptyset, \Sigma_{\mathfrak{A}_j}, A_j')$, where the production set $\Sigma_{\mathfrak{A}_j}$ is defined as follows:

$\Sigma_{\mathfrak{A}_j} = \{b_j \sigma_{j-1} \to \sigma_j(1, \ldots, \nu^j(\sigma_j))[\{1, \ldots, \nu^j(\sigma_j)\}, \beta_j]|$

There is an element $(\sigma_0, \ldots, \sigma_{j-1}, \sigma_j) \in V$ such that σ_{j-1} has the form $(b_{j-1} \ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varrho_{j-1}, W_{j-1}, \tau_{j-1})$,
σ_j has the form $(b_j \ldots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varrho_j, W_j, \tau_j)$.
There is a mapping ε_j: $S_{u_j} \to A_j[W_{j-1}]_{j-1}$ such that conditions i)—iv) in part (5).b of Definition 3.1 hold.
The mapping β_j: $\{1, \ldots, \nu^j(\sigma_j)\} \to A_j\{1, \ldots, \nu^{j-1}(\sigma_{j-1})\}$ is defined as follows:
Let ξ_{j-1}: $S_{u_{j-1}} \to \{1, \ldots, |S_{u_{j-1}}|\}$, ξ_j: $S_{u_j} \to \{1, \ldots, |S_{u_j}|\}$. For every $t_j \in S_{u_j}$, $\beta_j(\xi_j(t_j)) = c_j \xi_{j-1}(t_{j-1})$ $(c_j \in A_j, t_{j-1} \in S_{u_{j-1}})$ iff $\varrho_j(t_j) = (t_0, \ldots, t_{j-1}, t_j)$ and $\tau_j((t_0, \ldots, t_{j-1}, t_j)) = c_j \ldots c_1 t_0$.
(Thus for every $t_j \in S_{u_j}$, $\beta_j(\xi_j(t_j)) = c_j \xi_{j-1}(t_{j-1})$ $(c_j \in A_j, t_{j-1} \in S_{u_{j-1}})$ iff $\varepsilon_j(t_j) = c_j t_{j-1}$.)$\}$

Take the R-transducer $\mathfrak{A}_k = (\Sigma_{\mathfrak{B}}(k-1), \emptyset, A_k, G_k, Y_k, \Sigma_{\mathfrak{A}_k}, A_k')$ where the production set $\Sigma_{\mathfrak{A}_k}$ is defined as follows:

$\Sigma_{\mathfrak{A}_k} = \{a_k \sigma_{k-1} \to u_k[S_{u_k}, \beta_k]|$
There is an element $(\sigma_0, \ldots, \sigma_{k-1}, \sigma_k) \in V$ such that σ_{k-1} has the form $(b_{k-1} \ldots b_1, u_0, u_{k-1}[S_{u_{k-1}}, \varphi_{k-1}], \varrho_{k-1}, W_{k-1}, \tau_{k-1})$,

σ_k has the form $(b_k \dots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$. There is a mapping ε_k: $S_{u_k} \to A_k[W_{k-1}]_{k-1}$ such that conditions i)—iv) in part (5).b of Definition 3.3 hold. The mapping β_k: $S_{u_k} \to A_k\{1, \dots, v^{k-1}(\sigma_{k-1})\}$ is defined as follows: Let ξ_{k-1}: $S_{u_{k-1}} \to \{1, \dots, |S_{u_{k-1}}|\}$, ξ_k: $S_{u_k} \to S_{u_k}$. For every $t_k \in S_{u_k}$, $\beta_k(t_k) = c_k \xi_{k-1}(t_{k-1})$ iff $\varrho_k(t_k) = (t_0, \dots, t_{k-1}, t_k)$ and $\tau_k((t_0, \dots, t_{k-1}, t_k)) = c_k \dots c_1 t_0$. (Thus for every $t_k \in S_{u_k}$, $\beta_k(t_k) = c_k \xi_{k-1}(t_{k-1})$ $(c_k \in A_k, t_{k-1} \in S_{u_{k-1}})$ iff $\varepsilon_k(t_k) = c_k t_{k-1}$.)\}

We may assume without loss of generality that for

$$i = 2, \dots, k-1, A_i \cap T_{\Sigma_{\mathfrak{B}}(i-1)}(\emptyset) = \emptyset, \quad A_i \cap T_{\Sigma_{\mathfrak{B}}(i)}(\emptyset) = \emptyset$$

and that $A_1 \cap T_{\Sigma_{\mathfrak{B}}(0)}(\emptyset) = \emptyset$, $A_k \cap T_{\Sigma_{\mathfrak{B}}(k-1)}(\emptyset) = \emptyset$.

Thus $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ satisfy requirement (2) of Definition 1.10.

We shall prove that $\tau_{\mathfrak{B}} = \tau_{\mathfrak{A}_1} \circ \dots \circ \tau_{\mathfrak{A}_k}$. Let $\mathfrak{C}$ be the k-synchronized R-transducer that can be obtained from $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ by the construction of Theorem 3.5. In this case

$$\mathfrak{C} = (G_0, \Sigma_{\mathfrak{B}}(1), \dots, \Sigma_{\mathfrak{B}}(k-1), G_k, Y_0, \underbrace{\emptyset, \dots, \emptyset}_{k-1 \text{ times}}, Y_k, A_1, \dots, A_k, A_1', \dots, A_k', \Sigma_{\mathfrak{C}}, \bar{V}).$$

We may assume without loss of generality that for $i = 1, \dots, k$, $A_i \dots A_1 \cap T_{G_0}(Y_0) = \emptyset$ and that $A_k \dots A_1 \cap T_{G_k}(Y_k) = \emptyset$. Thus $\mathfrak{C}$ satisfies the requirements of Definition 3.1.

By Theorem 3.5, $\tau_{\mathfrak{C}} = \tau_{\mathfrak{A}_1} \circ \tau_{\mathfrak{A}_2} \circ \dots \circ \tau_{\mathfrak{A}_k}$, so it is sufficient to prove that $\tau_{\mathfrak{B}} = \tau_{\mathfrak{C}}$. In order to prove this equality we shall introduce bijective mappings γ_j: $\Sigma_{\mathfrak{B}}(j) \to \Sigma_{\mathfrak{C}}(j)$ for $j = 0, \dots, k-1$ and a surjective mapping γ_k: $\Sigma_{\mathfrak{B}}(k) \to \Sigma_{\mathfrak{C}}(k)$, and we shall show that for $j = 0, \dots, k$ the mappings $\gamma_0, \dots, \gamma_j$ satisfy assumption (1) and that for $j = 0, \dots, k$ the mapping γ_j satisfies assumption (2).

(1) There are two cases.

Case 1. $0 \leq j \leq k-1$. In this case if $(\sigma_0, \dots, \sigma_j) \in V_j$ then $(\gamma_0(\sigma_0), \dots, \gamma_j(\sigma_j)) \in \bar{V}_j$, and if $(\bar{\sigma}_0, \dots, \bar{\sigma}_j) \in \bar{V}_j$ then $(\gamma_0^{-1}(\bar{\sigma}_0), \dots, \gamma_j^{-1}(\bar{\sigma}_j)) \in V_j$.

Case 2. $j = k$. In this case if $(\sigma_0, \dots, \sigma_k) \in V_k$ then $(\gamma_0(\sigma_0), \dots, \gamma_k(\sigma_k)) \in \bar{V}_k$, and if $(\bar{\sigma}_0, \dots, \bar{\sigma}_{k-1}, \bar{\sigma}_k) \in \bar{V}_k$ then there is a unique $\sigma_k \in \Sigma_{\mathfrak{B}}(k)$ such that $\gamma_k(\sigma_k) = \bar{\sigma}_k$ and $(\gamma_0^{-1}(\bar{\sigma}_0), \dots, \gamma_{k-1}^{-1}(\bar{\sigma}_{k-1}), \sigma_k) \in V_k$.

(2) There are three cases.

Case 1. $j = 0$. In this case $\Sigma_{\mathfrak{C}}(0) = \Sigma_{\mathfrak{B}}(0)$ and γ_0 is the identity function.

Case 2. $1 \leq j \leq k-1$. Let $(\sigma_0, \dots, \sigma_j) \in V_j$ and $\sigma_0 = u_0$. Assume that σ_l $(1 \leq l \leq j)$ has the form $(b_l \dots b_1, u_0, u_l[S_{u_l}, \varphi_l], \varrho_l, W_l, \tau_l)$. Let ξ_0: $\{1, \dots, v(u_0)\} = [W_j]_0 \to \{1, \dots, v(u_0)\}$, ξ_l: $S_{u_l} = [W_j]_l \to \{1, \dots, |S_{u_l}|\}$ for $l = 1, \dots, j$. Then $\gamma_j(\sigma_j)$ has the form $\gamma_j(\sigma_j) = (b_j \dots b_1, u_0, \sigma_j(1, \dots, v^j(\sigma_j))[\{1, \dots, v^j(\sigma_j)\}, \bar{\varphi}_j], \bar{\varrho}_j, \bar{W}_j, \bar{\tau}_j)$, and the following hold:

i) $[\bar{W}_j]_0 = [W_j]_0$ and for $i = 1, \dots, j-1, [\bar{W}_j]_i = \{1, \dots, |[W_j]_i|\} = \{1, \dots, |S_{u_i}|\} = \operatorname{rg}(\xi_i)$.

ii) $(t_0, t_1, \dots, t_l) \in W_j$ iff $(\xi_0(t_0), \xi_1(t_1), \dots, \xi_l(t_l)) \in \bar{W}_j$ $(1 \leq l \leq j)$ $(t_0 \in N^*, t_1 \in N^*, \dots, t_l \in N^*)$.

iii) For every

$t_j \in S_{u_j}, \varphi_j(t_j) = c_j \dots c_1 t_0$ iff $\bar{\varphi}_j(\xi_j(t_j)) = c_j \dots c_1 \xi_0(t_0)$,

$(t_0 \in S_{u_0}, c_1 \in A_1, \dots, c_j \in A_j)$.

iv) For each

$t_j \in S_{u_j}, \varrho_j(t_j) = (t_0, t_1, \dots, t_j)$ iff $\bar{\varrho}_j(\xi_j(t_j)) = (\xi_0(t_0), \xi_1(t_1), \dots, \xi_j(t_j))$.

v) For every

$(t_0, t_1, \dots, t_l) \in W_j \quad (1 \leqq l \leqq j), \quad \tau_j((t_0, t_1, \dots, t_l)) = c_l \dots c_1 t_0$ iff

$\bar{\tau}_j((\xi_0(t_0), \xi_1(t_1), \dots, \xi_l(t_l))) = c_l \dots c_1 \xi_0(t_0)$,

$(t_0 \in \{1, \dots, \nu(u_0)\}, c_1 \in A_1, \dots, c_l \in A_l)$.

