# On the expected behaviour of the *NF* algorithm for a dual bin-packing problem

J. CSIRIK, G. GALAMBOS

## Introduction

The following version of dual bin-packing problems was first studied by Assmann et al [1]: there is given a list $L=\{a_1, a_2, \ldots, a_n\}$ of items (elements) and a size $s(a_i)$ for each item. Let us denote $C$ as a positive constant, $C \geqq \max_{1 \leqq i \leqq n} s(a_i)$. The aim is to pack the elements into a *maximum* number of bins so that the sum of the sizes in any given bin is *at least* $C$. (The name "dual" originates from the "classical" bin-packing problem, where the elements have to be packed into the *minimum* number of bins in such a way that the sum of the sizes of the elements in any given bin is *at most* $C$). This problem is *NP-hard* and hence the investigation of the performance of approximation algorithms is important. Without loss of generality, we may assume that $C=1$ and $0<s(a_i)\leqq 1$; if $a_i$ is real, then $s(a_i)=a_i$.

The worst-case behaviour of the well-known heuristic algorithms Next-Fit (*NF*) and Next-Fit Decreasing (*NFD*) was analysed in [1]. The *NF* algorithm places $a_1$ into the first bin ($B_1$). Let us suppose that $a_i$, $i>1$, is to be packed, and let $B_j (j \geqq 1)$ be the highest indexed non-empty bin. The algorithm places $a_i$ into $B_j$ if the sum of elements in this bin (so far) is smaller than 1; otherwise it closes the bin $B_j$, opens a new bin ($B_{j+1}$) and places the element $a_i$ into this newly-opened bin. The *NFD* algorithm differs from *NF* only in preordering the elements. The worst-case behaviour of an approximation algorithm may be characterized by means of the asymptotic worst-case ratio. To define this, let $OPT(L)$ be the maximum possible number of bins for a given instance $L$. For a given approximation algorithm $A$, let $A(L)$ denote the number of bins used by $A$ to pack $L$. Let

$R_A^N = \min \{A(L)/OPT(L) : L$ is an instance with $OPT(L)=N\}$.

The asymptotic worst-case ratio for $A$ is then defined as

$$R_A^\infty = \liminf_{N\to\infty} R_A^N$$

Assmann et al proved that

$$R_{NF}^\infty = R_{NFD}^\infty = 1/2$$

In the last part of that paper the average-case behaviour of these (and other) algorithms was investigated. The behaviours of the algorithms were compared in randomly generated instances, where the elements of $L$ were drawn from different distributions. In the conclusion of the paper it was suggested that the expected performance of these algorithms should be examined analytically as well. In [3] this analysis has been carried out for both algorithms. Csirik et al [3] showed that if the elements of $L=(a_1, a_2, \ldots, a_n)$ are identically distributed and drawn independently from a uniform distribution on $(0, 1/k]$ ($k$ is a positive integer), then

| | | $k=1$ | $k=2$ | $k=3$ | $k=4$ |
|---|---|---|---|---|---|
| | $R^{\infty}_{NF}$ | 0.735 | 0.8564 | 0.900 | 0.923 |
| and | $R^{\infty}_{NFD}$ | 0.710 | 0.840 | 0.891 | 0.918 |

On the other hand, Knödel [4] showed that the first-fit ($FF$) algorithm is asymptotically optimal for the "classical" bin-packing problem if the elements of the input list are drawn independently from the following distribution:

$$a_i = \begin{cases} 1/3 \text{ with probability } 1/3, \\ 2/3 \text{ with probability } 1/3, \\ \;1 \;\text{ with probability } 1/3. \end{cases}$$

(In the $FF$ algorithm we try to pack the element $a_i$ into all opened bins, i.e. into $B_1, B_2, \ldots, B_i$, and open a new bin only if none of them has enough room for it.)

Csirik [2] generalized this result for the input sequence:

$$a_i = \begin{cases} b \quad\;\; \text{with probability } 1/2, \\ 1-b \text{ with probability } 1/2, \end{cases} \tag{I}$$

where $0<b<1/2$ and it was proved that the $FF$ is asymptotically optimal for these sequences, too.

In this note we investigate the expected behaviour of the $NF$ algorithm at sequence (I) for the dual version of the bin-packing problem.

