A note on zero-congruences

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In this note, a theorem is proved about zero-congruences in iteration theories. If T is an iteration theory (see [BEW1], [BEW2], [Es] and θ is a theory congruence (not necessarily a dagger congruence), we call θ a **zero congruence** if $f\theta g$ for all morphisms $f, g\colon 1\to 0$ in T. Recall that a theory congruence θ on T is a family of equivalence relations $\theta_{n,p}$ on $T(n,p)$, $n, p\geqq 0$, which are preserved by composition and source pairing. A theory congruence θ is a dagger congruence if $f^\dagger\theta g^\dagger$ whenever $f\theta g$.

1. Theorem. *For any iteration theory T, the least zero congruence is a dagger congruence.*

The proof is constructive in the sense that the least zero congruence is described explicitly. The theorem is of interest for the following reason. For any set A, let Pfn (A) denote the iteration theory whose morphisms $n\to p$ are the partial functions $A\times[n]\to A\times[p]$. Let PFN denote the variety of all iteration theories generated by those theories of the form Pfn (A). In a forthcoming paper by Bloom and Ésik, it is shown that for any ranked set Γ, there is an iteration theory freely generated by Γ in the variety PFN. This theory may be described as the quotient of the theory Γtr freely generated by Γ in the variety of all iteration theories by the least zero congruence. In the course of the study of that argument, it was discovered that the least zero congruence on Γtr automatically preserved dagger. We wondered if this was a general phenomenon. The theorem shows that it is.

2. Definition. Let $\varrho=\varrho_{n,p}$ be the family of binary relations on $T(n,p)$ defined as follows: for any morphisms $f, f'\colon n\to p$, $f\varrho_{n,p}f'$ if there are morphisms $g\colon n\to k+p$ and $b, b'\colon k\to 0$ in T such that $f=g\cdot(b+1_p)$ and $f'=g\cdot(b'+1_p)$.

Note that ϱ is symmetric. Let ϱ^* denote the reflexive, transitive closure of ϱ, which is to say that $f\varrho^*f'$ iff there is a finite sequence $f_1, f_2, \ldots, f_n$ with $f=f_1, f'=f_n$, and $f_i\varrho f_{i+1}$ for $1\leqq i\leqq n-1$, $n\geqq 1$.

3. Lemma. *If f, f', g, g' are morphisms in T with the appropriate sources and targets, and if $f\varrho f'$, $g\varrho g'$, then*

a) $$\langle f, g\rangle\varrho\langle f', g'\rangle,$$

b) $$f\cdot g\,\varrho f'\cdot g',$$

and

c) $$f^\dagger \varrho\, g^\dagger$$

Proof of *a)*. Suppose that

$$f = F\cdot(b+1_p),\quad f' = F\cdot(b'+1_p),$$

where $F\colon n\to k+p$ and $b, b'\colon k\to 0$. Suppose further that

$$g = G\cdot(c+1_p),\quad g' = G\cdot(c'+1_p),$$

where $G\colon m\to r+p$ and $c, c'\colon r\to 0$. Then

$$\langle f, g\rangle = H\cdot(b+c+1_p) \quad\text{and}\quad \langle f', g'\rangle = H\cdot(b'+c'+1_p),$$

where $H=\langle F\cdot(\varkappa+1_p), G\cdot(\lambda+1_p)\rangle$, $\varkappa=1_k+0_r$, and $\lambda=0_k+1_r$.

Proof of *b)*. We assume

$$f = F\cdot(b+1_p),\quad f' = F\cdot(b'+1_p) \quad\text{and}\quad g = G\cdot(c+1_s),\quad g' = G\cdot(c'+1_s),$$

where $F\colon n\to k+p$ and $G\colon p\to r+s$. Then

$$f\cdot g = F\cdot(b+1_p)\cdot G\cdot(c+1_s) = F\cdot(1_k+G)\cdot(b+c+1_s)$$

and

$$f'\cdot g' = F\cdot(1_k+G)\cdot(b'+c'+1_s).$$

Proof of *c)*. Suppose that $f=F\cdot(b+1_{p+n})$ and $g=F\cdot(b'+1_{p+n})$, where $F\colon n\to k+p+n$ and $b, b'\colon k\to 0$ in T. Then $f^\dagger=F^\dagger\cdot(b+1_p)$ and $g^\dagger=F^\dagger(b'+1_p)$.