Case 3. $j=k$. Let $(\sigma_0, \sigma_1, \dots, \sigma_k) \in V_k$ and $\sigma_0 = u_0$.
Assume that σ_l $(1 \leqq l \leqq k)$ has the form

$$(b_l \dots b_1, u_0, u_l[S_{u_l}, \varphi_l], \varrho_l, W_l, \tau_l).$$

Let

$$\xi_0 \colon \{1, \dots, \nu(u_0)\} = [W_k]_0 \to \{1, \dots, \nu(u_0)\},$$

$$\xi_l \colon S_{u_l} = [W_k]_l \to \{1, \dots, |S_{u_l}|\} \text{ for } l = 1, \dots, k-1, \; \xi_k \colon S_{u_k} = [W_k]_k \to S_{u_k}.$$

Then $\gamma_k(\sigma_k)$ has the form $\gamma_k(\sigma_k) = (b_k \dots b_1, u_0, u_k[S_{u_k}, \bar{\varphi}_k], \bar{\varrho}_k, \bar{W}_k, \bar{\tau}_k)$ and the following hold:

i) $[\bar{W}_k]_0 = [W_k]_0$, for $i = 1, \dots, k-1$, $[\bar{W}_k]_i = \{1, \dots, [W_k]_i\} = \{1, \dots, |S_{u_i}|\} = \mathrm{rg}(\xi_i)$ and $[\bar{W}_k]_k = [W_k]_k = S_{u_k} = \mathrm{rg}(\xi_k)$.

ii) $(t_0, t_1, \dots, t_l) \in W_k$ iff $(\xi_0(t_0), \xi_1(t_1), \dots, \xi_l(t_l)) \in \bar{W}_k$

$(1 \leqq l \leqq k, t_0, t_1, \dots, t_l \in N^*)$.

iii) For every

$t_k \in S_{u_k}, \varphi_k(t_k) = c_k \dots c_1 t_0$ iff $\bar{\varphi}_k(\xi_k(t_k)) = c_k \dots c_1 \xi_0(t_0)$,

$(t_0 \in S_{u_0}, c_1 \in A_1, \dots, c_k \in A_k)$.

iv) For each

$t_k \in S_{u_k}, \varrho_k(t_k) = (t_0, t_1, \dots, t_k)$ iff $\bar{\varrho}_k(\xi_k(t_k)) = (\xi_0(t_0), \xi_1(t_1), \dots, \xi_k(t_k))$.

v) For every

$(t_0, t_1, \dots, t_l) \in W_k \quad (1 \leqq l \leqq k), \tau_k((t_0, t_1, \dots, t_l)) = c_l \dots c_1 t_0$ iff

$\bar{\tau}_k((\xi_0(t_0), \xi_1(t_1), \dots, \xi_l(t_l))) = c_l \dots c_1 t_0$,

$(t_0 \in \{1, \dots, \nu(u_0)\}, c_1 \in A_1, \dots, c_l \in A_l)$.

We shall define mappings $\gamma_j \colon \Sigma_{\mathfrak{B}}(j) \to \Sigma_{\mathfrak{C}}(j)$ according to the construction of $\Sigma_{\mathfrak{B}}(j)$ and $\mathfrak{A}_j$.

Let $j=0$. Since $\Sigma_{\mathfrak{C}}(0)=\Sigma_{\mathfrak{B}}(0)$, let γ_0 be the identity mapping.
Let $j=1$. In this case

$\Sigma_{\mathfrak{C}}(1) = \{(b_1, u_0, \sigma_1(1, \ldots, v^1(\sigma_1))[\{1, \ldots, v^1(\sigma_1)\}, \bar{\varphi}_1], \bar{\varrho}_1, \overline{W}_1, \bar{\tau}_1)|$

i) $(u_0, \sigma_1)\in V$ such that σ_1 has the form $(b_1, u_0, u_1[S_{u_1}, \varphi_1], \varrho_1, W_1, \tau_1)$.
ii) $\gamma_0(u_0)=u_0$, the production

$$b_1 u_0 \to \sigma_1(1, \ldots, v^1(\sigma_1))[\{1, \ldots, v^1(\sigma_1)\}, \beta_1]\in\Sigma_{\mathfrak{A}_1},$$

where the mapping β_1: $\{1, \ldots, v^1(\sigma_1)\}\to A_1\{1, \ldots, v(u_0)\}$ is defined as follows:
Let

$$\xi_0\colon \{1, \ldots, v(u_0)\} \to \{1, \ldots, v(u_0)\}, \quad \xi_1\colon S_{u_1} \to \{1, \ldots, |S_{u_1}|\}.$$

For each

$t_1\in S_{u_1}, \beta_1(\xi_1(t_1)) = c_1\xi_0(t_0)$ iff $\varrho_1(t_1) = (t_0, t_1)$ and $\tau_1((t_0, t_1)) = c_1 t_0$.

(Thus for each $t_1\in S_{u_1}$, $\beta_1(\xi_1(t_1))=c_1\xi_0(t_0)$ iff $\varphi_1(t_1)=c_1 t_0$.)

iii) $\bar{\varphi}_1 = \beta_1$,

iv) $\bar{\varrho}_1\colon \{1, \ldots, v^1(\sigma_1)\} \to \overline{W}_1$;

for every

$\xi_1(t_1)\in\{1, \ldots, v^1(\sigma_1)\}$, if $\beta_1(\xi_1(t_1)) = c_1\xi_0(t_0)$ $(c_1\in A_1, t_0\in\{1, \ldots, v(u_0)\})$ then $\bar{\varrho}_1(\xi_1(t_1))=(\xi_0(t_0), \xi_1(t_1))$.

v) $\bar{\tau}_1\colon \overline{W}_1\to A_1\{1, \ldots, v(u_0)\}$;

for every

$(\xi_0(t_0), \xi_1(t_1))\in\overline{W}_1$, if $\beta_1(\xi_1(t_1)) = c_1\xi_0(t_0)$ $(c_1\in A_1, t_0\in\{1, \ldots, v(u_0)\})$ then $\bar{\tau}_1((\xi_0(t_0), \xi_1(t_1))) = c_1\xi_0(t_0).\}$

It can be seen that

$\overline{V}_1 = \{(\gamma_0(\sigma_0), \bar{\sigma}_1)|\sigma_0 = u_0\in\Sigma_{\mathfrak{B}}(0), \bar{\sigma}_1\in\Sigma_{\mathfrak{C}}(1))$ has the form

$$(b_1, u_0, \sigma_1(1, \ldots, v^1(\sigma_1))[\{1, \ldots, v^1(\sigma_1)\}, \bar{\varphi}_1], \bar{\varrho}_1, \overline{W}_1, \bar{\tau}_1)$$

and $\bar{\sigma}_1$ is generated by the production

$$b_1 u_0 \to \sigma_1(1, \ldots, v^1(\sigma_1))[\{1, \ldots, v^1(\sigma_1)\}, \bar{\varphi}_1]\in\Sigma_{\mathfrak{B}}(1).\}$$

We define γ_1: $\Sigma_{\mathfrak{B}}(1)\to\Sigma_{\mathfrak{C}}(1)$ as follows:
Let $\sigma_1=(b_1, u_0, u_1[S_{u_1}, \varphi_1], \varrho_1, W_1, \tau_1)\in\Sigma_{\mathfrak{B}}(1)$, then by the construction of $\mathfrak{A}_1$ and $\mathfrak{C}$ there is a unique production $b_1u_0\to\sigma_1(1, \ldots, v^1(\sigma_1))[\{1, \ldots, v^1(\sigma_1)\}, \beta_1]\in\Sigma_{\mathfrak{A}_1}$ which generates a unique $\bar{\sigma}_1\in\Sigma_{\mathfrak{C}}(1)$. We define $\gamma_1(\sigma_1)$ to be $\bar{\sigma}_1$. One can see by the definition of $\Sigma_{\mathfrak{C}}(1)$ that γ_1 is onto, hence γ_1 is bijective.

It is routine work to check according to the construction of $\mathfrak{A}_1$ and $\Sigma_{\mathfrak{C}}(1)$ that γ_0, γ_1 satisfy condition (1) and that γ_1 satisfies condition (2).

Let j be an index between 2 and $k-1$. We can assume that $\Sigma_{\mathfrak{C}}(0), \Sigma_{\mathfrak{C}}(1), \ldots, \Sigma_{\mathfrak{C}}(j-1)$ and $\gamma_0, \ldots, \gamma_{j-1}$ are defined such that $\gamma_0, \ldots, \gamma_{j-1}$ satisfy condition (1) and that γ_{j-1} satisfies condition (2).