## Results

First we present our method for special lists. Let the elements of $L=(a_1, a_2, \ldots, a_n)$ be chosen independently of the following distribution:

$$a_i = \begin{cases} 1/3 \text{ with probability } 1/2, \\ 2/3 \text{ with probability } 1/2. \end{cases} \tag{1}$$

Let us denote by $E_n$ the expected number of full bins for the lists $L=(a_1, a_2, \ldots, a_n)$ (the elements are drawn independently from (1)), and by $E_{n,k}$ the expected number of bins for lists with a number $k$ of 2/3 elements (and so a number $(n-k)$ of 1/3 elements) if we pack $L$ by $NF$. Then

$$E_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} E_{n,k} \tag{2}$$

On the other hand, in the packing of $L$ by $NF$ the first bin will be full after packing $a_1$ and $a_2$ if at least one of them is a 2/3 element. If both of them are 1/3 elements, then the first bin is full after the packing of $a_3$. Thus, we have the following recursion:

$$E_{n,k} = \frac{k(k-1)}{n(n-1)}(E_{n-2,k-2}+1)+2\frac{k(n-k)}{n(n-1)}(E_{n-2,k-1}+1)+$$

$$+\frac{(n-k)(n-k-1)}{n(n-1)}\left(\frac{n-k-2}{n-2}E_{n-3,k}+\frac{k}{n-2}E_{n-3,k-1}+1\right), \quad \text{if} \quad n \geqq 3 \quad \text{and} \quad k \geqq 1. \tag{3}$$

It is easy to see that $E_{n,0}=\lfloor n/3\rfloor$, $E_{1,1}=0$, $E_{2,1}=1$, $E_{2,2}=1$.
Using (3), from (2) we get:

$$E_n = \frac{3}{4}E_{n-2}+\frac{1}{4}E_{n-3}+1 \tag{4}$$

and we know that $E_0=0$, $E_1=0$, $E_2=3/4$.
Let us search $E_n$ in the following form:

$$E_n = \frac{n}{2}(1-A)-B_n \tag{5}$$

Then from (4) we have

$$B_n = \frac{3}{4}B_{n-2}+\frac{1}{4}B_{n-3}+\frac{1}{8}(1-9A) \tag{6}$$

Our aim is to give the asymptotic behaviour of $E_n$. From (5) it would be enough to choose an appropriate $A$ so that $|B_n|<T$, where $T$ is a constant. If now $A=1/9$, then from (6)

$$B_n = \frac{3}{4}B_{n-2}+\frac{1}{4}B_{n-3} \tag{7}$$

and from (4) and (5)

$$B_1 = \frac{4}{9}, \quad B_2 = \frac{5}{36}, \quad B_3 = \frac{1}{3}$$

By induction on $i$, we can prove from (7) that for all $i \geqq 4$

$$5/36 \leqq B_i \leqq 4/9$$

and hence $B_i$ is bounded. But then from (5) we have the following

**Lemma.**

$$\lim_{n\to\infty}\frac{E_n}{n/2} = 8/9.$$

We now generalize our result for the following input sequences: let the elements of $L=(a_1, a_2, \ldots, a_n)$ be independent, identically distributed, random variables with distribution (I). Let $l_1=\lceil 1/b\rceil$. We use the notation $E_n$ in the above sense, and let $E_{n,k}$ denote the expected number of bins for the lists $L=(a_1, a_2, \ldots, a_n)$ with a number $k$ of elements $1-b$. Then (2) is true for these sequences as well, and our lemma is valid for $l_1=3$.

In the packing $L$ by $NF$ we have two cases:

1. If $a_1=1-b$, then the first bin is always full after the packing of $a_2$.

2. If $a_1=b$, then the first bin is full with the first element $1-b$ in the sequel $a_2, a_3, \ldots, a_{l_1-1}$. If all of $a_2, \ldots, a_{l_1-1}$ are equal $b$, then the first bin is full after packing of the element $a_{l_1}$.