It follows immediately from Lemma 3 that ϱ^* is a theory congruence.

4. Lemma. *ϱ^* is the least zero congruence.*

Proof. Let θ be any zero congruence. It is clear that if $f\varrho g$, then $f\theta g$. Thus $f\varrho^* g$ implies $f\theta g$. If θ is the least zero congruence, the converse also holds (i.e. if $f\theta g$, then $f\varrho^* g$).

The proof of the theorem follows from the preceding two facts.

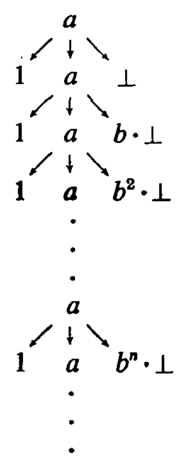

Figure 1

5. Remark. Aside from the fact that iteration theories are algebraic theories, the only property of iteration theories used in the above proof is the validity of the identity

$$[F\cdot(b+1_{p+n})]^{\dagger} = F^{\dagger}\cdot(b+1_p),$$

for any $F: n\to k+p+n$, $b: k\to 0$. Since this identity is also valid in all iterative theories, and in all (ordered) rational theories [ADJ], the theorem holds for these theories as well.

6. Example. Let T be the iteration theory of all Γ-trees (not just those of finite index [EBT]). Let $a: 1\to 3$, $b: 1\to 1$, $\perp : 1\to 0$ be atomic. Let f be the infinite tree indicated in Figure 1. Note that f has infinitely many subtrees $1\to 0$. If g is the tree indicated in Figure 2, then g is $h^{\dagger}$, where $h=a\cdot(1_2+\perp): 1\to 2$. Note that f and g are not congruent by the least zero congruence, since clearly f is not related by ϱ^* to g. Among the trees related by ϱ^* to f are those indicated in Figure 3.

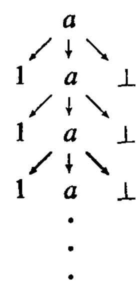

Figure 2

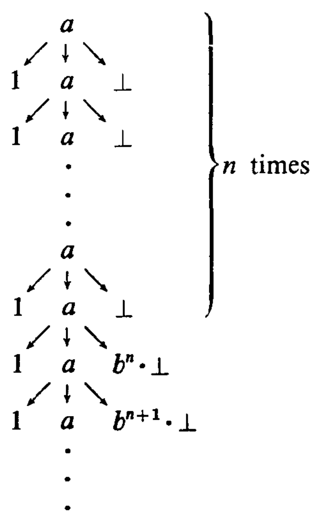

Figure 3

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References

[ADJ] J. B. Wright, J. W. Thatcher, E. G. Wagner, "Rational algebraic theories and fixed-point solutions", Proceedings 17th IEEE Symposium, Foundations of Computing, Houston (1976).

[BEW1] S. L. Bloom, C. C. Elgot, J. B. Wright, "Solutions of the iteration equation and extensions of the scalar iteration operation", SIAM J. Computing, 9 (1980), 25—45.

[BEW2] S. L. Bloom, C. C. Elgot, J. B. Wright, "Vector iteration in pointed iterative theories", SIAM J. Computing, 9 (1980), 525—540.

[EBT] C. Elgot, S. L. Bloom, R. Tindell, "On the algebraic structure of rooted trees", Journal of Computer and System Sciences, 16 (1978), 362—399.

[Es] Z. Ésik, "Identities in iterative and rational theories", Comput. Linguistics and Comput. Languages XIV (1980) 183—207.

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