We know that $\Sigma_{\mathfrak{C}}(j)$ is the set

$\Sigma_{\mathfrak{C}}(j) = \{(b_j \dots b_1, u_0, \sigma_j(1, \dots, v^j(\sigma_j))[\{1, \dots, v^j(\sigma_j)\}, \bar{\varphi}_j], \bar{\varrho}_j, \overline{W}_j, \bar{\tau}_j)|$

i) There is an element $(\sigma_0, \dots, \sigma_{j-1}, \sigma_j) \in V$ such that $\sigma_0 = u_0$, σ_{j-1} has the form

$$(b_{j-1} \dots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varrho_{j-1}, W_{j-1}, \tau_{j-1}),$$

σ_j has the form $(b_j \dots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varrho_j, W_j, \tau_j)$. There is a mapping $\varepsilon_j\colon S_{u_j} \to A_j[W_{j-1}]_{j-1}$ such that conditions i)—iv) in part (5).b of Definition 3.1 hold.

ii) $\gamma_{j-1}(\sigma_{j-1}) = (b_{j-1} \dots b_1, u_0, \sigma_{j-1}(1, \dots, v^{j-1}(\sigma_{j-1}))[\{1, \dots, v^{j-1}(\sigma_{j-1})\}, \bar{\varphi}_{j-1}],$
$\bar{\varrho}_{j-1}, \overline{W}_{j-1}, \bar{\tau}_{j-1}) \in \Sigma_{\mathfrak{C}}(j-1)$

and the production $b_j \sigma_{j-1} \to \sigma_j(1, \dots, v^j(\sigma_j))[\{1, \dots, v^j(\sigma_j)\}, \beta_j]$ is in $\Sigma_{\mathfrak{A}_j}$, where the mapping

$$\beta_j\colon \{1, \dots, v^j(\sigma_j)\} \to A_j\{1, \dots, v^{j-1}(\sigma_{j-1})\}$$

is defined as follows:

Let $\xi_{j-1}\colon S_{u_{j-1}} \to \{1, \dots, |S_{u_{j-1}}|\}$, $\xi_j\colon S_{u_j} \to \{1, \dots, |S_{u_j}|\}$. For every $t_j \in S_{u_j}$, $\beta_j(t_j) = c_j \xi_{j-1}(t_{j-1})$ iff $\varrho_j(t_j) = (t_0, \dots, t_{j-1}, t_j)$ and $\tau_j((t_0, \dots, t_{j-1}, t_j)) = c_j \dots c_1 t_0$. (Thus for each $t_j \in S_{u_j}$, $\beta_j(\xi_j(t_j)) = c_j \xi_{j-1}(t_{j-1})$ iff $\varepsilon_j(t_j) = c_j t_{j-1}$.)

iii) $\overline{W}_j = \{(\bar{t}_0, \dots, \bar{t}_{j-2}, \xi_{j-1}(t_{j-1}), \xi_j(t_j)) | \beta_j(\xi_j(t_j)) = c_j \xi_{j-1}(t_{j-1}),$
$\bar{\varrho}_{j-1}(\xi_{j-1}(t_{j-1})) = (\bar{t}_0, \dots, \bar{t}_{j-2}, \xi_{j-1}(t_{j-1}))\} \cup$
$\cup \{(\bar{t}_0, \dots, \bar{t}_{j-1}) \in \overline{W}_{j-1} |$ there are no $\bar{t}_j$ in $\{1, \dots, v^j(\sigma_j)\}$ and
$c_j \in A_j$ such that $\beta_j(\bar{t}_j) = c_j \bar{t}_{j-1}\} \cup$
$\cup \{(\bar{t}_0, \dots, \bar{t}_l) \in \overline{W}_{j-1} | 1 \leqq l \leqq j-2\}.$

iv) $\bar{\varrho}_j\colon \{1, \dots, v^j(\sigma_j)\} \to \overline{W}_j$ satisfies the following requirement: for every
$t_j \in S_{u_j}$ if $\beta_j(\xi_j(t_j)) = c_j \xi_{j-1}(t_{j-1})$ and
$\bar{\varrho}_{j-1}(\xi_{j-1}(t_{j-1})) = (\xi_0(t_0), \dots, \xi_{j-1}(t_{j-1}))$ then
$(\xi_0(t_0), \dots, \xi_{j-1}(t_{j-1}), \xi_j(t_j)) \in \overline{W}_j$ and
$\bar{\varrho}_j(\xi_j(t_j)) = (\xi_0(t_0), \dots, \xi_{j-1}(t_{j-1}), \xi_j(t_j)).$

v) For $\bar{\tau}_j\colon \overline{W}_j \to A_j \dots A_1\{1, \dots, v(u_0)\}$,
$\bar{\tau}_j|_{\overline{W}_j \cap \overline{W}_{j-1}} = \bar{\tau}_{j-1}|_{\overline{W}_j \cap \overline{W}_{j-1}}$ and if
$(t_0, \dots, t_{j-2}, \xi_{j-1}(t_{j-1}), \xi_j(t_j)) \in \overline{W}_j,$
$\beta_j(\xi_j(t_j)) = c_j \xi_{j-1}(t_{j-1})$ and $\bar{\varrho}_{j-1}(\xi_{j-1}(t_{j-1})) = (\bar{t}_0, \dots, \bar{t}_{j-2}, \xi_{j-1}(t_{j-1}))$
then $\bar{\tau}_j((\bar{t}_0, \dots, \bar{t}_{j-2}, \xi_{j-1}(t_{j-1}), \xi_j(t_j))) = c_j \bar{\tau}_{j-1}((\bar{t}_0, \dots, \bar{t}_{j-2}, \xi_{j-1}(t_{j-1}))).$

vi) $\bar{\varphi}_j = \bar{\varrho}_j \circ \bar{\tau}_j$.}

It can be seen that

$\bar{V}_j = \{(\gamma_0(\sigma_0), \ldots, \gamma_{j-1}(\sigma_{j-1}), \bar{\sigma}_j) |$ for $i = 0, \ldots, j-1$

$\sigma_i \in \Sigma_{\mathfrak{B}}(i), (\gamma_0(\sigma_0), \ldots, \gamma_{j-1}(\sigma_{j-1})) \in \bar{V}_{j-1}$,

σ_0 has the form u_0, and σ_{j-1} has the form $(b_{j-1} \ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varrho_{j-1}, W_{j-1}, \tau_{j-1})$. There is an element

$$\sigma_j = (b_j \ldots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varrho_j, W_j, \tau_j) \in \Sigma_{\mathfrak{B}}(j)$$

such that $(\sigma_0, \ldots, \sigma_{j-1}, \sigma_j) \in V_j$, and there is a mapping ε_j: $S_{u_j} \to A_j[W_{j-1}]_{j-1}$ such that conditions i)—iv) in part (5).b of Definition 3.1. hold.

$$\bar{\sigma}_j = (b_j, \ldots, b_1, u_0, \sigma_j(1, \ldots, v^j(\sigma_j))[\{1, \ldots, v^j(\sigma^j)\}, \bar{\varphi}_j], \bar{\varrho}_j, \bar{W}_j, \bar{\tau}_j)$$

satisfies the requirements ii)—vi) of $\Sigma_{\mathfrak{C}}(j)$.}

We define γ_j: $\Sigma_{\mathfrak{B}}(j) \to \Sigma_{\mathfrak{C}}(j)$ as follows: Let us consider the set

$\Gamma_j = \{\gamma: \Sigma_{\mathfrak{B}}(j) \to \Sigma_{\mathfrak{C}}(j) |$ for each $\sigma_j \in \Sigma_{\mathfrak{B}}(j), \gamma(\sigma_j) = \bar{\sigma}_j$

has the form

$$(b_j \ldots b_1, u_0, \sigma_j(1, \ldots, v^j(\sigma_j))[\{1, \ldots, v^j(\sigma^j)\}, \bar{\varphi}_j], \bar{\varrho}_j, \bar{W}_j, \bar{\tau}_j)$$

and there is a vector $(\sigma_0, \ldots, \sigma_{j-1}, \sigma_j) \in V_j$ such that σ_0 has the form u_0, σ_{j-1} has the form $(b_{j-1} \ldots b_1, u_0, u_{j-1}[S_{u_{j-1}}, \varphi_{j-1}], \varrho_{j-1}, W_{j-1}, \tau_{j-1}), \sigma_j$ has the form $(b_j \ldots b_1, u_0, u_j[S_{u_j}, \varphi_j], \varrho_j, W_j, \tau_j)$, and there is a mapping ε_j: $S_{u_j} \to A_j[W_{j-1}]_{j-1}$ such that conditions i)—iv) in part (5).b of Definition 3.1 hold, and $\bar{\sigma}_j$ satisfies the requirements ii)—vi) of $\Sigma_{\mathfrak{C}}(j)$.}

One can see that if $\bar{\gamma}_j \in \Gamma_j$ then $\bar{\gamma}_j$ is injective and $\bar{\gamma}_j$ satisfies condition (2) because of the construction of $\mathfrak{A}_j$ and $\Sigma_{\mathfrak{C}}(j)$. Using this fact one can see that $|\Gamma_j| = 1$. Let γ_j be the only element of Γ_j. One can see that γ_j is bijective, and $\gamma_0, \ldots, \gamma_{j-1}, \gamma_j$ satisfy condition (1).

Let $j = k$. We can assume that $\Sigma_{\mathfrak{C}}(0), \Sigma_{\mathfrak{C}}(1,) \ldots, \Sigma_{\mathfrak{C}}(k-1)$ and $\gamma_0, \ldots, \gamma_{k-1}$ are defined such that $\gamma_0, \ldots, \gamma_{k-1}$ satisfy condition (1) and that γ_{k-1} satisfies condition (2).

We know that $\Sigma_{\mathfrak{C}}(k)$ is the set

$\Sigma_{\mathfrak{C}}(k) = \{(b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \bar{\varrho}_k, \bar{W}_k, \bar{\tau}_k) |$

i) there is an element $(\sigma_0, \ldots, \sigma_{k-1}, \sigma_k) \in V_k$ such that $\sigma_0 = u_0, \sigma_{k-1}$ has the form $(b_{k-1} \ldots b_1, u_0, u_{k-1}[S_{u_{k-1}}, \varphi_{k-1}], \varrho_{k-1}, W_{k-1}, \tau_{k-1})$, σ_k has the form $(b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$. There is a mapping ε_k: $S_{u_k} \to A_k[W_{k-1}]_{k-1}$ such that conditions i)—iv) in part (5).b of Definition 3.1 hold.

ii) $\gamma_{k-1}(\sigma_{k-1}) = (b_{k-1} \ldots b_1, u_0, \sigma_{k-1}(1, \ldots, v^{k-1}(\sigma_{k-1}))[\{1, \ldots, v^{k-1}(\sigma_{k-1})\}, \bar{\varphi}_{k-1}],$
$\bar{\varrho}_{k-1}, \bar{W}_{k-1}, \bar{\tau}_{k-1}) \in \Sigma_{\mathfrak{B}}(k-1)$

and the production

$b_k \sigma_{k-1} \to u_k \; [S_{u_k}, \beta_k]$ is in $\Sigma_{\mathfrak{A}_k}$,

where the mapping

$\beta_k \colon S_{u_k} \to A_k \{1, \dots, \nu^{k-1}(\sigma_{k-1})\}$

is defined as follows: Let

$\xi_{k-1} \colon S_{u_{k-1}} \to \{1, \dots, |S_{u_{k-1}}|\}$, $\xi_k \colon S_{u_k} \to S_{u_k}$.