Similarly to (3), from these two cases we have

$$E_{n,k} = \frac{k(k-1)}{n(n-1)}(E_{n-2,k-2}+1)+2\frac{k(n-k)}{n(n-1)}(E_{n-2,k-1}+1)+$$

$$+\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}(E_{n-3,k-1}+1)+\ldots+\frac{(n-k)(n-k-1)\ldots(n-k-l_1+3)k}{n(n-1)\ldots(n-l_1+2)}\times$$

$$\times(E_{n-l_1+1,k-1}+1)+\frac{(n-k)(n-k+1)\ldots(n-k-l_2+2)}{n(n-1)\ldots(n-l_1+2)}\times$$

$$\times\left(\frac{n-k-l_1+1}{n-l_1+1}E_{n-l_1,k}+\frac{k}{n-l_1+1}E_{n-l_1,k-1}+1\right) \tag{8}$$

and hence from (2)

$$E_n = \frac{3}{4}E_{n-2}+\frac{1}{2^3}E_{n-3}+\ldots+\frac{1}{2^{l_1-1}}E_{n-l_1+1}+\frac{1}{2^{l_1-1}}E_{n-l_1}+1 \tag{9}$$

We look for $E_n$ again in the form given in (5). Then

$$B_n = \frac{3}{4}B_{n-2}+\frac{1}{2^3}B_{n-3}+\ldots+\frac{1}{2^{l_1-1}}B_{n-l_1+1}+\frac{1}{2^{l_1-1}}B_{n-l_1}+$$

$$+(1-A)\left[\frac{n}{2}\left(1-\frac{3}{4}-\frac{1}{2^3}-\ldots-\frac{1}{2^{l_1-1}}-\frac{1}{2^{l_1-1}}\right)\right]+$$

$$+(1-A)\left(\frac{3}{4}\cdot\frac{2}{2}+\frac{1}{2^3}\cdot\frac{3}{2}+\frac{1}{2^4}\cdot\frac{4}{2}+\ldots+\frac{1}{2^{l_1-1}}\frac{l_1-1}{2}+\frac{1}{2^{l_1-1}}\frac{1}{2}\right)-1 =$$

$$= \frac{3}{4}B_{n-2}+\frac{1}{2^3}B_{n-3}+\ldots+\frac{1}{2^{l_1-1}}B_{n-l_1+1}+\frac{1}{2^{l_1-1}}B_{n-l_1}+\frac{2^{l_1-2}-1}{2^{l_1}}-A\frac{2^{l_1}+2^{l_1-2}-1}{2^{l_1}}. \tag{10}$$

If we now choose

$$A = \frac{2^{l_1-2}-1}{2^{l_1}+2^{l_1-2}-1}$$

then

$$B_n = \frac{3}{4}B_{n-2}+\frac{1}{2^3}B_{n-3}+\ldots+\frac{1}{2^{l_1-1}}B_{n-l_1+1}+\frac{1}{2^{l_1-1}}B_{n-l_1}$$

and thus $B_n$ is again a bounded sequence. Accordingly, we have proved our main result:

**Theorem.** Let the elements of $L=(a_1, a_2, \dots, a_n)$ be independent, identically distributed, random variables with distribution

$$a_i = \begin{cases} b & \text{with probability } 1/2, \\ 1-b & \text{with probability } 1/2, \end{cases}$$

where $0<b<1/2$. Let $l_1=\lceil 1/b \rceil$. If we pack the list $L$ by the *NF* algorithm and if $E_n$ denotes the expected number of filled bins, then

$$\lim_{n\to\infty} \frac{E_n}{n/2} = \frac{2^{l_1}}{2^{l_1}+2^{l_1-2}-1}$$

J. CSIRIK
DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1.
SZEGED
HUNGARY

G. GALAMBOS
KALMÁR LABORATORY
OF CYBERNETICS
ÁRPÁD TÉR 2.
SZEGED
HUNGARY

## References

[1] ASSMANN, S. F., JOHNSON, D. S., KLEITMAN, D. J., LEUNG, J. Y. T.: On a dual version of the one-dimensional Bin-packing problem, Journal of Algorithm, 5 (1984), 505.
[2] CSIRIK, J.: Bin packing as a random walk: a note on Knödel's paper, Op. Res. Letters, 5 (1986), 161.
[3] CSIRIK, J., FRENK, J. B. G., GALAMBOS, G., RINNOOY KAN, A. H. G.: Expected performance of simple algorithms for a dual bin-packing problem, to be publised.
[4] KNÖDEL, W.: Über das mittlere Verhalten von On-Line Packungs Algorithmen, EIK 19 (1983), 427.

*(Received Apr. 6, 1986)*