For every

$t_k \in S_{u_k}$, $\beta_k(t_k) = c_k \xi_{k-1}(t_{k-1})$ iff $\varrho_k(t_k) = (t_0, \dots, t_{k-1}, t_k)$ and

$\tau_k((t_0, \dots, t_{k-1}, t_k)) = c_k \dots c_1 t_0$.

(Thus for each $t_k \in S_{u_k}$, $\beta_k(\xi_k(t_k)) = c_k \xi_{k-1}(t_{k-1})$ iff $\varepsilon_k(t_k) = c_k t_{k-1}$.)

iii) $\overline{W}_k = \{(\bar{t}_0, \dots, \bar{t}_{k-2}, \xi_{k-1}(t_{k-1}), \xi_k(t_k)) | \beta_k(\xi_k(t_k)) = c_k \xi_{k-1}(t_{k-1}),$

$\bar{\varrho}_{k-1}(\xi_{k-1}(t_{k-1})) = (\bar{t}_0, \dots, \bar{t}_{k-2}, \xi_{k-1}(t_{k-1}))\} \cup$

$\cup \{(\bar{t}_0, \dots, \bar{t}_{k-2}, \bar{t}_{k-1}) \in W_{k-1} |$

there are no $t_k \in S_{u_k}$ and $c_k \in A_k$ such that $\beta_k(\bar{t}_k) = c_k \bar{t}_{k-1}\} \cup$

$\cup \{(\bar{t}_0, \dots, \bar{t}_{l-1}, \bar{t}_l) \in W_{k-1} | 1 \leqq l \leqq k-2\}$.

iv) $\bar{\varrho}_k \colon S_{u_k} \to \overline{W}_k$ satisfies the following requirement: for every

$t_k \in S_{u_k}$, if $\beta_k(\xi_k(t_k)) = c_k \xi_{k-1}(t_{k-1})$ and

$\bar{\varrho}_{k-1}(\xi_{k-1}(t_{k-1})) = (\bar{t}_0, \dots, \bar{t}_{k-2}, \xi_{k-1}(t_{k-1}))$, then

$(\bar{t}_0, \dots, \bar{t}_{k-2}, \xi_{k-1}(t_{k-1}), \xi_k(t_k)) \in W_k$ and

$\bar{\varrho}_k(\xi_k(t_k)) = (\bar{t}_0, \dots, \bar{t}_{k-2}, \xi_{k-1}(t_{k-1}), \xi_k(t_k))$.

v) For

$\bar{\tau}_k \colon \overline{W}_k \to (A_k \dots A_2 A_1 \cup \dots \cup A_2 A_1 \cup A_1)\{1, \dots, \nu(u_0)\}$, $\bar{\tau}_k|_{W_k \cap W_{k-1}} = \bar{\tau}_{k-1}|_{W_k \cap W_{k-1}}$

and if

$(\bar{t}_0, \dots, \bar{t}_{k-2}, \xi_{k-1}(t_{k-1}), \xi_k(t_k)) \in \overline{W}_k$, $\beta_k(\xi_k(t_k)) = c_k \xi_{k-1}(t_{k-1})$, and

$\bar{\varrho}_{k-1}(\xi_{k-1}(t_{k-1})) = (\bar{t}_0, \dots, \bar{t}_{k-2}, \xi_{k-1}(t_{k-1}))$ then

$\bar{\tau}_k((\bar{t}_0, \dots, \bar{t}_{k-2}, \xi_{k-1}(t_{k-1}), \xi_k(t_k)) = c_k \tau_{k-1}((\bar{t}_0, \dots, \bar{t}_{k-2}, \xi_{k-1}(t_{k-1})))$.

vi) $\varphi_k = \bar{\varrho}_k \circ \bar{\tau}_k$.}

It can be seen that

$\overline{V}_k = \{(\gamma_0(\sigma_0), \dots, \gamma_{k-1}(\sigma_{k-1}), \bar{\sigma}_k) |$ for $i = 0, \dots, k-1$, $\sigma_i \in \Sigma_{\mathfrak{B}}(i)$,

$(\gamma_0(\sigma_0), \dots, \gamma_{k-1}(\sigma_{k-1})) \in V_{k-1}$, σ_0 has the form u_0, and σ_{k-1} has the form $(b_{k-1}\dots b_1, u_0, u_{k-1}[S_{u_{k-1}}, \varphi_{k-1}], \varrho_{k-1}, W_{k-1}, \tau_{k-1})$. There is an element $\sigma_k=(b_k\dots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)\in\Sigma_{\mathfrak{B}}(k)$ such that $(\sigma_0, \dots, \sigma_k)\in V_k$, and there is a mapping $\varepsilon_k\colon S_{u_k}\to A_k[W_{k-1}]_{k-1}$ such that conditions i)—iv) in part (5).b of Definition 3.1 hold, and $\bar\sigma_k=(b_k\dots b_1, u_0, u_k[S_{u_k}, \bar\varphi_k], \bar\varrho_k, \overline{W}_k, \bar\tau_k)$ satisfies the requirements ii)—vi) of $\Sigma_{\mathfrak{C}}(k)$.}

We define $\gamma_k\colon \Sigma_{\mathfrak{B}}(k)\to\Sigma_{\mathfrak{C}}(k)$ as follows: Let us consider the set

$\Gamma_k=\{\gamma\colon \Sigma_{\mathfrak{B}}(k)\to\Sigma_{\mathfrak{C}}(k)|$ for each $\sigma_k\in\Sigma_{\mathfrak{B}}(k)$, $\gamma(\sigma_k)=\bar\sigma_k$

has the form $(b_k\dots b_1, u_0, u_k[S_{u_k}, \varphi_k], \bar\varrho_k, \overline{W}_k, \bar\tau_k)$ and there is a vector $(\sigma_0, \dots, \sigma_{k-1}, \sigma_k)\in V_k$ such that σ_0 has the form u_0, σ_{k-1} has the form

$$(b_{k-1}\dots b_1, u_0, u_{k-1}[S_{u_{k-1}}, \varphi_{k-1}], \varrho_{k-1}, W_{k-1}, \tau_{k-1}),$$

σ_k has the form $(b_k\cdots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$ and there is a mapping $\varepsilon_k\colon S_{u_k}\to A_k[W_{k-1}]_{k-1}$ such that conditions i)—iv) in part (5).b of Definition 3.1 hold, and $\bar\sigma_k$ satisfies the requirements ii)—vi) of $\Sigma_{\mathfrak{C}}(k)\}$.

One can see that if $\bar\gamma_k\in\Gamma_k$ then $\bar\gamma_k$ satisfies condition (2) because of the constructions of $\mathfrak{A}_k$ and $\Sigma_{\mathfrak{C}}(k)$. Using this fact one can see that $|\Gamma_k|=1$. Let γ_k be the only element of Γ_k. One can see that γ_k is surjective. Using the fact that γ_k satisfies condition (2), one can easily prove that the mappings $\gamma_0, \dots, \gamma_k$ satisfy condition (1).

Finally we shall prove, using the fact that for $j=0, \dots, k$ the mappings $\gamma_0, \dots, \gamma_j$ satisfy condition (1) and for $j=0, \dots, k$ the mapping γ_j satisfies condition (2), that $\tau_{\mathfrak{B}}=\tau_{\mathfrak{C}}$.

Assume that $K_0=(e[\{e\}, \psi_0\colon e\to bp], \Theta^0, Z^0, \Omega^0)$ is a starting configuration of $\mathfrak{B}$ and that for a configuration $K_1=(q_1[S_{q_1}, \psi^1], \Theta^1, Z^1, \Omega^1)$, $K_0\Rightarrow^*_{\mathfrak{B}}K_1$ holds. Then K_0 is a starting configuration of $\mathfrak{C}$ as well. We shall show that there is a configuration

$$\overline{K}_1=(q^1[S_{q^1}, \psi^1], \overline{\Theta}^1, \overline{Z}^1, \overline{\Omega}^1)$$

of $\mathfrak{C}$ with bijective correspondences

$$(*)\qquad \begin{aligned} &\alpha_0\colon [Z^1]_0\to[\overline{Z}^1]_0\\ &\quad\vdots\\ &\alpha_k\colon [Z^1]_k\to[\overline{Z}^1]_k \end{aligned}$$

such that α_0 and α_k is the identity function and $K_0\Rightarrow^*_{\mathfrak{C}}\overline{K}_1$ holds, moreover

i) for every $s_k\in S_{q^1}$, $\Theta^1(s_k)=(s_0, s_1, \dots, s_k)$ iff $\Theta^1(\alpha_k(s_k))=(\alpha_0(s_0), \alpha_1(s_1), \dots, \alpha_k(s_k))$ and

ii) $(s_0, s_1, \dots, s_j)\in Z^1$ iff $(\alpha_0(s_0), \alpha_1(s_1), \dots, \alpha_j(s_j))\in\overline{Z}^1$

$(1\leqq j\leqq k, (s_0, s_1, \dots, s_j)\in(N^*)^j)$ and

iii) for every

$(s_0, s_1, \dots, s_j)\in Z^1$ $(1\leqq j\leqq k)$, $\Omega^1((s_0, s_1, \dots, s_j))=\overline{\Omega}^1(\alpha_0(s_0), \alpha_1(s_1), \dots, \alpha_j(s_j)))$.

Conversely, if $K_0\Rightarrow^*_{\mathfrak{C}}\overline{K}_1$ holds then there is a configuration K_1 of $\mathfrak{B}$ and there are bijective functions $(*)$ such that α_0 and α_k are identity functions and i), ii), iii) hold. Hence if K_1 is final then $\overline{K}_1$ is final and vice versa, thus $\tau_{\mathfrak{B}}=\tau_{\mathfrak{C}}$ follows.

First we shall prove the first part of the statement, the second part can be proved similarly. We prove by induction on the length of the transition $K_0 \Rightarrow_{\mathfrak{B}}^* K_1$.

a) The length of $K_0 \Rightarrow_{\mathfrak{B}}^* K_1$ is zero, $(K_1 = K_0)$. Trivial.

b) Assume that the statement is true for $K_0 \Rightarrow_{\mathfrak{B}}^* K_1$, $K_0 \Rightarrow_{\mathfrak{C}}^* \bar{K}_1$ and for the functions $(*)$ and that $K_1 \Rightarrow_{\mathfrak{B}} K_2 = (q^2[S_{q^2}, \psi^2], \Theta^2, Z^2, \Omega^2)$ holds.

By the definition of the relation $\Rightarrow_{\mathfrak{B}}$, there are mappings $\varkappa_i$: $[Z^1]_i \to \Sigma_{\mathfrak{B}}(i)$ for $i=0, 1, \ldots, k$ such that for every $(s_0, s_1, \ldots, s_j) \in Z^1$ $(1 \leqq j \leqq k)$ if

$$\Omega^1((s_0, s_1, \ldots, s_j)) = b_j \ldots b_1 u_0(1, \ldots, \nu(u_0))[\{1, \ldots, \nu(u_0)\}, \vartheta_0]$$

$$(b_j \in A_j, \ldots, b_1 \in A_1, u_0 \in G_0 \cup Y_0, \vartheta_0\colon \{1, \ldots, \nu(u_0)\} \to T_{G_0}(Y_0))$$

then $\varkappa_0(s_0) = u_0$, $\varkappa_i(s_i)$ has the form $(b_i \ldots b_1, u_0, u_i[S_{u_i}, \varphi_i], \varrho_i, W_i, \tau_i)$ for $i=1, \ldots, j$, moreover $(\varkappa_0(s_0), \varkappa_1(s_1), \ldots, \varkappa_j(s_j)) \in V_j$. Take the mappings $\bar{\varkappa}_i$: $[\bar{Z}^1]_i \to \Sigma_{\mathfrak{C}}(i)$ for $i=0, \ldots, k$ defined by $\bar{\varkappa}_i(\alpha_i(s_i)) = \gamma_i(\varkappa_i(s_i))$ for each $s_i \in [Z^1]_i$. Notice, that $\bar{\varkappa}_i$ is well defined, because α_i and γ_i are bijective. By the induction hypothesis for each $(s_0, s_1, \ldots, s_j) \in Z^1$ $(1 \leqq j \leqq k)$, $\Omega^1((s_0, s_1, \ldots, s_j)) = \bar{\Omega}_1((\alpha_0(s_0), \alpha_1(s_1), \ldots \ldots, \alpha_j(s_j)))$. Since $\varkappa_0 = \bar{\varkappa}_0$ and for each $s_i \in [Z^1]_i$ the first two components of $\varkappa_i(s_i)$ are equal to the first two components of $\bar{\varkappa}_i(\alpha_i(s_i))$ for $i=1, \ldots, k$, moreover for every $(\sigma_0, \sigma_1, \ldots, \sigma_j) \in V_j$ $(1 \leqq j \leqq k)$, $(\gamma_0(\sigma_0), \gamma_1(\sigma_1), \ldots, \gamma_j(\sigma_j)) \in \bar{V}_j$ it follows that the mappings $\bar{\varkappa}_i$ $(i=0, 1, \ldots, k)$ satisfy condition (1) in Definition 3.3. The mappings $\bar{\varkappa}_i$ $(i=0, 1, \ldots, k)$ uniquely determine a configuration $\bar{K}_2 = (\bar{q}^2[S_{\bar{q}^2}, \bar{\psi}^2], \bar{\Theta}^2, \bar{Z}^2, \bar{\Omega}^2)$ of $\mathfrak{C}$ such that $\bar{K}_1 \Rightarrow_{\mathfrak{C}} \bar{K}_2$ holds. First we show that $q^2[S_{q^2}, \psi^2] = \bar{q}^2[S_{\bar{q}^2}, \bar{\psi}^2]$. By the transition $K_1 \Rightarrow_{\mathfrak{B}} K_2$ we know that $q^2 = q^1[S_{q^1}, \delta]$, where δ: $S_{q^1} \to T_{G_k}(Y_k \cup N^*)$ satisfies the following formula: for each $s_k \in S_{q^1}$ if $\varkappa_k(s_k) = (b_k \ldots b_1 u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$ then $\delta(s_k) = \omega_{s_k}(u_k)$. By the induction hypothesis and the transition $\bar{K}_1 \Rightarrow_{\mathfrak{C}} \bar{K}_2$ we obtain that $\bar{q}^2 = q^1[S_{q^1}, \bar{\delta}]$, where $\bar{\delta}$: $S_{q^1} \to T_{G_k}(Y_k \cup N^*)$ satisfies the following formula: for every $s_k \in S_{q^1}$ if $\varkappa_k(s_k) = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$ and $\delta(s_k) = \omega_{s_k}(u_k)$ then $\bar{\varkappa}_k(\alpha_k(s_k)) = \bar{\varkappa}_k(s_k) = \gamma_k(\varkappa_k(s_k)) = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \bar{\varrho}_k, \bar{W}_k, \bar{\tau}_k)$ for some $\bar{\varrho}_k, \bar{W}_k$ and $\bar{\tau}_k$, moreover $\bar{\delta}(s_k) = \omega_{s_k}(u_k) = \delta(s_k)$, thus $q^2 = \bar{q}^2$.

Again by the transition $K_1 \Rightarrow_{\mathfrak{B}} K_2$ and $\bar{K}_1 \Rightarrow_{\mathfrak{C}} \bar{K}_2$ we have that ψ^2 and $\bar{\psi}^2$: $S_{q^2} \to A_k \ldots A_1 T_{G_0}(Y_0)$ satisfy the following conditions:

Let $\bar{s}_k \in S_{q^2}$ be arbitrary and consider its unique decomposition $\bar{s}_k = s_k t_k$, where $s_k \in S_{q^1}$, $\delta(s_k) = \omega_{s_k}(u_k)$ for some $u_k (\in P_{G_k}(Y_k))$, $t_k \in S_{u_k}$ and $\varkappa_k(s_k)$ has the form $\varkappa_k(s_k) = (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$. Then if $\varphi_k(t_k) = c_k \ldots c_1 t_0$, $(c_k \ldots \ldots c_1 \in A_k \ldots A_1$, $t_0 \in \{1, \ldots, \nu(u_0)\})$ and $\psi^1(s_k) = u_0(1, \ldots, \nu(u_0))[\{1, \ldots, \nu(u_0)\}, \vartheta_0]$, $(u_0 \in G_0 \cup Y_0, \vartheta_0\colon \{1, \ldots, \nu(u_0)\} \to T_{G_0}(Y_0))$ then $\psi^2(s_k t_k) = c_k \ldots c_1 \vartheta_0(t_0)$.

We know that $\bar{s}_k$ has the same decomposition using $\bar{\delta} = \delta$ and $\bar{\varkappa}_k$, because $\bar{\varkappa}_k(s_k)$ has the form $(b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \bar{\varrho}_k, \bar{W}_k, \bar{\tau}_k)$. Since $\varphi_k(t_k) = c_k \cdots c_1 t_0$ and $\psi^1(s_k) = u_0(1, \ldots, \nu(u_0))[\{1, \ldots, \nu(u_0)\}, \vartheta_0]$ thus $\bar{\psi}^2(s_k t_k) = c_k \ldots c_1 \vartheta_0(t_0)$. We have obtained that $\psi^2 = \bar{\psi}^2$.

$$Z^2 = \{(s_0 t_0, \ldots, s_j t_j) | (s_0, \ldots, s_l) \in Z^1, \quad j \leqq l \leqq k,$$

$$\varkappa_l(s_l) = (b_l \ldots b_1, u_0, u_l[S_{u_l}, \varphi_l], \varrho_l, W_l, \tau_l) \text{ and } (t_0, t_1, \ldots, t_j) \in W_l\}.$$

$$\bar{Z}^2 = \{(\alpha_0(s_0) \bar{t}_0, \ldots, \alpha_j(s_j) \bar{t}_j) | (\alpha_0(s_0), \ldots, \alpha_l(s_l)) \in \bar{Z}^1, \quad j \leqq l \leqq k,$$

$$\text{and } (\bar{t}_0, \ldots, \bar{t}_j) \text{ is a member of the fifth component of } \bar{\varkappa}_l(\alpha_l(s_l)).\}$$

Now we define the mappings α_i^2: $[Z^2]_i \to [\bar{Z}^2]_i$, $(i=0, \ldots, k)$ as follows: Let α_0^2 and α_k^2 be identity mappings, and for $i=1, \ldots, k-1$ take an arbitrary element $s_i t_i \in [Z^2]_i$, where $\varkappa_i(s_i)=(b_i \ldots b_1, u_0, u_i[S_{u_i}, \varphi_i], \varrho_i, W_i, \tau_i)$, then we define $\alpha_i^2(s_i t_i)$ to be $\alpha_i(s_i)\xi_i(t_i)$, where ξ_i: $S_{u_i} \to \{1, \ldots, |S_{u_i}|\}$.

We have to show that for $i=1, \ldots, k-1$ α_i^2 is a bijective function. Let $s_i t_i = \bar{s}_i \bar{t}_i (\in Z_i^2)$ and assume that $s_i \neq \bar{s}_i$. Then one of s_i or $\bar{s}_i$ is a proper initial segment of the other one, which contradicts the definition of the configuration, thus α_i^2 is a well defined function.

Assume that $\alpha_i^2(s_i t_i)=\alpha_i^2(\bar{s}_i \bar{t}_i)$ such that $s_i \neq \bar{s}_i$ or $t_i \neq \bar{t}_i$. If $s_i \neq \bar{s}_i$ then $\alpha_i(s_i) \neq \alpha_i(\bar{s}_i)$ and ξ_i is a function whose range is N thus $\alpha_i(s_i)\xi_i(t_i) \neq \alpha_i(\bar{s}_i)\xi_i(\bar{t}_i)$. If $s_i=\bar{s}_i$ and $t_i \neq \bar{t}_i$ then $\alpha_i^2(s_i t_i)=\alpha_i(s_i)\xi_i(t_i) \neq \alpha_i(s_i)\xi_i(\bar{t}_i)=\alpha^2(\bar{s}_i \bar{t}_i)$ since $\xi_i(t_i) \neq \xi_i(\bar{t}_i)$ thus we obtained that α_i^2 is injective.

Let $\bar{s}_i \bar{t}_i \in [Z^2]_i$, then there is an element $(\bar{s}_0 \bar{t}_0, \ldots, \bar{s}_i \bar{t}_i, \ldots, \bar{s}_j \bar{t}_j) \in Z^2$, where $j \geqq i$. By the construction of Z^2, $(\bar{s}_0, \ldots, \bar{s}_i, \ldots, \bar{s}_j, \ldots, \bar{s}_l) \in Z^1$ for some $\bar{s}_{j+1}, \ldots, \bar{s}_l (\in N^*)$, $1 \leqq j \leqq l \leqq k$, and

$$\bar{\varkappa}_l(\bar{s}_l) = \begin{cases} (b_l \ldots b_1, \sigma_0, \sigma_l(1, \ldots, v^l(\sigma_l))[\{1, \ldots, v^l(\sigma_l)\}, \bar{\varphi}_l], \bar{\varrho}_l, \bar{W}_l, \bar{\tau}_l) & \text{if } l \leqq k, \\ (b_k \ldots b_1, \sigma_0, u_k[S_{u_k}, \varphi_k], \bar{\varrho}_k, \bar{W}_k, \bar{\tau}_k) & \text{if } l = k, \end{cases}$$

and $(\bar{t}_0, \ldots, \bar{t}_i, \ldots, \bar{t}_j) \in W_l$. By the induction hypothesis there is an element $(s_0, \ldots, s_i, \ldots, s_j, \ldots, s_l)$ of Z^1 such that

$$(\alpha_0(s_0), \ldots, \alpha_i(s_i), \ldots, \alpha_j(s_j), \ldots, \alpha_l(s_l)) = (\bar{s}_0, \ldots, \bar{s}_i, \ldots, \bar{s}_j, \ldots, \bar{s}_l).$$

Since $\bar{\varkappa}_l(s_l)=\gamma_l(\varkappa_l(s_l))$ by definition, we can apply condition (2) (ii) stated for γ_l, which tells us that $(\bar{t}_0, \ldots, \bar{t}_i, \ldots, \bar{t}_j) \in \overline{W}_l$ iff there is a $(\xi_0^{-1}(\bar{t}_0), \ldots, \xi_i^{-1}(\bar{t}_i), \ldots, \xi_j^{-1}(\bar{t}_j)) \in W_l$ for $\xi_0, \ldots, \xi_j$ defined in the condition. Thus

$$(s_0 \xi_0^{-1}(\bar{t}_0), \ldots, s_i \xi_i^{-1}(\bar{t}_i), \ldots, s_j \xi_j^{-1}(\bar{t}_j)) \in Z^2,$$

moreover

$$\alpha_i^2(s_i \xi_i^{-1}(\bar{t}_i)) = \alpha_i(s_i)\xi_i(\xi_i^{-1}(\bar{t}_i)) = \alpha_i(s_i)\bar{t}_i = \bar{s}_i \bar{t}_i,$$

hence α_i^2 is surjective $(i=1, \ldots, k-1)$. Thus we have proved that α_i^2 is bijective $(i=0, \ldots, k)$.

Let $\bar{s}_k \in S_{q^2}$ be arbitrary and consider its unique decomposition $\bar{s}_k = s_k t_k$ where $s_k \in S_{q^1}$, $\varkappa_k(s_k)$ has the form $(b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k)$, $\delta(s_k)=\omega_{s_k}(u_k)$, $t_k \in S_{u_k}$. In this case $\bar{\varkappa}_k(s_k)(=\gamma_k(\varkappa_k(s_k)))$ has the form $(b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \bar{\varrho}_k, \overline{W}_k, \bar{\tau}_k)$. Using condition (2) (iv) stated for γ_k, $\varrho_k(t_k)=(t_0, t_1, \ldots, t_k)$ iff $\bar{\varrho}_k(t_k) = (\xi_0(t_0), \xi_1(t_1), \ldots, \xi_k(t_k))$ for $\xi_0, \xi_1, \ldots, \xi_k$ defined in the condition.

Using the induction hypothesis $\Theta^1(s_k)=(s_0, s_1, \ldots, s_k)$ iff $\bar{\Theta}^1(s_k)=(\alpha_0(s_0), \alpha_1(s_1), \ldots, \alpha_k(s_k))$. By the definition of Θ^2 and $\bar{\Theta}^2$ we obtain that

$$\Theta^2(\bar{s}_k) = (s_0 t_0, s_1 t_1, \ldots, s_k t_k) \quad \text{iff}$$

$$\bar{\Theta}^2(\bar{s}_k) = (\alpha_0(s_0)\xi_0(t_0), \alpha_1(s_1)\xi_1(t_1), \ldots, \alpha_k(s_k)\xi_k(t_k)) =$$

$$= (\alpha_0^2(s_0 t_0), \alpha_1^2(s_1 t_1), \ldots, \alpha_k^2(s_k t_k)).$$

Thus we have proved that condition i) holds for the mappings $\alpha_0^2, \ldots, \alpha_k^2$.

Let $(s_0 t_0, \ldots, s_j t_j) \in Z^2$ be arbitrary, where $1 \leqq j \leqq k$ and $(s_0, \ldots, s_j, \ldots, s_l) \in Z^1$ for some $s_{j+1}, \ldots, s_l \in (N^*)$, $(j \leqq l \leqq k)$, moreover $\varkappa_l(s_l)=(b_l \ldots b_1, u_0, u_l[S_{u_l}, \varphi_l],$

$\varrho_l, W_l, \tau_l)$ and $(t_0, \ldots, t_j)\in W_l$. By the induction hypothesis, $(\alpha_0(s_0), \ldots, \alpha_l(s_l))\in \bar{Z}^1$. By the definition of $\bar{\varkappa}_l, \bar{\varkappa}_l(\alpha_l(s_l))=\gamma_l(\varkappa_l(s_l))$, i.e.,

$$\bar{\varkappa}_l(\alpha_l(s_l)) = \begin{cases} (b_l \ldots b_1, u_0, \sigma_l(1, \ldots, v^l(\sigma_l))[\{1, \ldots, v^l(\sigma_l)\}, \bar{\varphi}_l], \bar{\varrho}_l, \overline{W}_l, \bar{\tau}_l) & \text{if } l < k, \\ (b_k \ldots b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k) & \text{if } l = k. \end{cases}$$

We can apply condition (2) (ii) stated for γ_l which tells us that $(\xi_0(t_0), \ldots, \xi_j(t_j))\in\overline{W}_l$ iff $(t_0, \ldots, t_j)\in W_l$ for the mappings $\xi_0, \ldots, \xi_j$ defined in the condition. Thus

$$(\alpha_0^2(s_0 t_0), \ldots, \alpha_j^2(s_j t_j)) = (\alpha_0(s_0)\xi_0(t_0), \ldots, \alpha_j(s_j)\xi_j(t_j))\in Z^2.$$

Conversely, let $(v_0, \ldots, v_j)\in \bar{Z}^2$ be arbitrary. By the construction of the set Z^2 there are two vectors $(\bar{s}_0, \ldots, \bar{s}_j, \ldots, \bar{s}_l)\in Z^1$ $(1\leq j\leq l\leq k)$ and $(\bar{t}_0, \ldots, \bar{t}_j)\in ((N^*)^j)$ such that $v_i=\bar{s}_i\bar{t}_i$ for $i=0, \ldots, j$, and $(\bar{t}_0, \ldots, \bar{t}_j)$ is in the fifth component of $\bar{\varkappa}_l(\bar{s}_l)$. By the induction hypothesis, $(\alpha_0^{-1}(\bar{s}_0), \ldots, \alpha_j^{-1}(\bar{s}_j), \ldots, \alpha_l^{-1}(\bar{s}_l))\in Z^1$. We know that $\bar{\varkappa}_l(\bar{s}_l)= =\gamma_l(\varkappa_l(\alpha_l^{-1}(\bar{s}_l)))$. According to condition (2)(ii) stated for γ_l we obtain that $(\xi_0^{-1}(\bar{t}_0), \ldots, \xi_j^{-1}(\bar{t}_j))$ is in the fifth component of $\varkappa_l(\bar{s}_l)$ for the mappings $\xi_0, \ldots, \xi_j$ defined in the condition. Thus $(\alpha_0^{-1}(\bar{s}_0)\xi_0^{-1}(\bar{t}_0), \ldots, \alpha_j^{-1}(\bar{s}_j)\xi_j^{-1}(\bar{t}_j))\in Z^2$, moreover

$$\alpha_i^2(\alpha_i^{-1}(\bar{s}_i)\xi_i^{-1}(\bar{t}_i)) = \bar{s}_i\bar{t}_i \quad \text{for} \quad i = 0, \ldots, j.$$

We have proved that condition ii) holds for the mappings $\alpha_0^2, \ldots, \alpha_k^2$.

It has remained to prove that condition iii) holds for $\alpha_0^2, \ldots, \alpha_2^k$. Let $(s_0 t_0, \ldots \ldots, s_j t_j)\in Z^2$ be arbitrary, where $1\leq j\leq k$, $(s_0, \ldots, s_j, \ldots, s_l)\in Z^1$ for some $s_{j+1}, \ldots \ldots, s_l(\in N^*)$, $j\leq l\leq k$, and $\varkappa_l(s_l)=(b_l \ldots b_1, u_0, u_l[S_{u_l}, \varphi_l], \varrho_l, W_l, \tau_l)$ and $(t_0, \ldots \ldots, t_j)\in W_l$. We know that $\Omega^1((s_0, \ldots, s_j, \ldots, s_l))=b_l \ldots b_1 u_0(1, \ldots, v(u_0))[\{1, \ldots \ldots, v(u_0)\}, \vartheta_0]$ for some ϑ_0: $\{1, \ldots, v(u_0)\}\to T_{G_0}(Y_0)$, and $\tau_l((t_0, \ldots, t_j))=c_j \ldots c_1 t_0$ for some $c_j\in A_j, \ldots, c_1\in A_1, t_0\in\{1, \ldots, v(u_0)\}$, thus $\Omega^2((s_0 t_0, \ldots, s_j t_j))=c_j \ldots c_1\vartheta_0(t_0)$. By the induction hypothesis, $(\alpha_0(s_0), \ldots, \alpha_j(s_j), \ldots, \alpha_l(s_l))\in Z^1$ and

$$\bar{\varkappa}_l(\alpha_l(s_l)) = \gamma_l(\varkappa_l(s_l)) = \begin{cases} (b_l \ldots b_1, u_0, u_l[S_{u_l}, \varphi_l], \varrho_l, W_l, \tau_l) \quad \text{if} \quad l = k, \\ (b_l \ldots b_1, u_0, \varkappa_l(s_l)(1, \ldots, v^l(\varkappa_l(s_l)))[\{1, \ldots, v^l(\varkappa_l(s_l))\}, \bar{\varphi}_l], \\ \bar{\varrho}_l, \overline{W}_l, \bar{\tau}_l) \quad \text{if} \quad l < k. \end{cases}$$

We can apply condition 2(v) stated for γ_l which tells us that $\tau_l((t_0, \ldots, t_j))=c_j \ldots c_1 t_0$ iff $\bar{\tau}_l((\xi_0(t_0), \ldots, \xi_j(t_j)))=c_j \ldots c_1\xi_0(t_0)$ for the mappings $\xi_0, \ldots, \xi_j$ defined in the condition.

Thus $\bar{\tau}_l((\xi_0(t_0), \ldots, \xi_j(t_j)))=c_j \ldots c_1\xi_0(t_0)$ holds. By the induction hypothesis, $\Omega^1((\alpha_0(s_0), \ldots, \alpha_j(s_j), \ldots, \alpha_l(s_l)))=b_l \ldots b_1 u_0(1, \ldots, v(u_0))[\{1, \ldots, v(u_0)\}, \vartheta_0]$. By the definition of $\bar{\Omega}^2$ and α_i^2 $(i=0, \ldots, k)$,

$$\bar{\Omega}^2((\alpha_0(s_0)\xi_0(t_0), \ldots, \alpha_j(s_j)\xi_j(t_j))) =$$

$$= \bar{\Omega}^2((\alpha_0^2(s_0 t_0), \ldots, \alpha_j^2(s_j t_j))) = c_j \ldots c_1\vartheta_0(t_0).$$

Thus $\Omega^2((s_0 t_0, \ldots, s_j t_j))=\bar{\Omega}^2((\alpha_0^2(s_0 t_0), \ldots, \alpha_j^2(s_j t_j)))$ holds. The proof of the first part of the statement is complete. The second part of the statement can be proved by induction on the length of the transition $K_0\Rightarrow_{\mathfrak{C}}^*\bar{K}_1$.

a) The length of $K_0\Rightarrow_{\mathfrak{C}}^*\bar{K}_1$ is zero, $(\bar{K}_1=K_0)$. Trivial.

b) Assume that the statement is true for $K_0\Rightarrow_{\mathfrak{C}}^*\bar{K}_1$, $K_0\Rightarrow_{\mathfrak{B}}^*K_1$ and for the functions (*) and that $\bar{K}_1\Rightarrow_{\mathfrak{C}}^*\bar{K}_2$ holds. By the definition of the relation $\Rightarrow_{\mathfrak{C}}$ there are mappings

$\bar{\varkappa}_i$: $[\bar{Z}^1]_i \to \Sigma_{\mathfrak{B}}(i)$ for $i=0, 1, ..., k$ such that for every $(\bar{s}_0, \bar{s}_1, ..., \bar{s}_j) \in \bar{Z}^1$ $(1 \leqq j \leqq k)$ if $\bar{\Omega}^1((\bar{s}_0, \bar{s}_1, ..., \bar{s}_j)) = b_j ... b_1 u_0(1, ..., v(u_0))[\{1, ..., v(u_0)\}, \vartheta_0]$ $(b_j \in A_j, ..., b_1 \in A_1,\ u_0 \in G_0 \cup Y_0,\ \vartheta_0$: $\{1, ..., v(u_0)\} \to T_{G_0}(Y_0))$ then $\bar{\varkappa}_0(s_0) = u_0$, for $i=1, ..., j$, $\varkappa_i(s_i)$ has the form

$$\begin{cases} (b_i ... b_1, u_0, \sigma_i(1, ..., v^i(\sigma^i))[\{1, ..., v^i(\sigma_i)\}, \bar{\varphi}_i], \bar{\varrho}_i, \bar{W}_i, \bar{\tau}_i) & \text{if} \quad 1 \leqq i \leqq k-1, \\ (b_k ... b_1, u_0, u_k[S_{u_k}, \varphi_k], \varrho_k, W_k, \tau_k) & \text{if} \quad i = k, \end{cases}$$

moreover $(\bar{\varkappa}_0(\bar{s}_0), ..., \bar{\varkappa}_j(\bar{s}_j)) \in \bar{V}_j$.

Take the mappings $\varkappa_i$: $[Z^1]_i \to \Sigma_{\mathfrak{B}}(i)$ for $i=0, ..., k-1$ defined by $\varkappa_i(s_i) = \gamma_i^{-1}(\bar{\varkappa}_i(\alpha_i(s_i)))$ for each $s_i \in [Z^1]_i$. Notice that $\varkappa_i$ is well defined, because α_i and γ_i are bijective. According to Definition 3.2 for each $\bar{s}_k \in [Z^1]_k$, $\Theta^1(s_k)$ is the only element of $\bar{Z}^1$ which has the form $(\bar{s}_0, ..., \bar{s}_{k-1}, \bar{s}_k)$ for some $\bar{s}_0, ..., \bar{s}_{k-1} \in N^*$. We know that $(\bar{\varkappa}_0(\bar{s}_0), ..., \bar{\varkappa}_{k-1}(\bar{s}_{k-1}), \bar{\varkappa}_k(\bar{s}_k)) \in \bar{V}_k$. We can apply condition (1) stated for $\gamma_0, ..., \gamma_{k-1}, \gamma_k$, which tells us that there is a unique $\sigma_k \in \Sigma_{\mathfrak{B}}(k)$ such that $\gamma_k(\sigma_k) = \bar{\varkappa}_k(\bar{s}_k)$ and $(\gamma_0^{-1}(\varkappa_0(s_0)), ..., \gamma_{k-1}^{-1}(\varkappa_{k-1}(s_{k-1})), \sigma_k) \in V_k$. Let $\varkappa_k(\bar{s}_k)$ be σ_k. By the induction hypothesis for each $(s_0, s_1, ..., s_j) \in Z^1$ $(1 \leqq j \leqq k)$, $\Omega^1((s_0, s_1, ..., s_j)) = \bar{\Omega}^1((\alpha_0(s_0)\ \alpha_1(s_1), ..., \alpha_j(s_j)))$. Since $\varkappa_0 = \bar{\varkappa}_0$ and for each $s_i \in [Z^1]_i$ the first two components of $\varkappa_i(s_i)$ are equal to the first two components of $\bar{\varkappa}_i(\alpha_i(s_i))$ for $i=1, ..., k$, moreover for every $(s_0, s_1, ..., s_j) \in Z^1$ $(1 \leqq j \leqq k)$, $(\varkappa_0(s_0), \varkappa_1(s_1), ..., \varkappa_j(s_j)) \in V_j$ it follows that the mappings $\varkappa_i$ $(i=0, 1, ..., k)$ satisfy condition (1) in the Definition 3.3.

The mappings $\varkappa_i$ $(i=0, 1, ..., k)$ uniquely determine a configuration $K_2 = (q^2[S_{q^2}, \psi^2], \Theta^2, Z^2, \Omega^2)$ such that $K_1 \Rightarrow_{\mathfrak{C}} K_2$ holds.

From now on the proof of the second part of the statement is similar to the proof of the first part.

The proof of the theorem is complete.

4. Example

Let us consider the folloving two R-transducers:

$$\begin{aligned}
\mathfrak{A}_1 &= (G_0, Y_0, A_1, G_1, Y_1, A_1', \Sigma_{\mathfrak{A}_1}), \quad \text{where} \\
G_0 &= G_0^2 = \{g_0\},\ Y_0 = \{x_0\}, \\
G_1 &= G_1^2 = \{g_1\},\ Y_1 = \{x_1, y_1\}, \\
A_1 &= \{a_1, b_1, c_1\},\ A_1' = \{a_1\}, \\
\Sigma_{\mathfrak{A}_1} &= \{b_1 x_0 \to y_1,\ b_1 x_0 \to x_1, \\
&\quad a_1 g_0 \to g_1(1, 2)[\{1, 2\}, \varphi_{11}\colon 1 \mapsto b_1 1;\ \varphi_{11}\colon 2 \mapsto b_1 2], \\
&\quad a_1 g_0 \to g_1(1, 2)[\{1, 2\}, \varphi_{12}\colon 1 \mapsto b_1 1;\ \varphi_{12}\colon 2 \mapsto c_1 2]\}. \\
\tau_{\mathfrak{A}_1} &= \{(g_0(x_0, x_0), g_1(y_1, y_1)),\ (g_0(x_0, x_0), g_1(x_1, x_1)), \\
&\quad (g_0(x_0, x_0), g_1(x_1, y_1)), (g_0(x_0, x_0), g_1(y_1, x_1))\}. \\
\mathfrak{A}_2 &= (G_1, Y_1, A_2, G_2, Y_2, A_2', \Sigma_{\mathfrak{A}_2}), \quad \text{where}
\end{aligned}$$

$$G_2 = G_2^2 = \{g_2\},\ Y_2 = \{x_2, y_2, z_2\},$$
$$A_2 = \{a_2, b_2\},\ A_2' = \{a_2\}.$$
$$\Sigma_{\mathfrak{A}_2} = \{a_2 g_1 \to g_2(1,2)[\{1,2\}, \varphi_{21}\colon 1 \mapsto b_2 1;\ \varphi_{22}\colon 2 \mapsto b_1 1],$$
$$b_2 x_1 \to y_2,\ b_2 x_1 \to z_2,\ b_2 y_1 \to x_2\}.$$

One can see that

$$\tau_{\mathfrak{A}_1} \circ \tau_{\mathfrak{A}_2} = \{(g_0(x_0, x_0), g_2(x_2, x_2)),\ (g_0(x_0, x_0), g_2(y_2, y_2)),$$
$$(g_0(x_0, x_0), g_2(y_2, z_2)),\ (g_0(x_0, x_0), g_2(z_2, y_2)),$$
$$(g_0(x_0, x_0), g_2(z_2, z_2))\}.$$

We construct the 2-synchronized R-transducer $\mathfrak{B}$ according to the Theorem 3.5:

$$\mathfrak{B} = (G_0, G_1, Y_0, Y_1, A_1, A_2, A_1', A_2', \Sigma_{\mathfrak{B}}, V),\quad \text{where}$$
$$\Sigma_{\mathfrak{B}}(0) = V_0 = G_0 \cup Y_0,$$
$$\Sigma_{\mathfrak{B}}(1) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\},\quad \text{where}\quad \sigma_1 = (b_1, x_0, y_1, \emptyset, \emptyset, \emptyset),$$
$$\sigma_2 = (b_1, x_0, x_1, \emptyset, \emptyset, \emptyset),$$
$$\sigma_3 = (a_1, g_0, g_1(1,2)[\{1,2\}, \varphi_3\colon 1 \mapsto b_1 1;\ \varphi_3\colon 2 \mapsto b_1 2],$$
$$\varrho_3\colon 1 \mapsto (1,1);\ \varrho_3\colon 2 \mapsto (2,2), \{(1,1),(2,2)\},$$
$$\tau_3\colon (1,1) \mapsto b_1 1;\ \tau_3\colon (2,2) \mapsto b_1 2),$$
$$\sigma_4 = (a_1, g_0, g_1(1,2)[\{1,2\}, [\varphi_4\colon 1 \mapsto b_1 1;\ \varphi_4\colon 2 \mapsto c_1 2],$$
$$\varrho_4\colon 1 \mapsto (1,1);\ \varrho_4\colon 2 \mapsto (2,2), \{(1,1),(2,2)\},$$
$$\tau_4\colon (1,1) \mapsto b_1 1;\ \tau_4\colon (2,2) \mapsto c_1 2).$$
$$V_1 = \{(x_0, \sigma_1), (x_0, \sigma_2), (g_0, \sigma_3), (g_0, \sigma_4)\}.$$
$$\Sigma_{\mathfrak{B}}(2) = \{\sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9\},\quad \text{where}\quad \sigma_5 = (b_2 b_1, x_0, x_2, \emptyset, \emptyset, \emptyset),$$
$$\sigma_6 = (b_2 b_1, x_0, y_2, \emptyset, \emptyset, \emptyset),\ \sigma_7 = (b_2 b_1, x_0, z_2, \emptyset, \emptyset, \emptyset),$$
$$\sigma_8 = (a_2 a_1, g_0, g_2(1,2)[\{1,2\}, \varphi_8\colon 1 \mapsto b_2 b_1 1;\ \varphi_8\colon 2 \mapsto b_2 b_1 1],$$
$$\varrho_8\colon 1 \mapsto (1,1,1);\ \varrho_8\colon 2 \mapsto (1,1,2), \{(1,1,1),(1,1,2),(2,2)\},$$
$$\tau_8\colon (1,1,1) \mapsto b_2 b_1 1;\ \tau_8\colon (1,1,2) \mapsto b_2 b_1 1;\ \tau_8\colon (2,2) \mapsto b_2 1),$$
$$\sigma_9 = (a_2 a_1, g_0, g_2(1,2)[\{1,2\}, \varphi_9\colon 1 \mapsto b_2 b_1 1;\ \varphi_9\colon 2 \mapsto b_2 b_1 1],$$
$$\varrho_9\colon 1 \mapsto (1,1,1);\ \varrho_9\colon 2 \mapsto (1,1,2), \{(1,1,1),(1,1,2),(2,2)\},$$
$$\tau_9\colon (1,1,1) \mapsto b_2 b_1 1;\ \tau_9\colon (1,1,2) \mapsto b_2 b_1 1;\ \tau_9\colon (2,2) \mapsto c_1 1).$$
$$V_2 = \{(x_0, \sigma_1, \sigma_5), (x_0, \sigma_2, \sigma_6), (x_0, \sigma_2, \sigma_7), (g_0, \sigma_3, \sigma_8), (g_0, \sigma_4, \sigma_9)\}.$$

Let us consider configurations $K_0, K_1, K_2, K_3, K_4, K_5, K_6$ of $\mathfrak{B}$, where K_0 is a starting configuration, K_2, K_3, K_4, K_5, K_6 are final configurations.

$$K_0 = (a_2 a_1 g_0(x_0, x_0), \Theta_0\colon e \mapsto (e, e, e), \{(e, e, e)\}, \quad \Omega_0\colon (e, e, e) \mapsto a_2 a_1 g_0(x_0, x_0)),$$

$$K_1 = (g_2(b_2 b_1 x_0, b_2 b_1 x_0), \Theta_1\colon 1 \mapsto (1, 1, 1);\ \Theta_1\colon 2 \mapsto (1, 1, 2),$$
$$\{(1, 1, 1), (1, 1, 2), (2, 2)\}, \Omega_1\colon (1, 1, 1) \mapsto b_2 b_1 x_0;$$
$$\Omega_1\colon (1, 1, 2) \mapsto b_2 b_1 x_0;\ \Omega_1\colon (2, 2) \mapsto b_1 x_0),$$

$$K_2 = (g_2(x_2, x_2), \emptyset, \emptyset, \emptyset),$$
$$K_3 = (g_2(y_2, y_2), \emptyset, \emptyset, \emptyset),$$
$$K_4 = (g_2(y_2, z_2), \emptyset, \emptyset, \emptyset),$$
$$K_5 = (g_2(z_2, y_2), \emptyset, \emptyset, \emptyset),$$
$$K_6 = (g_2(z_2, z_2), \emptyset, \emptyset, \emptyset).$$

All the transitions from configuration K_0 in $\mathfrak{B}$ which are ended by final configuration are the following:

$$K_0 \Rightarrow_{\mathfrak{B}} K_1 \Rightarrow_{\mathfrak{B}} K_2,$$
$$K_0 \Rightarrow_{\mathfrak{B}} K_1 \Rightarrow_{\mathfrak{B}} K_3,$$
$$K_0 \Rightarrow_{\mathfrak{B}} K_1 \Rightarrow_{\mathfrak{B}} K_4,$$
$$K_0 \Rightarrow_{\mathfrak{B}} K_1 \Rightarrow_{\mathfrak{B}} K_5,$$
$$K_0 \Rightarrow_{\mathfrak{B}} K_1 \Rightarrow_{\mathfrak{B}} K_6.$$

The transition $K_0 \Rightarrow_{\mathfrak{B}} K_1$ is determined by the mappings:

$$\varkappa_0\colon \{e\} \to \Sigma_{\mathfrak{B}}(0);\ \varkappa_0(e) = g_0,$$
$$\varkappa_1\colon \{e\} \to \Sigma_{\mathfrak{B}}(1);\ \varkappa_1(e) = \sigma_3,$$
$$\varkappa_2\colon \{e\} \to \Sigma_{\mathfrak{B}}(2);\ \varkappa_2(e) = \sigma_8.$$

The transition $K_1 \Rightarrow_{\mathfrak{B}} K_2$ is determined by the mappings:

$$\varkappa_0\colon \{1, 2\} \to \Sigma_{\mathfrak{B}}(0);\ \varkappa_0(1) = x_0;\ \varkappa_0(2) = x_0,$$
$$\varkappa_1\colon \{1, 2\} \to \Sigma_{\mathfrak{B}}(1);\ \varkappa_1(1) = \sigma_1;\ \varkappa_1(2) = \sigma_2,$$
$$\varkappa_2\colon \{1, 2\} \to \Sigma_{\mathfrak{B}}(2);\ \varkappa_2(1) = \sigma_5;\ \varkappa_2(2) = \sigma_5.$$

The transition $K_1 \Rightarrow_{\mathfrak{B}} K_3$ is determined by the mappings:

$$\varkappa_0\colon \{1, 2\} \to \Sigma_{\mathfrak{B}}(0);\ \varkappa_0(1) = x_0;\ \varkappa_0(2) = x_0,$$
$$\varkappa_1\colon \{1, 2\} \to \Sigma_{\mathfrak{B}}(1);\ \varkappa_1(1) = \sigma_2;\ \varkappa_1(2) = \sigma_1,$$
$$\varkappa_2\colon \{1, 2\} \to \Sigma_{\mathfrak{B}}(2);\ \varkappa_2(1) = \sigma_6;\ \varkappa_2(2) = \sigma_6.$$

The transition $K_1 \Rightarrow_{\mathfrak{B}} K_4$ is determined by the mappings:

$\varkappa_0$: $\{1, 2\} \to \Sigma_{\mathfrak{B}}(0)$; $\varkappa_0(1) = x_0$; $\varkappa_0(2) = x_0$,

$\varkappa_1$: $\{1, 2\} \to \Sigma_{\mathfrak{B}}(1)$; $\varkappa_1(1) = \sigma_2$; $\varkappa_1(2) = \sigma_2$,

$\varkappa_2$: $\{1, 2\} \to \Sigma_{\mathfrak{B}}(2)$; $\varkappa_2(1) = \sigma_6$; $\varkappa_2(2) = \sigma_7$.

The transition $K_1 \Rightarrow_{\mathfrak{B}} K_5$ is determined by the mappings:

$\varkappa_0$: $\{1, 2\} \to \Sigma_{\mathfrak{B}}(0)$; $\varkappa_0(1) = x_0$; $\varkappa_0(2) = x_0$,

$\varkappa_1$: $\{1, 2\} \to \Sigma_{\mathfrak{B}}(1)$; $\varkappa_1(1) = \sigma_2$; $\varkappa_1(2) = \sigma_2$,

$\varkappa_2$: $\{1, 2\} \to \Sigma_{\mathfrak{B}}(2)$; $\varkappa_2(1) = \sigma_7$; $\varkappa_2(2) = \sigma_6$.

The transition $K_1 \Rightarrow_{\mathfrak{B}} K_6$ is determined by the mappings:

$\varkappa_0$: $\{1, 2\} \to \Sigma_{\mathfrak{B}}(0)$; $\varkappa_0(1) = x_0$; $\varkappa_0(2) = x_0$,

$\varkappa_1$: $\{1, 2\} \to \Sigma_{\mathfrak{B}}(1)$; $\varkappa_1(1) = \sigma_2$; $\varkappa_1(2) = \sigma_2$,

$\varkappa_2$: $\{1, 2\} \to \Sigma_{\mathfrak{B}}(2)$; $\varkappa_2(1) = \sigma_7$; $\varkappa_2(2) = \sigma_7$.

One can see that $\tau_{\mathfrak{B}} = \tau_{\mathfrak{A}_1} \circ \tau_{\mathfrak{A}_2}$.

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